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Asymptotic integration of linear differential-algebraic equations

Vu Hoang Linh ✉ [1] and **Nguyen Ngoc Tuan**[2]

[1] Faculty of Mathematics, Mechanics and Informatics, Vietnam National University, Hanoi, Vietnam
[2] Department of Mathematics, Hung Yen University of Technology and Education, Hung Yen, Vietnam

Dedicated to the memory of Professor Katalin Balla (1947–2005)

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Abstract. This paper is concerned with the asymptotic behavior of solutions of linear differential-algebraic equations with asymptotically constant coefficients. Some results of asymptotic integration which are well known for ordinary differential equations (ODEs) are extended to differential-algebraic equations (DAEs).

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1 Introduction

Linear differential-algebraic equations (DAEs) are equations of the form

$$E(t)x'(t) = A(t)x(t), \quad t \in \mathbb{I}, \tag{1.1}$$

where $E, A \in C(\mathbb{I}, \mathbb{C}^{n\times n})$ with $n \in \mathbb{N}$, $\mathbb{I} = [t_0, \infty)$, and $E(t)$ is assumed to be singular for all $t \in \mathbb{I}$. Linear systems of the form (1.1) may occur when one linearizes a general nonlinear system of DAEs

$$F(t, x(t), x'(t)) = 0, \quad t \in \mathbb{I}, \tag{1.2}$$

along a particular solution $x^*(t)$, where $F\colon \mathbb{I} \times \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is assumed to be sufficiently smooth.

Differential-algebraic equations are also called singular differential equations which are generalizations of ordinary differential equations (ODEs). They play an important role in mathematical modeling arising in multibody mechanics, electrical circuits, prescribed path control, chemical engineering, etc., see [4, 16, 25].

The qualitative theory and numerical analysis of DAEs are more difficult than ODEs because the equations cannot be solved explicitly for the derivative and hidden algebraic constraints may be involved. The difficulties are usually characterized by different index notions.

✉ Corresponding author. Email: linhvh@vnu.edu.vn

In the last two decades, the existence and uniqueness theory, the stability analysis, and the numerical treatment for DAEs, particularly for lower-index systems, have already been fairly well established, see [16, 18, 24].

In many problems, detailed information about the asymptotic behavior of solutions nearby singular points is useful. For example, it becomes desirable when one tries to formulate an approximate initial or boundary condition in the neighbourhood of singular points. The first asymptotic integration results for ODEs were given a long time ago by Levinson and others, see [6, 15, 19]. Later, further extensions of these classical results were carried out by many authors [2, 11, 13, 14, 22, 27]. Recently, there have been many contributions to the stability and the asymptotic behavior of solutions of DAEs, e.g. see [1, 3, 5, 7, 8, 9, 10, 17, 20, 21, 26] and references therein. However, up to our knowledge, asymptotic integration results are still missing in the DAE literature. Therefore, the purpose of this paper is to extend classical asymptotic integration results from linear ODEs to linear DAEs.

In this paper, we consider linear asymptotically constant coefficient differential-algebraic equations of the form

$$[E + F(t)]x'(t) = [A + B(t) + R(t)]x(t), \quad t \geq t_0, \tag{1.3}$$

where $E, A \in \mathbb{C}^{n\times n}$, F, B, $R \in C(\mathbb{I}; \mathbb{C}^{n\times n})$, and constant matrix E is assumed to be singular. Typically, the terms F, B and R play the role of perturbations which may arise, for example, in the linearization process or in the course of modeling. The main question is that if perturbations F, B and R are supposed to be sufficiently small in some sense, how certain solutions of (1.3) are related to those of the unperturbed DAEs, which are with constant coefficients and quite well understood. In particular, the behavior of solutions as t tends to infinity is of interest.

In order to characterize the asymptotic behavior of solutions of (1.3), we first transform the system into the semi-implicit form, i.e., the system is transformed into a coupled system consisting of an implicit differential equation and an algebraic one. Here, we use the decomposing procedure for index-1 DAEs, e.g. see [23], and the well-known Kronecker–Weierstraß canonical form [4, 12, 16] for the higher index case. Then, conditions for perturbations F, B and R are given so that asymptotic formulas for solutions of (1.3) are explicitly obtained, which show the asymptotic equivalence between the solutions of (1.3) and those of the corresponding constant-coefficient DAEs. These results generalize the well-known asymptotic integration results for linear ODEs. In addition, we show that perturbations arising in the leading term and for higher-index DAEs must be of appropriate structure. Otherwise, the asymptotic behavior of solutions of perturbed DAEs may be completely different from that of solutions of unperturbed DAEs. This is the main difference between the asymptotic integration results for ODEs and those for DAEs.

The paper is organized as follows. In the next section, we summarize some basic results from the theory of DAEs. In Section 3, we present the main result on the asymptotic integration for index-1 DAEs with perturbations arising only on the right hand side. Then, extensions to the case of the perturbed leading term and to the case of higher index DAEs are investigated in Sections 4 and 5. Some examples are also included for illustration. We close the paper by a conclusion and a suggestion for future works.

2 Preliminaries

Consider linear constant-coefficient DAEs

$$Ex'(t) = Ax(t), \quad t \in \mathbb{I}, \tag{2.1}$$

where E, A are as in (1.3).

The matrix pencil $\{E, A\}$ is said to be *regular* if there exists $\lambda \in \mathbb{C}$ such that the determinant $\det(\lambda E - A)$ is nonzero. Otherwise, if $\det(\lambda E - A) = 0$, for all $\lambda \in C$, then we say that $\{E, A\}$ is irregular or non-regular. If $\{E, A\}$ is regular, then $\lambda \in C$ is a (generalized finite) eigenvalue of $\{E, A\}$ and a nonzero vector ζ is the associated eigenvector if $\lambda E\zeta = A\zeta$. It is known that the system (2.1) is solvable if and only if the matrix pencil $\{E, A\}$ is regular [4, 12, 16]. The following theorem is known as the Kronecker–Weierstraß canonical form, which plays an important role in the analysis of linear constant-coefficient DAEs.

Theorem 2.1. *Suppose that $\{E, A\}$ is a regular pencil. Then, there exist nonsingular matrices G and H such that*

$$GEH = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad GAH = \begin{bmatrix} J_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix}, \tag{2.2}$$

where $n_1 + n_2 = n$, J_{n_1} is a $n_1 \times n_1$ matrix and N is a matrix of nilpotency index k, i.e., $N^k = 0$, but $N^{k-1} \neq 0$. If N is a zero matrix, then we define $k = 1$.

