Oscillation of a time fractional partial differential equation

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Abstract. We consider a time fractional partial differential equation subject to the Neumann boundary condition. Several sufficient conditions are established for oscillation of solutions of such equation by using the integral averaging method and a generalized Riccati technique. The main results are illustrated by examples.

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1 Introduction

In recent years differential equations with fractional order derivatives have attracted many researchers because of their applications in many areas of science and engineering. The need for fractional order differential equations originates in part from the fact that many phenomena cannot be modeled by differential equations with integer derivatives. Analytical and numerical techniques have been developed to study such equations. The fractional calculus has allowed the operations of integration and differentiation to be applied. Recently, the theory of fractional differential equations and their applications have been attracting more and more attention in the literature [1, 3, 4, 11, 12, 16–18, 20, 24, 25, 27]. Fractional differential equations are generalizations of classical differential equations of integer order and have gained considerable importance due to their various applications in viscoelasticity, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electroanalytical chemistry, optics and signal processing, control theory, electrical networks, probability and statistics and economics, etc.

Nowadays the interest in the study of fractional-order differential equations lies in the fact that fractional-order models are more accurate than integer-order ones, that is, there are more degrees of freedom in the fractional-order models. Fractional-order differential equations are

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also better in the description of hereditary properties of various materials and processes than integer-order differential equations. It is found that various applications can be elegantly modeled with the help of the fractional differential equations [6, 10, 15]. Also fractional differential and integral equations provide in some cases more accurate models of systems under consideration.

The study of oscillation theory for various equations like ordinary and partial differential equations, difference equation, dynamics equation on time scales and fractional differential equations is an interesting area of research and much effort has been made to establish oscillation criteria for these equations [7, 9, 14, 19, 21, 22, 26, 28]. Recently the research on fractional differential equation is a hot topic and only very few publications paid the attention to oscillation of fractional differential equation; see for example [2, 5, 7, 13, 23].

However, to the best of our knowledge, very little is known regarding the oscillatory behavior of fractional differential equations. But the study of oscillatory behavior of fractional partial differential equation is initiated in this paper. To develop the qualitative properties of fractional partial differential equations, it is of great interest to study the oscillatory behavior of fractional partial differential equation. In this paper, we establish several oscillation criteria for fractional partial differential equation by applying a generalized Riccati transformation technique and certain parameter functions. These results are considered essentially new. We also provide two examples to illustrate the results.

In this paper, we consider the time fractional partial differential equation of the form

$$\frac{\partial}{\partial t}\left(r(t)D^\alpha_{+,t}u(x,t)\right) + q(x,t)f\left(\int_0^t (t-v)^{-\alpha}u(x,v)\,dv\right) = a(t)\Delta u(x,t),$$
$$(x,t) \in G = \Omega \times \mathbb{R}_+,$$

(1.1)

with the Neumann boundary condition

$$\frac{\partial u(x,t)}{\partial N} = 0,\ (x,t) \in \partial\Omega \times \mathbb{R}_+,$$

(1.2)

where $\alpha \in (0,1)$ is a constant, $D^\alpha_{+,t}u$ is the Riemann–Liouville fractional derivative of order α of u with respect of t, Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian operator and N is the unit exterior normal vector to $\partial\Omega$.

Throughout this paper, we assume that the following conditions hold:

(A_1) $r(t) \in C^1([0,\infty); [0,\infty))$, $a \in C([0,\infty); \mathbb{R}_+)$;

(A_2) $q(x,t) \in C(\overline{G}; [0,\infty))$ and $\min_{x\in\Omega} q(x,t) = Q(t)$;

(A_3) $f: \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(u)/u \geq \mu$ for certain constant $\mu > 0$ and for all $u \neq 0$.

By a solution of equation (1.1) we mean a function $u(x,t) \in C^{1+\alpha}(\overline{\Omega} \times [0,\infty))$ such that $\int_0^t (t-v)^{-\alpha}u(x,v)\,dv \in C^1(\overline{G}; \mathbb{R})$, $D^\alpha_{+,t}u(x,t) \in C^1(\overline{G}; \mathbb{R})$ and satisfies (1.1) on $\overline{G}$.