Without loss of generality, we may assume that N and J_{n_1} are given in the Jordan canonical form. The *index* of the pencil $\{E, A\}$ is defined by the nilpotency index of the matrix N in (2.2).

For index-1 DAEs, the following reduction of (2.1) can be realized in practice, e.g., see [23]. Let the matrix E in (2.1) satisfy $\operatorname{rank}(E) = n_1$, where $1 \leq n_1 < n$ and let the matrices $U \in \mathbb{C}^{n \times n_2}$ and $V \in \mathbb{C}^{n \times n_2}$ be such that their columns form (minimal) bases for the left and right null-spaces of E, respectively, i.e.,

$$U^T E = 0, \quad EV = 0. \tag{2.3}$$

Then, we define the matrices

$$\mathbb{U} = \begin{bmatrix} U^\perp & U \end{bmatrix}, \quad \mathbb{V} = \begin{bmatrix} V^\perp & V \end{bmatrix}, \tag{2.4}$$

where $U^\perp$ and $V^\perp$ are the bases of the orthogonal subspaces associated with U and V. Letting

$$x = \mathbb{V} \begin{bmatrix} u^T & v^T \end{bmatrix}^T,$$

where $u(t) \in \mathbb{C}^{n_1}$ and $v(t) \in \mathbb{C}^{n_2}$, and multiplying (2.1) by $\mathbb{U}^T$, we obtain

$$\begin{aligned} E_{11}u' &= A_{11}u + A_{12}v, \\ 0 &= A_{21}u + A_{22}v, \end{aligned} \tag{2.5}$$

where

$$E_{11} = U^{\perp^T} E V^\perp, \tag{2.6}$$

and

$$A_{11} = U^{\perp^T} A V^\perp, \; A_{12} = U^{\perp^T} A V, \; A_{21} = U^T A V^\perp, \; A_{22} = U^T A V. \tag{2.7}$$

The matrix E_{11} is invertible since $\operatorname{rank}(E_{11}) = \operatorname{rank}(\mathbb{U}^T E \mathbb{V}) = \operatorname{rank}(E) = n_1$. In practice, the transformation matrices $\mathbb{U}$ and $\mathbb{V}$ can be computed from the singular value decomposition (SVD) of E. Namely, their columns are left and right singular vectors of E, respectively. Thus, the transformation matrices $\mathbb{U}$ and $\mathbb{V}$ are orthogonal.

It is easy to see that $\{E, A\}$ is regular of index-1 if and only if the matrix A_{22} is nonsingular. In this case, then from the second equation of the system (2.5), we imply that

$$v = -A_{22}^{-1} A_{21} u. \tag{2.8}$$

Substituting the equation (2.8) into the first equation of the system (2.5) and then multiplying by E_{11}^{-1}, we obtain an ODE

$$u' = E_{11}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})u, \tag{2.9}$$

which is called the *essential underlying* ODE. The asymptotic integration of ODEs under small perturbations is a well-established topic of the qualitative theory. In the next section, by using the transformed system (2.5), first we extend the classical ODE results of asymptotic integration, e.g. see [6], to index-1 DAEs of the form (2.1) with perturbations arising on the right hand side.

3 Asymptotic solutions for index-1 DAEs

In this section, first we consider the perturbed DAEs of the form

$$Ex'(t) = [A + R(t)]x(t), \quad t \in \mathbb{I}, \tag{3.1}$$

where $E, A \in \mathbb{C}^{n\times n}$, the pencil $\{E, A\}$ is of index-1, and $R \in C(\mathbb{R}_+; \mathbb{C}^{n\times n})$. We will show that if R is sufficiently small in some sense, then the asymptotic behavior of the solutions of (3.1) is determined by the solutions of the unperturbed system (2.1).

Let the matrices U, V, $\mathbb{U}$, and $\mathbb{V}$ be defined by (2.3) and (2.4) in Section 2. Multiplying (3.1) by $\mathbb{U}^T$ and substituting

$$x = \mathbb{V}\begin{bmatrix} u^T & v^T \end{bmatrix}^T,$$

we obtain

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u' \\ v' \end{bmatrix} = \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} R_{11}(t) & R_{12}(t) \\ R_{21}(t) & R_{22}(t) \end{bmatrix}\right)\begin{bmatrix} u \\ v \end{bmatrix}, \tag{3.2}$$

where E_{11}, A_{ij}, $i, j = 1, 2$, are defined as in (2.6) and (2.7), and

$$R_{11}(t) = U^{\perp^T} R(t)V^{\perp},\ R_{12}(t) = U^{\perp^T} R(t)V,\ R_{21}(t) = U^T R(t)V^{\perp},\ R_{22}(t) = U^T R(t)V. \tag{3.3}$$

Since matrix E_{11} is invertible, then we obtain

$$\begin{aligned} u' &= E_{11}^{-1}(A_{11} + R_{11}(t))u + E_{11}^{-1}(A_{12} + R_{12}(t))v, \\ 0 &= (A_{21} + R_{21}(t))u + (A_{22} + R_{22}(t))v, \end{aligned} \tag{3.4}$$

which is a DAE system in semi-explicit form. In order to investigate the asymptotic behavior of solutions of equation (3.1), we make some assumptions.

Assumption 3.1. Suppose that $\sup_{t\ge t_0} \|R_{22}(t)\| < \|A_{22}^{-1}\|^{-1}$ holds.

Then, it is easy to see that $(A_{22} + R_{22}(t))$ is invertible for all $t \ge t_0$ and the inverse is uniformly bounded. From now on, we omit the argument t of the coefficients for simplicity, where no confusion arises.

It follows from the second equation of (3.4) that

$$v = -(A_{22} + R_{22})^{-1}(A_{21} + R_{21})u. \tag{3.5}$$

By reformulating

$$(A_{22} + R_{22})^{-1} = A_{22}^{-1} - (A_{22} + R_{22})^{-1}R_{22}A_{22}^{-1},$$

the equation (3.5) can be rewritten as

$$v = -\left[A_{22}^{-1}A_{21} + \widetilde{R}_{21}(t)\right]u, \tag{3.6}$$

where $\widetilde{R}_{21}(t) = A_{22}^{-1}R_{21} - (A_{22}+R_{22})^{-1}R_{22}A_{22}^{-1}(A_{21}+R_{21})$.

Substituting (3.6) into the first equation of system (3.4), we obtain the following ODE for the differential component u

$$u' = E_{11}^{-1}\Big[A_{11} - A_{12}A_{22}^{-1}A_{21} + R_{11} - A_{12}A_{22}^{-1}R_{21} - R_{12}A_{22}^{-1}(A_{21}+R_{21}) \\ +(A_{12}+R_{12})(A_{22}+R_{22})^{-1}R_{22}A_{22}^{-1}(A_{21}+R_{21})\Big]\,u.$$

Let us denote

$$\widetilde{A}_{11} = E_{11}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21}),$$

and

$$\widetilde{R}_{11} = E_{11}^{-1}\left[R_{11} - A_{12}A_{22}^{-1}R_{21} - \left(R_{12}A_{22}^{-1} - (A_{12}+R_{12})(A_{22}+R_{22})^{-1}R_{22}A_{22}^{-1}\right)(A_{21}+R_{21})\right].$$

Then we obtain

$$u' = [\widetilde{A}_{11} + \widetilde{R}_{11}(t)]u. \tag{3.7}$$

Assumption 3.2. Let $R_{2j}(t) \to 0$, $j = 1,2$, as $t \to \infty$.

Assumption 3.3. Let the matrix function R be absolutely integrable on $[0;\infty)$, i.e.,

$$\int_{t_0}^{\infty} \|R(t)\|\,dt < \infty. \tag{3.8}$$

Theorem 3.4. *Let Assumptions 3.1, 3.2, and 3.3 hold and the matrix $\widetilde{A}_{11}$ be similar to a diagonal matrix J. Suppose that ξ_j is an eigenvector associated with an eigenvalue μ_j of the pencil $\{E, A\}$, i.e., $\mu_j E\xi_j = A\xi_j$. Then, the system (3.1) has a solution $\varphi_j(t)$ such that*

$$\lim_{t\to\infty} \varphi_j(t)e^{-\mu_j t} = \xi_j.$$

Proof. Let

$$\xi_j = \mathbb{V}\left[\ \xi_j^{1^T}\xi_j^{2^T}\ \right]^T,$$

where $\xi_j^1 \in \mathbb{C}^{n_1}$ and $\xi_j^2 \in \mathbb{C}^{n_2}$. From the equality $\mu_j E\xi_j = A\xi_j$, we imply that

$$\mu_j \mathbb{U}^T E\mathbb{V}\begin{bmatrix} \xi_j^1 \\ \xi_j^2 \end{bmatrix} = \mathbb{U}^T A\mathbb{V}\begin{bmatrix} \xi_j^1 \\ \xi_j^2 \end{bmatrix}$$

or equivalently,

$$\begin{aligned} \mu_j E_{11}\xi_j^1 &= A_{11}\xi_j^1 + A_{12}\xi_j^2, \\ 0 &= A_{21}\xi_j^1 + A_{22}\xi_j^2. \end{aligned} \tag{3.9}$$

Since the matrix A_{22} is invertible, and from the second equation of the system (3.9), we obtain

$$\xi_j^2 = -A_{22}^{-1}A_{21}\xi_j^1. \tag{3.10}$$

Substituting (3.10) into the first equation of the system (3.9), we have

$$\mu_j \xi_j^1 = E_{11}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})\xi_j^1. \tag{3.11}$$

Hence, μ_j is an eigenvalue and ξ_j^1 is an associated eigenvector of the matrix $\widetilde{A}_{11} = E_{11}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})$.

Now, we consider the essential underlying system (3.7). It is easy to show that, taking into account the formula of $\widetilde{R}_{11}$, Assumptions 3.1 and 3.3 imply that $\widetilde{R}_{11}$ is absolutely integrable. It follows from [6, p. 104, Prob. 29] that the system (3.7) has a solution $u_j(t)$ such that

$$\lim_{t\to\infty} u_j(t)e^{-\mu_j t} = \xi_j^1.$$

Under Assumptions 3.1 and 3.2, it is easy to check that $\widetilde{R}_{21}(t) \to 0$ as $t \to \infty$. Therefore, from the equality (3.10), the corresponding algebraic component $v_j(t)$ determined by

$$v_j(t) = -\big(A_{22}^{-1}A_{21} + \widetilde{R}_{21}(t)\big)u_j(t)$$

satisfies

$$\lim_{t\to\infty} v_j(t)e^{-\mu_j t} = -\lim_{t\to\infty} \big(A_{22}^{-1}A_{21} + \widetilde{R}_{21}(t)\big)u_j(t)e^{-\mu_j t} = -A_{22}^{-1}A_{21}\xi_j^1 = \xi_j^2.$$

Thus, the function

$$\varphi_j(t) = \mathbb{V}\begin{bmatrix} u_j(t) \\ v_j(t) \end{bmatrix}$$

is a solution of equation (3.1) and it satisfies

$$\lim_{t\to\infty} \varphi_j(t)e^{-\mu_j t} = \lim_{t\to\infty} \mathbb{V}\begin{bmatrix} u_j(t)e^{-\mu_j t} \\ v_j(t)e^{-\mu_j t} \end{bmatrix} = \mathbb{V}\begin{bmatrix} \xi_j^1 \\ \xi_j^2 \end{bmatrix} = \xi_j.$$

The proof of Theorem 3.4 is complete. □

Remark 3.5. It is well known that $\widetilde{A}_{11}$ is similar to a diagonal matrix if and only if the number of linearly independent eigenvectors of index-1 pencil $\{E, A\}$ is exactly n_1, the rank of matrix E. This holds true, for example, if all the eigenvalues of $\{E, A\}$ are distinct. Further, Assumptions 3.2 and 3.3 may be relaxed somewhat. Namely, it is sufficient to give the analogous conditions for $\widetilde{R}_{21}$ and $\widetilde{R}_{11}$. However, here we aim to formulate as-simple-as-possible sufficient conditions for the asymptotic integration.

If the matrix pencil $\{E,\ A\}$ has multiple eigenvalues and the matrix $\widetilde{A}_{11}$ is similar to a block diagonal matrix J with Jordan blocks J_k, $1 \le k \le l$ and the maximal size of the Jordan blocks J_k is $r+1$, $r \ge 1$, then we need the following stronger assumption on R in order to obtain asymptotic formulas for the solutions of (3.1).