A solution u of (1.1) is said to be oscillatory in G if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

2 Preliminaries

In this section, we give the definitions of fractional derivatives and integrals and a lemma which are useful throughout this paper.

There are several kinds of definitions of fractional derivatives and integrals. In this paper, we use the Riemann–Liouville left-sided definition on the half-axis $\mathbb{R}_+$. The following notations tions will be used for our convenience

$$v(t) = \int_\Omega u(x,t)\,dx.$$

Definition 2.1. The Riemann–Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a function $u(x,t)$ is given by

$$(D_{+,t}^\alpha u)(x,t) := \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-v)^{-\alpha}u(x,v)\,dv \tag{2.1}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$ where Γ is the gamma function.

Definition 2.2. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y\colon \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$(I_+^\alpha y)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1}y(v)\,dv \ \text{ for } \ t > 0 \tag{2.2}$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$.

Definition 2.3. The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $y\colon \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by

$$
\begin{aligned}
(D_+^\alpha y)(t) &:= \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\left(I_+^{\lceil\alpha\rceil-\alpha}y\right)(t) \\
&= \frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\int_0^t (t-v)^{\lceil\alpha\rceil-\alpha-1}y(v)\,dv \ \text{ for } \ t > 0
\end{aligned}
\tag{2.3}
$$

provided the right hand side is pointwise defined on $\mathbb{R}_+$ where $\lceil\alpha\rceil$ is the ceiling function of α.

Lemma 2.4. *Let y be a solution of* (1.1) *and*

$$G(t) := \int_0^t (t-v)^{-\alpha}y(v)\,dv \text{ for } \alpha \in (0,1) \text{ and } t > 0. \tag{2.4}$$

Then

$$G'(t) = \Gamma(1-\alpha)(D_+^\alpha y)(t). \tag{2.5}$$

Proof. From (2.3) and (2.4), for $\alpha \in (0,1)$ and $t > 0$, we obtain

$$
\begin{aligned}
G'(t) &= \Gamma(1-\alpha)\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-v)^{-\alpha}y(v)\,dv \\
&= \Gamma(1-\alpha)\left[\frac{1}{\Gamma(\lceil\alpha\rceil-\alpha)}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\int_0^t (t-v)^{\lceil\alpha\rceil-\alpha-1}y(v)\,dv\right] \\
&= \Gamma(1-\alpha)(D_+^\alpha y)(t).
\end{aligned}
$$

The proof is complete. $\qquad\square$

3   Main results

Theorem 3.1. *If the fractional differential inequality*

$$\frac{d}{dt}\left[r(t)D_+^\alpha v(t)\right] + Q(t)f(G(t)) \le 0 \tag{3.1}$$

has no eventually positive solution, then every solution of (1.1) *and* (1.2) *is oscillatory in G.*

Proof. Suppose that u is a nonoscillatory solution of (1.1) and (1.2). Without loss of generality we may assume that $u(x,t) > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Integrating (1.1) over Ω, we obtain

$$\frac{d}{dt}\left[r(t)\left(\int_\Omega (D_{+,t}^\alpha u)(x,t)dx\right)\right] + \int_\Omega q(x,t)f\left(\int_0^t (t-v)^{-\alpha}u(x,v)dv\right)dx$$
$$= a(t)\int_\Omega \Delta u(x,t)\,dx. \tag{3.2}$$

Using Green's formula, it is obvious that

$$\int_\Omega \Delta u(x,t)dx \le 0, \ t \ge t_1. \tag{3.3}$$

By using Jensen's inequality and (A_2), we have

$$\int_\Omega q(x,t)f\left(\int_0^t (t-v)^{-\alpha}u(x,v)\,dv\right)dx$$
$$\ge Q(t)f\left(\int_\Omega \left(\int_0^t (t-v)^{-\alpha}u(x,v)\,dv\right)dx\right)$$
$$= Q(t)f\left(\int_0^t (t-v)^{-\alpha}\left(\int_\Omega u(x,v)\,dx\right)dv\right). \tag{3.4}$$