Assumption 3.6. Let the matrix $R(t)$ satisfy that

$$\int_{t_0}^{\infty} t^r \|R(t)\|\, dt < +\infty. \tag{3.12}$$

Theorem 3.7. *Assume that the matrix $\widetilde{A}_{11}$ is similar to a block diagonal matrix J with Jordan blocks J_k, $1 \le k \le l$ and the maximal size of the Jordan blocks J_k is $r+1$, $r \ge 1$. Assume also that Assumptions 3.1, 3.2, and 3.6 hold. Let μ_j be an eigenvalue of the matrix pencil $\{E, A\}$ and let the unperturbed DAE system (2.1) have a solution of the form*

$$e^{\mu_j t} t^m c + O\big(e^{\mu_j t} t^{m-1}\big), \tag{3.13}$$

where c is a vector and $0 \le m \le r$. Then, system (3.1) has a solution $\varphi_j(t)$ such that

$$\lim_{t\to\infty} \big[\varphi_j(t) e^{-\mu_j t} t^{-m} - c\big] = 0.$$

Proof. As we show in the proof of Theorem 3.4, μ_j is also an eigenvalue of matrix $\widetilde{A}_{11}$. Denote

$$c = \mathbb{V} \begin{bmatrix} c^{1^T} & c^{2^T} \end{bmatrix}^T,$$

where $c^1 \in \mathbb{C}^{n_1}$ and $c^2 \in \mathbb{C}^{n_2}$.

We again consider the EUODE system (3.7). From the assumption (3.13) on the solution of the unperturbed DAE, the corresponding unperturbed EUODE system has a solution of the form $e^{\mu_j t} t^m c^1 + O(e^{\mu_j t} t^{m-1})$. Furthermore, $c^2 = -A_{22}^{-1} A_{21} c^1$ holds. Under Assumptions 3.1 and 3.6, it can be shown that $\widetilde{R}_{11}$ satisfies $\int_{t_0}^{\infty} t^r \|\widetilde{R}_{11}(t)\| dt < +\infty$. Hence, by the result of [6, p. 106, Problem 35], the system (3.7) has a solution $u_j(t)$ such that

$$\lim_{t\to\infty} \big[u_j(t) e^{-\mu_j t} t^{-m} - c^1\big] = 0.$$

On the other hand, again using (3.6), the corresponding algebraic component $v_j(t)$ satisfies

$$\lim_{t\to\infty} v_j(t) e^{-\mu_j t} = -\lim_{t\to\infty} \big(A_{22}^{-1} A_{21} + \widetilde{R}_{21}(t)\big) u_j(t) e^{-\mu_j t} t^{-m} = -A_{22}^{-1} A_{21} c^1 = c^2.$$

Thus,

$$\varphi_j(t) = \mathbb{V} \begin{bmatrix} u_j(t) \\ v_j(t) \end{bmatrix}$$

is a solution of system (3.1), and

$$\lim_{t\to\infty} \varphi_j(t) e^{-\lambda_j t} t^{-m} = \lim_{t\to\infty} \mathbb{V} \begin{bmatrix} u_j(t) e^{-\lambda_j t} t^{-m} \\ v_j(t) e^{-\lambda_j t} t^{-m} \end{bmatrix} = \mathbb{V} \begin{bmatrix} c^1 \\ c^2 \end{bmatrix} = c.$$

The proof of Theorem 3.7 is complete. □

Remark 3.8. Assumption 3.3 (or 3.6) cannot be replaced by the condition $\lim_{t\to\infty} R(t) = 0$ since it is known that even in the ODE case that the statements of Theorems 3.4 and 3.7 fail under this relaxed condition. Further, if E is nonsingular, then the results of Theorems 3.4 and 3.7 are reduced to the well-known results for ODEs [6].

In many applications, perturbations arising in the systems can be decomposed into two parts: one tends to zero as $t \to \infty$ and the other is absolutely integrable. Now, consider the DAEs of the form

$$Ex'(t) = [A + B(t) + R(t)]x(t) \tag{3.14}$$

where $E, A \in \mathbb{C}^{n\times n}$, and B, $R \in C(\mathbb{R}_+; \mathbb{C}^{n\times n})$ which are assumed to be sufficiently small in some sense.

Applying again the transformation with $\mathbb{U}$ and $\mathbb{V}$ to (3.14) as above, we obtain

$$\begin{aligned} E_{11}u' &= (A_{11} + B_{11}(t) + R_{11}(t))u + (A_{12} + B_{12}(t) + R_{12}(t))v, \\ 0 &= (A_{21} + B_{21}(t) + R_{21}(t))u + (A_{22} + B_{22}(t) + R_{22}(t))v, \end{aligned} \tag{3.15}$$

where E_{11}, A_{ij}, and R_{ij}, $j = 1, 2$, are defined as in (2.6), (2.7), and (3.3), and

$$B_{11}(t) = U^{\perp^T} B(t) V^{\perp},\ B_{12}(t) = U^{\perp^T} B(t) V,\ B_{21} = U^T B V^{\perp},\ B_{22} = U^T B(t) V. \tag{3.16}$$

In order to investigate the asymptotic behavior of solutions of equation (3.14), we present the following assumptions.

Assumption 3.9. Suppose that $\sup_{t \geq t_0} (\|B_{22}(t)\| + \|R_{22}(t)\|) < \|A_{22}^{-1}\|^{-1}$ holds.

Assumption 3.9 implies that the inverse matrix $(A_{22} + B_{22} + R_{22})^{-1}$ exists and it is uniformly bounded. From the second equation of the system (3.15), we find that

$$v = -(A_{22} + B_{22}(t) + R_{22}(t))^{-1}(A_{21} + B_{21}(t) + R_{21}(t))u. \tag{3.17}$$

By an elementary reformulation, we have

$$(A_{22} + B_{22}(t) + R_{22}(t))^{-1} = (A_{22} + B_{22}(t))^{-1} + C(t),$$

where $C(t) = -(A_{22} + B_{22}(t) + R_{22}(t))^{-1} R_{22}(t)(A_{22} + B_{22}(t))^{-1}$.

Substituting (3.17) into the first equation of system (3.15), we obtain an ODE system for u of the form

$$u' = [\widetilde{A}_{11} + \widetilde{B}_{11}(t) + \widehat{R}_{11}(t)]u, \tag{3.18}$$

where $\widetilde{A}_{11} = E_{11}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})$, and

$$\widetilde{B}_{11} = E_{11}^{-1}\left[B_{11} - B_{12}A_{22}^{-1}A_{21} - (A_{12} + B_{12})\left(A_{22}^{-1}B_{21} - (A_{22} + B_{22})^{-1}B_{22}A_{22}^{-1}(A_{21} + B_{21})\right)\right],$$

and

$$\begin{aligned} \widehat{R}_{11} = E_{11}^{-1}\Big[&R_{11} - (A_{12} + B_{12})(A_{22} + B_{22})^{-1}R_{21} \\ &+ \Big((A_{12} + B_{12})C(t) + R_{12}\big((A_{22} + B_{22})^{-1} + C(t)\big)\Big)(A_{21} + B_{21} + R_{21})\Big]. \end{aligned}$$

On the other hand, (3.17) is equivalent to

$$v = -\left[(A_{22} + B_{22})^{-1} + C(t)\right](A_{21} + B_{21} + R_{21})u$$

and thus it can be rewritten in the form

$$v = -\left(A_{22}^{-1}A_{21} + \widetilde{B}_{21}(t)\right)u, \tag{3.19}$$

where $\widetilde{B}_{21}(t) = A_{22}^{-1}(B_{21} + R_{21}) - ((A_{22}^{-1} + B_{22})^{-1}B_{22}A_{22}^{-1} + C(t))(A_{21} + B_{21} + R_{21})$.

Assumption 3.10. Let $B(t)$ be differentiable on $[t_0, \infty)$ such that

$$\int_{t_0}^{\infty} \|B'(t)\|\, dt < \infty \tag{3.20}$$

and $B(t) \to 0$ as $t \to \infty$.

Assumption 3.11. Let the matrix $R(t)$ be absolutely integrable on $[t_0, \infty)$, i.e.,

$$\int_{t_0}^{\infty} \|R(t)\| \, dt < \infty, \tag{3.21}$$

and let $R_{2j}(t) \to 0$ as $t \to \infty$ hold, $j = 1, 2$.

Lemma 3.12. *Let Assumptions 3.9, 3.10, and 3.11 hold. Then the following statements are true.*

(i) $\widetilde{B}_{21}(t) \to 0$ *as* $t \to \infty$*;*

(ii) $\widetilde{B}_{11}(t)$ *is differentiable on* $[t_0, \infty)$ *and satisfies* $\int_{t_0}^{\infty} \|\widetilde{B}'_{11}(t)\| \, dt < \infty$*; Furthermore,* $\widetilde{B}_{11}(t) \to 0$ *as* $t \to \infty$*;*

(iii) $\widehat{R}_{11}$ *is absolutely integrable on* $[t_0, \infty)$*, i.e.,* $\int_{t_0}^{\infty} \|\widehat{R}_{11}(t)\| \, dt < \infty$.

Proof. By taking into account the explicit formulas of $\widetilde{B}_{21}$, $\widetilde{B}_{11}$, $\widehat{R}_{11}$, the verifications are straightforward. □

Theorem 3.13. *Let Assumptions 3.9, 3.10, and 3.11 hold and let the matrix pencil* $\{E, A\}$ *have distinct eigenvalues* μ_j*,* $j = 1, 2, \ldots, n_1$*. Furthermore, let* $\lambda_j(t)$ *be the roots of* $\det(A + B(t) - \lambda E) = 0$*. Clearly, by reordering the* μ_j *if necessary, we have* $\lim_{t\to\infty} \lambda_j(t) = \mu_j$*. For a given k, let*

$$D_{kj}(t) = \mathrm{Re}(\lambda_k(t) - \lambda_j(t)).$$

Suppose that each j falls into one of two classes I_1 *and* I_2*, where* $j \in I_1$ *if* $\int_0^t D_{kj}(\tau) \, d\tau \to +\infty$ *as* $t \to \infty$ *and*

$$\int_{t_1}^{t_2} D_{kj}(\tau) \, d\tau > -K \quad (t_2 \geq t_1 \geq 0), \tag{3.22}$$

$j \in I_2$ *if*

$$\int_{t_1}^{t_2} D_{kj}(\tau) \, d\tau < K \quad (t_2 \geq t_1 \geq 0), \tag{3.23}$$

where k is fixed and where K is a constant. Let ξ_k *be the eigenvector associated with* μ_k *of the pencil* $\{E, A\}$*, so that* $\mu_k E \xi_k = A \xi_k$*. Then equation (3.14) has a solutions* φ_k *and there exists* t_1 $(t_0 \leq t_1 \leq \infty)$ *such that*

$$\lim_{t\to\infty} \varphi_k(t) e^{-\int_{t_1}^{t} \lambda_k(\tau) \, d\tau} = \xi_k.$$

Proof. Let again

$$\xi_k = \mathbb{V} \begin{bmatrix} \xi_k^{1^T} & \xi_k^{2^T} \end{bmatrix}^T,$$

where $\xi_k^1 \in \mathbb{C}^{n_1}$ and $\xi_k^2 \in \mathbb{C}^{n_2}$, as in the proof of Theorem 3.4. Then, from the equality $\mu_k E \xi_k = A \xi_k$, we again have the formulas (3.10) and (3.11), which means that ξ_k^1 is an eigenvector associated with the eigenvalue μ_k of the matrix $\widetilde{A}_{11} = E_{11}^{-1}(A_{11} - A_{12} A_{22}^{-1} A_{21})$. Now, let $p_k(t)$ be an eigenvector associated with the eigenvalue $\lambda_k(t)$ of the pencil $\{E, A + B(t)\}$, i.e., $\lambda_k E p_k(t) = (A + B(t)) p_k(t)$ for all $t \geq t_0$ with a sufficiently large t_0. Let

$$p_k(t) = \mathbb{V} \begin{bmatrix} p_k^{1^T}(t) & p_k^{2^T}(t) \end{bmatrix}^T,$$

where $p_k^1(t) \in \mathbb{C}^{n_1}$ and $p_k^2(t) \in \mathbb{C}^{n_2}$. From the definition, we have

$$\lambda_k(t) \mathbb{U}^T E \mathbb{V} p_k(t) = \mathbb{U}^T (A + B(t)) \mathbb{V} p_k(t),$$

or equivalently

$$\begin{aligned} \lambda_k E_{11} p_k^1 &= (A_{11}+B_{11})p_k^1 + (A_{12}+B_{12})p_k^2, \\ 0 &= (A_{21}+B_{21})p_k^1 + (A_{22}+B_{22})p_k^2. \end{aligned} \tag{3.24}$$

From Assumption 3.9, it follows that $\|B_{22}(t)\| < (\|A_{22}^{-1}\|)^{-1}$ for all $t \geq t_0$. Hence, there exists the inverse matrix $(A_{22}+B_{22})^{-1}$ and we have

$$(A_{22}+B_{22})^{-1} = A_{22}^{-1} - (A_{22}+B_{22})^{-1} B_{22} A_{22}^{-1}.$$

From the second equation of the system (3.24), we obtain

$$p_k^2 = -(A_{22}+B_{22})^{-1}(A_{21}+B_{21})p_k^1,$$

or equivalently

$$p_k^2 = -\left[A_{22}^{-1}A_{21} + A_{22}^{-1}B_{21} - (A_{22}+B_{22})^{-1}B_{22}A_{22}^{-1}(A_{21}+B_{21})\right]p_k^1. \tag{3.25}$$