Combining (3.2)–(3.4) and using definitions, we have

$$\frac{d}{dt}\left[r(t)D_+^\alpha v(t)\right] + Q(t)f\left(G(t)\right) \le 0. \tag{3.5}$$

Therefore $v(t)$ is an eventually positive solution of (3.1). This contradicts the hypothesis and completes the proof. □

Theorem 3.2. *Suppose that the conditions (A_1)–(A_3) and*

$$\int_{t_0}^\infty \frac{1}{r(t)}\,dt = \infty \tag{3.6}$$

hold. Furthermore, assume that there exists a positive function $c \in C^1[t_0, \infty)$ such that

$$\limsup_{t\to\infty}\int_{t_1}^t \left[\mu\, c(s)Q(s) - \frac{1}{4}\frac{(c'(s))^2\, r(s)}{c(s)\Gamma(1-\alpha)}\right]ds = \infty. \tag{3.7}$$

Then every solution of (3.1) *is oscillatory.*

Proof. Suppose that $v(t)$ is a nonoscillatory solution of (3.1). Without loss of generality we may assume that v is an eventually positive solution of (3.1). Then there exists $t_1 \geq t_0$ such that $v(t) > 0$ and $G(t) > 0$ for $t \geq t_1$. Then it is obvious that

$$[r(t)D_+^\alpha(v(t))]' \leq -Q(t)f(G(t)) < 0, \ t \geq t_0. \tag{3.8}$$

Thus $D_+^\alpha v(t) \geq 0$ or $D_+^\alpha v(t) < 0$, $t \geq t_1$ for some $t_1 \geq t_0$. We now claim that $(D_+^\alpha v(t)) \geq 0$ for $t \geq t_1$. Suppose not, then $(D_+^\alpha v(t)) < 0$ and there exists $T \geq t_1$ such that $(D_+^\alpha v(T)) < 0$. Since $[r(t)(D_+^\alpha v(t))]' < 0$ for $t \geq t_1$, it is clear that $r(t)(D_+^\alpha v(t))e < r(T)(D_+^\alpha v(T))$ for $t \geq T$. Therefore, from (2.4), we have

$$\frac{G'(t)}{\Gamma(1-\alpha)} = (D_+^\alpha v(t)) \leq \frac{r(T)(D_+^\alpha v(T))}{r(t)}.$$

Integrating the above inequality from T to t, we have

$$\frac{G(t) - G(T)}{\Gamma(1-\alpha)} = r(T)(D_+^\alpha v(T)) \int_T^t \frac{1}{r(s)} \, ds$$

$$G(t) = G(T) - \Gamma(1-\alpha)r(T)(D_+^\alpha v(T)) \int_T^t \frac{1}{r(s)} \, ds$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} G(t) \leq -\infty$ which is a contradiction. Hence $(D_+^\alpha v(t)) \geq 0$ for $t \geq t_1$ holds.

Define the function w by the generalized Riccati substitution

$$w(t) = c(t)\frac{r(t)(D_+^\alpha v(t))}{G(t)} \text{ for } t \geq t_1 \tag{3.9}$$

Then we have $w(t) > 0$ for $t \geq t_1$. From (A_3), (2.4), (3.1) and (3.9), it follows that

$$w'(t) = c'(t)\left[\frac{r(t)(D_+^\alpha v(t))}{G(t)}\right] + c(t)\left[\frac{(r(t)(D_+^\alpha v(t)))'}{G(t)} - \frac{G'(t)r(t)(D_+^\alpha v(t))}{G^2(t)}\right]$$

$$\leq \frac{c'(t)}{c(t)}w(t) - \frac{c(t)Q(t)f(G(t))}{G(t)} - \frac{c(t)r(t)\Gamma(1-\alpha)(D_+^\alpha v(t))^2}{G^2(t)}$$

$$\leq \frac{c'(t)}{c(t)}w(t) - \mu\, c(t)Q(t) - \frac{\Gamma(1-\alpha)}{c(t)r(t)}w^2(t) \tag{3.10}$$

$$= -\mu\, c(t)Q(t) - \left(\sqrt{\frac{\Gamma(1-\alpha)}{c(t)r(t)}}w(t) - \frac{1}{2}\sqrt{\frac{c(t)r(t)}{\Gamma(1-\alpha)}}\frac{c'(t)}{c(t)}\right)^2 + \frac{1}{4}\frac{(c'(t))^2 r(t)}{c(t)\Gamma(1-\alpha)}$$

$$\leq -\mu\, c(t)Q(t) + \frac{1}{4}\frac{(c'(t))^2 r(t)}{c(t)\Gamma(1-\alpha)}.$$