Substituting (3.25) into the first equation of the system (3.24), we obtain

$$\begin{aligned} \lambda_k p_k^1 = E_{11}^{-1} \Big[& A_{11} - A_{12}A_{22}^{-1}A_{21} + B_{11} - B_{12}A_{22}^{-1}A_{21} \\ & -(A_{12}+B_{12})(A_{22}^{-1}B_{21} - (A_{22}+B_{22})^{-1}B_{22}A_{22}^{-1}(A_{21}+B_{21}))\Big] p_k^1, \end{aligned}$$

or equivalently

$$\lambda_k p_k^1 = \left[\widetilde{A}_{11} + \widetilde{B}_{11}(t)\right]p_k^1. \tag{3.26}$$

This means that $\lambda_k(t)$ is also an eigenvalue of the matrix $\widetilde{A}_{11} + \widetilde{B}_{11}(t)$ and p_k^1 is an eigenvector associated with $\lambda_k(t)$. By the assumption on the eigenvalues of $\{E, A\}$, the matrix $\widetilde{A}_{11}$ has distinct eigenvalues. On the other hand, by Lemma 3.12, we have that $\widetilde{B}_{11}(t) \to 0$ as $t \to \infty$. Thus, $\lambda_k(t) \to \mu_k$ as $t \to \infty$ (by reordering if necessary). Furthermore, the conditions of [6, Chapter 3, Theorem 8.1] are satisfied by the underlying ODE system (3.18). It follows that there exists $t_1 \geq t_0$ such that (3.18) has a solution $u_k(t)$ satisfying

$$\lim_{t\to\infty} u_k(t) e^{-\int_{t_1}^t \lambda_k(\tau)\,d\tau} = \xi_k^1.$$

By the equality (3.19), there exists $v_k(t)$ defined by

$$v_k(t) = -\big(A_{22}^{-1}A_{21} + \widetilde{B}_{21}(t)\big)u_k(t),$$

which fulfills

$$\lim_{t\to+\infty} v_k(t) e^{-\int_{t_1}^t \lambda_k(\tau)\,d\tau} = -\lim_{t\to+\infty} \big(A_{22}^{-1}A_{21} + \widetilde{B}_{21}(t)\big)u_k(t) e^{-\int_{t_1}^t \lambda_k(\tau)\,d\tau} = -A_{22}^{-1}A_{21}\xi_k^1 = \xi_k^2.$$

Let us define

$$\varphi_k(t) = \mathbb{V} \begin{bmatrix} u_k(t) \\ v_k(t) \end{bmatrix}.$$

By its construction, $\varphi_k(t)$ is obviously a solution of system (3.14) and it satisfies

$$\begin{aligned} \lim_{t\to+\infty} \varphi_k(t) e^{-\int_{t_1}^t \lambda_k(\tau)d\tau} &= \lim_{t\to+\infty} \mathbb{V} \begin{bmatrix} u_k(t) e^{-\int_{t_1}^t \lambda_k(\tau)d\tau} \\ v_k(t) e^{-\int_{t_1}^t \lambda_k(\tau)d\tau} \end{bmatrix} \\ &= \mathbb{V} \begin{bmatrix} \xi_k^1 \\ \xi_k^2 \end{bmatrix} = \xi_k. \end{aligned}$$

This completes the proof of Theorem 3.13. □

4 The case of perturbed leading coefficient

In this section, we extend the results obtained in Section 3 to DAEs with perturbed leading coefficient

$$[E + F(t)]x'(t) = [A + R(t)]x(t), \tag{4.1}$$

where $E, A \in \mathbb{C}^{n\times n}$ and $F,\ R \in C(\mathbb{I}; \mathbb{C}^{n\times n})$.

We again suppose that the matrix E is singular, but the pencil $\{E, A\}$ is regular of index one. First, we introduce the concept of allowable perturbations, see [3].

Definition 4.1. The perturbation F arising in the leading term is said to be allowable if $\ker(E + F(t)) = \ker E$ for all $t \in \mathbb{I}$. Otherwise we say F is not allowable.

The following example shows that if $\ker(E + F(t)) \neq \ker E$, then the asymptotic behavior of solutions of the perturbed DAE (4.1) and the asymptotic behavior of solution of the unperturbed one may be quite different, even if the perturbation F is small, e.g., it is convergent to 0 as $t \to \infty$ and absolutely integrable.

Example 4.2. Consider the index-1 DAE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad t \geq 1.$$

It is easy to obtain the solution $x_1(t) = e^{t-1}x_1(1)$ and $x_2(t) = 0$. After that, we consider the following perturbed DAE

$$\begin{bmatrix} 1 & 0 \\ 0 & \dfrac{1}{3t^2} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The first component x_1 is unchanged. However, the second component $x_2(t) = e^{t^3-1}x_2(1)$, which tends to ∞ as $t \to \infty$. That is, a small perturbation in the leading coefficient can completely change the behavior of the solutions. For a related result, see the stability analysis of DAEs containing a small parameter in [9].

In the remainder part of this section we assume that F is allowable. Let us apply to (4.1) the transformation with the same $\mathbb{U}$ and $\mathbb{V}$ as in Section 2. Then, we obtain the transformed DAE

$$\begin{aligned} (E_{11} + F_{11}(t))u' &= (A_{11} + R_{11}(t))u + (A_{12} + R_{12}(t))v, \\ F_{21}(t)u' &= (A_{21} + R_{21}(t))u + (A_{22} + R_{22}(t))v, \end{aligned} \tag{4.2}$$

where $E_{11},\ A_{11},\ A_{12},\ A_{21},\ A_{22}$ are defined in (2.6) and (2.7), $R_{11},\ R_{12},\ R_{21},\ R_{22}$ are defined in (3.3). Further, we have

$$F_{11} = U^{\perp^T} F V^{\perp}, \quad F_{21} = U^T F V^{\perp}. \tag{4.3}$$

Note that under the assumption on F, we have $U^{\perp^T} F V = 0$ and $U^T F V = 0$.

We make the following set of assumptions.

Assumption 4.3. Let $\sup_{t\geq t_0} \|F_{11}(t)\| < (\|E_{11}^{-1}\|)^{-1}$ hold.

Assumption 4.4. Let $R_{2j}(t) \to 0$, $j = 1, 2$, and $F_{21}(t) \to 0$, as $t \to \infty$. Further, let $\sup_{t\geq t_0} \|R_{1j}(t)\| < \infty$, $j = 1, 2$.

Assumption 4.5. Let $\sup_{t\ge t_0}\|\overline{R}_{22}(t)\| < \|A_{22}^{-1}\|^{-1}$ hold, where

$$\overline{R}_{22}(t) = -F_{21}(t)(E_{11}+F_{11}(t))^{-1}(A_{12}+R_{12}(t)) + R_{22}(t),$$

provided that $(E_{11}+F_{11}(t))^{-1}$ exists for all $t \in \mathbb{I}$.

Assumption 4.6. Let both R and F be absolutely integrable on $\mathbb{I}$.

We will show that under these assumptions, the perturbed DAE (4.1) can be transformed into the form (3.1). Then, the theorems in Section 3 can be applied.

Theorem 4.7. *Consider the DAE system (4.1) with index-1 pencil $\{E, A\}$. Let Assumptions 4.3–4.6 hold and let $\widetilde{A}_{11} = E_{11}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})$ is similar to a diagonal matrix J. Suppose that μ_j is an eigenvalue of the pencil $\{E, A\}$ and ξ_j is an eigenvector associated with μ_j, i.e., $\mu_j E\xi_j = A\xi_j$. Then, the perturbed DAE system (4.1) has a solution $\varphi_j(t)$ such that*

$$\lim_{t\to\infty} \varphi_j(t)e^{-\mu_j t} = \xi_j.$$

Proof. Assumption 4.3 implies that matrices $(E_{11}+F_{11}(t))$ and $(I+F_{11}(t)E_{11}^{-1})$ are invertible for all $t \in \mathbb{I}$ and the inverse matrices are uniformly bounded. We have $E_{11}+F_{11} = (I+F_{11}E_{11}^{-1})E_{11}$ and $(I+F_{11}(t)E_{11}^{-1})^{-1} = I - (I+F_{11}(t)E_{11}^{-1})F_{11}E_{11}^{-1}$. In order to bring (4.2) into the form (3.2), we first multiply the first equation of (4.2) by $-F_{21}(E_{11}+F_{11})^{-1}$ add the obtained result to the second equation of (4.2). Then, we scale the first equation of (4.2) by multiplying it by $(I+F_{11}(t)E_{11}^{-1})^{-1}$. As the result, we obtain a new DAE system as follows

$$\begin{aligned} E_{11}u' &= (A_{11}+\overline{R}_{11}(t))u + (A_{12}+\overline{R}_{12}(t))v, \\ 0 &= (A_{21}+\overline{R}_{21}(t))u + (A_{22}+\overline{R}_{22}(t))v, \end{aligned} \tag{4.4}$$

where

$$\overline{R}_{1j} = (I+F_{11}E_{11}^{-1})^{-1}(F_{11}E_{11}^{-1}A_{1j}+R_{1j}),\ \overline{R}_{2j} = -F_{21}(E_{11}+F_{11})^{-1}(A_{1j}+R_{1j}) + R_{2j},$$

for $j = 1,2$. Under Assumptions 4.3–4.6, it is not difficult to verify that $\overline{R}_{ij}$, $i,j = 1,2$, satisfy Assumptions 3.1–3.3 in Section 3. Thus, the conditions of Theorem 3.4 are fulfilled. Applying Theorem 3.4 to (4.4), the proof is complete. □

Similarly, by invoking Theorem 3.2, the following theorem is obtained for the case when the matrix $\widetilde{A}_{11}$ is not diagonalizable.

Theorem 4.8. *Consider the DAE system (4.1) with index-1 pencil $\{E, A\}$. Let Assumptions 4.3–4.5 hold and let $\widetilde{A}_{11}$ be similar to the block diagonal matrix J with Jordan blocks J_k, $1 \le k \le l$. Assume that $r+1$ is the maximal number of rows in any matrices J_k, $1 \le k \le l$. Here $r \ge 1$ is considered. Let R and F satisfy*

$$\int_{t_0}^{\infty} t^r\|R(t)\|\,dt < \infty, \quad \int_{t_0}^{\infty} t^r\|F(t)\|\,dt < \infty. \tag{4.5}$$

We suppose that μ_j is an eigenvalue of the pencil $\{E, A\}$ and that the corresponding unperturbed DAE has a solution of the form

$$e^{\mu_j t}t^m c + O(e^{\mu_j t}t^{m-1}),$$

where c is a nonzero vector and $0 \le m \le r$. Then, system (4.1) has a solution $\varphi_j(t)$ such that

$$\lim_{t\to\infty}[\varphi_j(t)e^{-\mu_j t}t^{-m} - c] = 0.$$

By analogue, the result of Theorem 3.13 can be extended to DAEs of the form (4.2), as well.

5 The case of higher-index DAEs

In this section, we revisit the DAEs of the form (2.1) and (3.1), but now we assume that the pencil $\{E, A\}$ has index $k \geq 2$ and the Weierstraß–Kronecker canonical form (2.2) holds with a pair of nonsingular matrices G and H.

Multiplying both sides of (2.1) by G and introducing a variable transformation $x = Hy$, $y = (u^T \ v^T)^T$, where $u(t) \in \mathbb{C}^{n_1}$ and $v(t) \in \mathbb{C}^{n_2}$, we obtain

$$\begin{aligned} u' &= [J_{n_1} + R_{11}(t)]u + R_{12}(t)v, \\ Nv' &= R_{21}(t)u + [I_{n_2} + R_{22}(t)]v, \end{aligned} \tag{5.1}$$

where

$$GRH = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}.$$

Exploiting the nilpotency of N, it is easy to verify that the unperturbed system associated with (5.1) has solution $u(t) = e^{J_{n_1}(t-t_0)}u(t_0)$, $v(t) = 0$ for $t \geq t_0$. The analysis of perturbed system (5.1) is more complicated than the index-1 case.

In general, if the perturbation $R(t)$ is not well-structured, then the asymptotic behavior of solutions is not preserved even under such small perturbations discussed in Section 3.

Example 5.1. First, consider the following DAE

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad t \geq 1.$$

The solution is $x_1(t) = e^{2(t-1)}x_1(1)$, $x_2(t) = x_3(t) = 0$. Let the system be perturbed as follows

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3t^2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The solution of the perturbed system is $x_1(t) = e^{2(t-1)}x_1(1)$, $x_2(t) = 3t^2e^{t^3-1}x_3(1)$, $x_3(t) = e^{t^3-1}x_3(1)$. Clearly, $x_2(t) \to \infty$ and $x_3(t) \to \infty$ as $t \to \infty$ except for $x_3(1) = 0$.

Similarly, if we consider another perturbed system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\sin t^3}{t^2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

then, it is easy to see that $x_1(t)$ remains the same, but both

$$x_3(t) = \frac{\sin t^3}{t^2}e^{2(t-1)}x_1(1) \quad \text{and} \quad x_2(t) = \left(3\cos t^3 e^{2t} - 2\frac{\sin t^3}{t^3}e^{2t} + \frac{\sin t^3}{t^2}2e^{2t}\right)x_1(1)$$

are unboundedly oscillating.