Integrating both sides from t_1 to t, we have

$$w(t) \leq w(t_1) - \int_{t_1}^t \left[\mu\, c(s)Q(s) - \frac{1}{4}\frac{(c'(s))^2\, r(s)}{c(s)\Gamma(1-\alpha)}\right] ds. \tag{3.11}$$

Letting $t \to \infty$, we get $\lim_{t\to\infty} w(t) \leq -\infty$ which contradicts (3.7) and completes the proof. $\square$

Corollary 3.3. *Let assumption* (3.7) *in Theorem* 3.2 *be replaced by*

$$\limsup_{t\to\infty} \int_{t_1}^t c(s)Q(s) \, ds = \infty \tag{3.12}$$

$$and \quad \limsup_{t\to\infty} \int_{t_1}^t \frac{r(s)(c'(s))^2}{c(s)} \, ds < \infty. \tag{3.13}$$

Then every solution of (3.1) *oscillates.*

From Theorem 3.2, by choosing the function c appropriately, we obtain different sufficient conditions for oscillation of (3.1) and if we define a function c by $c(t) = 1$ and $c(t) = t$, we have the following oscillation results.

Corollary 3.4. *Suppose that* (3.6) *holds. If*

$$\limsup_{t\to\infty} \int_{t_1}^{t} Q(s)\,ds = \infty, \tag{3.14}$$

then every solution of (3.1) *oscillates.*

Corollary 3.5. *Suppose that* (3.6) *holds. If*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\mu\, sQ(s) - \frac{1}{4}\frac{r(s)}{s\Gamma(1-\alpha)} \right] ds = \infty, \tag{3.15}$$

then every solution of (3.1) *oscillates.*

For the following theorem, we introduce a class of functions $\mathcal{R}$. Let

$$\mathbb{D}_0 = \{\, (t,s) : t > s \geq t_0 \,\}, \quad \mathbb{D} = \{\, (t,s) : t \geq s \geq t_0 \,\}.$$

The function $H \in C(\mathbb{D}, \mathbb{R})$ is said to belong to the class $\mathcal{R}$, if

(i) $H(t,t) = 0$, for $t \geq t_0$, $H(t,s) > 0$, for $(t,s) \in \mathbb{D}_0$;

(ii) H has a continuous and non-positive partial derivative $\dfrac{\partial H(t,s)}{\partial s}$ on $\mathbb{D}_0$ with respect to s.

We assume that $\xi(t)$ for $t \geq t_0$ are given continuous functions such that $\xi(t) \geq 0$ and differentiable and define

$$\theta(t) = \frac{c'(t)}{c(t)} + 2\Gamma(1-\alpha)\xi(t), \quad \chi(t) = c(t)[r(t)\xi(t)]' - \Gamma(1-\alpha)c(t)r(t)\xi^2(t).$$

Theorem 3.6. *Suppose that the conditions* (A_1)–(A_4) *and* (3.6) *hold. Furthermore assume that there exists* $H \in \mathcal{R}$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[(\mu\, c(s)q(s) - \chi(s))H(t,s) - \frac{1}{4}\frac{c(s)r(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)} \right] ds = \infty. \tag{3.16}$$

Then every solution of (3.1) *is oscillatory.*

Proof. Suppose that $v(t)$ is a nonoscillatory solution of (3.1). Without loss of generality we may assume that v is an eventually positive solution of (3.1). Then there exists $t_1 \geq t_0$ such that $v(t) > 0$ and $G(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.2, we obtain $(D_+^\alpha v(t)) \geq 0$ for $t \geq t_1$. Now we define the Riccati substitution w by

$$w(t) = c(t)\left[\frac{r(t)(D_+^\alpha v)(t)}{G(t)} + r(t)\xi(t) \right], \tag{3.17}$$