Definition 5.2. Consider the perturbed DAE (5.1) and assume that N is nilpotent of index $k \geq 2$. The perturbation R is said to be allowable for the higher-index case if the component v of (5.1) is identically zero for $t \geq t_0$, i.e., the solution for v is preserved under perturbation.

Here we give sufficient conditions for allowable perturbations.

Lemma 5.3. *Suppose that $R_{21} = 0$ and R_{22} is such that $I + R_{22}$ is invertible and $NR_{22} = 0$ or $R_{22}N = 0$. Then, the perturbation in (5.1) is allowable.*

Proof. First, suppose that $NR_{22} = 0$. Under the assumptions and multiplying the second equation by N, we obtain $N^2v' = Nv$. Differentiation yields $N^2v'' = Nv' = (I + R_{22})v$. Multiplying the last equation by N, we have $N^3v'' = Nv$. Induction simply yields $N^k v^{(k-1)} = Nv$. Due to the nilpotency of N, it follows that $Nv = 0$. Hence, $Nv' = 0$ and we obtain $(I + R_{22})v = 0$, too. Since $I + R_{22}$ is invertible, we conclude $v \equiv 0$.

Second, assume that $R_{22}N = 0$. Under the assumptions, it is easy to see that $(I+R_{22})^{-1}N = N$. Thus, the second equation of (5.1) is equivalent to $Nv' = v$, which implies that v is identically zero, too. □

Thus, from now on we assume that the perturbation R satisfies the conditions of Lemma 5.3 and we consider only the system of the form

$$\begin{aligned} u' &= [J_{n_1} + R_{11}(t)]u + R_{12}(t)v, \\ Nv' &= [I_{n_2} + R_{22}(t)]v. \end{aligned} \tag{5.2}$$

Due to Lemma 5.3, the component v is identically zero. Consequently, it suffices to consider the reduced equation

$$u' = [J_{n_1} + R_{11}(t)]u. \tag{5.3}$$

We immediately obtain the following results for the perturbed DAE (5.1).

Theorem 5.4. *Let R satisfy the assumptions of Lemma 5.3 and let $R_{11}(t)$ be absolutely integrable on $[t_0, \infty)$, i.e.,*

$$\int_{t_0}^{\infty} \|R_{11}(t)\| \, dt < \infty. \tag{5.4}$$

Suppose that J_{n1} is diagonal and ξ_j is an eigenvector associated with an eigenvalue μ_j of the pencil $\{E, A\}$, i.e., $\mu_j E\xi_k = A\xi_j$. Then, the equation (5.1) has a solution $\varphi_j(t)$ such that

$$\lim_{t\to\infty} \varphi_j(t)e^{-\mu_j t} = \xi_j.$$

Proof. Suppose that ξ_j is an eigenvalue corresponding to eigenvalue μ_j of the pencil $\{E, A\}$, i.e., $\mu_j E\xi_j = A\xi_j$. Denote $\xi_j = H\widetilde{\xi}_j$, $\widetilde{\xi}_j = (\xi^{1^T}\xi^{2^T})^T$. From the quality $\mu_j E\xi_j = A\xi_j$, we imply that $\mu_j GEH\widetilde{\xi}_j = GAH\widetilde{\xi}_j$, or

$$\mu_j \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \xi_j^1 \\ \xi_j^2 \end{bmatrix} = \begin{bmatrix} J_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} \xi_j^1 \\ \xi_j^2 \end{bmatrix}.$$

Hence, we have

$$\begin{aligned} \mu_j \xi_j^1 &= J_{n_1}\xi_j^1, \\ \mu_j N\xi_j^2 &= \xi_j^2. \end{aligned} \tag{5.5}$$

From the second equation of the system (5.5), it is straightforward to verify that $\xi_j^2 = 0$. The first equation of the system (5.5) means exactly that ξ_j^1 is an eigenvector associated with eigenvalue

μ_j of the matrix J_{n_1}. Invoking [6, p. 104, Prob. 29], we conclude that (5.3) has a solution $u_j(t)$ such that

$$\lim_{t\to\infty} u_j(t)e^{-\mu_j t} = \xi_j^1.$$

Let $v_j(t) = 0$ be the second solution component, we have that

$$\varphi_j(t) = H \begin{bmatrix} u_j(t) \\ v_j(t) \end{bmatrix}$$

is a solution of equation (5.1). Hence, it follows that

$$\lim_{t\to\infty} \varphi_j(t)e^{-\mu_j t} = \lim_{t\to\infty} H \begin{bmatrix} u_j(t)e^{-\mu_j t} \\ v_j(t)e^{-\mu_j t} \end{bmatrix} = H \begin{bmatrix} \xi_j^1 \\ 0 \end{bmatrix} = H\widetilde{\xi}_j = \xi_j.$$

The proof of Theorem 5.4 is complete. □

As an analogue of Theorem 3.7, we obtain the following theorem for the case of non-diagonal J_{n_1}.

Theorem 5.5. *Let the matrix J_{n_1} be a Jordan matrix consisting of blocks J_k, $k \geq 1$ and $r+1$ is the maximal number of rows in any block J_k, $k \geq 1$. Here $r \geq 1$ is considered. Let R satisfy the same assumption as in Theorem 5.4. Let μ_j be an eigenvalue of $\{E, A\}$ and the equation (2.1) has a solution of the form*

$$e^{\mu_j t} t^m c + O(e^{\mu_j t} t^{m-1}),$$

where c is a vector and $0 \leq m \leq r$. Then the perturbed system (5.1) has a solution $\varphi_j(t)$ such that

$$\lim_{t\to\infty} \left[\varphi_j(t)e^{-\mu_j t} t^{-m} - c\right] = 0.$$

Proof. Using similar arguments as those in the proofs of Theorem 3.7 and Theorem 5.4, the proof is straightforward. □

Remark 5.6. An analogue of Theorem 3.13 can be obtained in the higher-index case as well. The asymptotic integration results can also be extended to DAEs with perturbation in the leading term. However, more restrictive structure of perturbation should hold.

6 Conclusion

In this paper we have extended the classical asymptotic integration results from ODEs to linear DAEs. Checkable conditions are given so that the asymptotic formulas for solution of asymptotically constant coefficient DAE systems are obtained explicitly. It has been shown that the solutions of the perturbed DAEs behave asymptotically like the corresponding solutions of the unperturbed DAEs as time t tends to ∞. As future works, asymptotic results for linear time-varying DAEs, for DAEs with nonlinear perturbations, and for delay DAEs would be of interest.

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