Then we have

$$
\begin{aligned}
w'(t) = {} & c'(t)\left[\frac{r(t)(D_+^\alpha v)(t)}{G(t)} + r(t)\xi(t) \right] \\
& + c(t)\left[\frac{(r(t)(D_+^\alpha v)(t))'}{G(t)} - \frac{r(t)G'(t)(D_+^\alpha v)(t)}{G^2(t)} + (r(t)\xi(t))' \right] \\
\leq {} & \frac{c'(t)}{c(t)}w(t) + c(t)[r(t)\xi(t)]' - \mu\, c(t)Q(t) - \Gamma(1-\alpha)\frac{c(t)}{r(t)}\left[\frac{w(t)}{c(t)} - r(t)\xi(t) \right]^2. \tag{3.18}
\end{aligned}
$$

Let $A = \frac{w(t)}{c(t)}$, $B = r(t)\xi(t)$. By applying the inequality [8],

$$A^{(1+\alpha)/\alpha} - (A - B)^{(1+\alpha)/\alpha} \leq B^{1/\alpha} \left[\left(1 + \frac{1}{\alpha}\right) A - \frac{1}{\alpha}B \right], \text{ for } \alpha = \frac{\text{odd}}{\text{odd}} \geq 1,$$

we see that

$$\left[\frac{w(t)}{c(t)} - r(t)\xi(t) \right]^2 = \left[\frac{w(t)}{c(t)} \right]^2 + [r(t)\xi(t)]^2 - \frac{2r(t)\xi(t)}{c(t)}w(t). \tag{3.19}$$

Substituting (3.19) into (3.18), we have

$$w'(t) \leq \left[\frac{c'(t)}{c(t)} + 2\Gamma(1-\alpha)\xi(t) \right] w(t) - \frac{\Gamma(1-\alpha)}{c(t)r(t)}w^2(t) - \mu\, c(t)Q(t)$$
$$+ \left[c(t)[r(t)\xi(t)]' - \Gamma(1-\alpha)c(t)r(t)\xi^2(t) \right]$$
$$\leq \theta(t)w(t) + \chi(t) - \mu\, c(t)Q(t) - \frac{\Gamma(1-\alpha)}{c(t)r(t)}w^2(t).$$

Multiplying both sides by $H(t,s)$ and integrating from t_1 to t, for $t \geq t_1$, we have

$$\int_{t_1}^{t} [\mu\, c(s)Q(s) - \chi(s)]H(t,s)\, ds \leq -\int_{t_1}^{t} H(t,s)w'(s)ds + \int_{t_1}^{t} H(t,s)\theta w(s)\, ds$$
$$- \int_{t_1}^{t} \frac{\Gamma(1-\alpha)}{c(s)r(s)}w^2(s)H(t,s)\, ds. \tag{3.20}$$

Using the integration by parts formula, we get

$$-\int_{t_1}^{t} H(t,s)w'(s)\, ds = -[H(t,s)w(s)]_{t_1}^{t} + \int_{t_1}^{t} H'_s(t,s)w(s)\, ds$$
$$< H(t,t_1)w(t_1) + \int_{t_1}^{t} H'_s(t,s)w(s)\, ds. \tag{3.21}$$

Substituting (3.21) into (3.20), we have

$$\int_{t_1}^{t} [\mu\, c(s)Q(s) - \chi(s)]H(t,s)ds$$
$$\leq H(t,t_1)w(t_1) + \int_{t_1}^{t} \left[[H'_s(t,s) + H(t,s)\theta(s)]w(s) - \frac{\Gamma(1-\alpha)H(t,s)}{c(s)r(s)}w^2(s) \right] ds$$
$$\leq H(t,t_1)w(t_1) + \int_{t_1}^{t} \left[h(t,s)w(s) - \frac{\Gamma(1-\alpha)H(t,s)}{c(s)r(s)}w^2(s) \right] ds$$
$$\leq H(t,t_1)w(t_1) + \int_{t_1}^{t} \left[\sqrt{\frac{\Gamma(1-\alpha)H(t,s)}{c(s)r(s)}}w(s) - \frac{1}{2}\sqrt{\frac{c(s)r(s)}{\Gamma(1-\alpha)H(t,s)}}h(t,s) \right]^2 ds$$
$$+ \frac{1}{4}\int_{t_1}^{t} \frac{c(s)r(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)}\, ds$$
$$\leq H(t,t_1)w(t_1) + \frac{1}{4}\int_{t_1}^{t} \frac{c(s)r(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)}\, ds,$$

which yields

$$\int_{t_1}^{t} \left[[\mu\, c(s)Q(s) - \chi(s)]H(t,s) - \frac{1}{4}\int_{t_1}^{t} \frac{c(s)r(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)} \right] ds \leq H(t,t_1)w(t_1).$$

Since $0 < H(t,s) \le H(t,t_1)$ for $t > s \le t_1$, we have $0 < \frac{H(t,s)}{H(t,t_1)} \le 1$ for $t > s \le t_1$. Hence we have

$$\frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[(\mu \, c(s)q(s) - \chi(s))H(t,s) - \frac{1}{4} \frac{c(s)r(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)} \right] ds \le w(t_1).$$

Letting $t \to \infty$, we have

$$\limsup_{t \to \infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[(\mu \, c(s)q(s) - \chi(s))H(t,s) - \frac{1}{4} \frac{c(s)r(s)h^2(t,s)}{\Gamma(1-\alpha)H(t,s)} \right] ds \le w(t_1)$$

which contradicts (3.16) and completes the proof. □

In Theorem 3.6, if we choose $H(t,s) = (t-s)^\lambda$, $t \ge s \ge t_1$, where $\lambda > 1$ is a constant, then we obtain the following corollaries.

Corollary 3.7. *Under the conditions of Theorem 3.6, if*

$$\limsup_{t \to \infty} \frac{1}{(t-t_1)^\lambda} \int_{t_1}^{t} \left[(\mu c(s)Q(s) - \chi(s))(t-s)^\lambda - \frac{1}{4} \frac{c(s)r(s)((t-s)\theta(s) - \lambda)}{\Gamma(1-\alpha)(t-s)} \right] ds < \infty,$$

then every solution of (3.1) is oscillatory.

4 Examples

Example 4.1. Consider the time-fractional partial differential equation

$$\frac{\partial}{\partial t} (D_{+,t}^\alpha u(x,t)) + \frac{e^x}{t^2} \left(\int_0^t (t-v)^{-\alpha} u(x,v) dv \right) = \frac{e^t}{4} \Delta u(x,t), \; (x,t) \in (0,\pi) \times (0,\infty), \quad (4.1)$$

with the boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0,$$

where $\alpha \in (0,1)$, In (4.1), $r(t) = 1$, $Q(t) = \min_{x \in \Omega} q(x,t) = \min_{x \in (0,\pi)} \frac{e^x}{t^2} = \frac{1}{t^2}$, $a(t) = \frac{e^t}{4}$ and $f(u) = u$. Take $t_0 > 0$ and $\mu = 1$. Thus all the conditions of the theorem (3.6) hold. Therefore every solution of (4.1) is oscillatory.

Example 4.2. Consider the time-fractional partial differential equation

$$\frac{\partial}{\partial t} (t^2 D_{+,t}^\alpha u(x,t)) + \frac{2}{t^2} \exp \left(\int_0^t (t-v)^{-\alpha} u(x,v) \, dv \right) \cdot \left(\int_0^t (t-v)^{-\alpha} u(x,v) \, dv \right)$$
$$= \frac{t}{2} \Delta u(x,t), \; (x,t) \in (0,\pi) \times (0,\infty), \quad (4.2)$$

with the boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0,$$

where $\alpha \in (0,1)$, In (4.2), $r(t) = t^2$, $Q(t) = \min_{x \in \Omega} q(x,t) = \min_{x \in (0,\pi)} \frac{2}{t^2} = \frac{2}{t^2}$, $a(t) = \frac{t}{2}$ and $f(u) = e^u u$. Take $t_0 > 0$ and $\mu = 1$. Thus all the conditions of the theorem (3.6) hold. Therefore every solution of (4.2) is oscillatory.

Acknowledgements

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