

# On Shuffle Ideals of General Algebras

Ville Piirainen\*

## Abstract

We extend a word language concept called shuffle ideal to general algebras. For this purpose, we introduce the relation  $\mathcal{SH}$  and show that there exists a natural connection between this relation and the homeomorphic embedding order on trees. We establish connections between shuffle ideals, monotonically ordered algebras and automata, and piecewise testable tree languages.

## 1 Introduction and preliminaries

This work is a part of an ongoing study on piecewise testability and related matters for tree languages. Piecewise testable languages and their algebraic properties have been approached from various directions, and offer a wide field of interesting notions for study from the tree language viewpoint. In addition to the ingenious combinatorial approach of Simon [10], there have been a few approaches with a more algebraic flavour, and this work is inspired most importantly by the papers by Straubing and Thérien [12], and Henckell and Pin [5]. These works concern, of course, word languages, subsets of a free monoid  $X^*$ , and obviously are not directly generalizable for tree languages, subsets of a term algebra  $\mathcal{T}_\Sigma(X)$ . However, all these papers contain many algebraic insights that can be considered in the tree language setting. We are much indebted to the work on ordered monoids in these papers, as well as to the related work on varieties of ordered algebras by Bloom [2], and Petković and Salehi [6].

The shuffle operation is a natural operation to consider for the elements of a free monoid. Using this operation one obtains so called shuffle ideals, which are subsets of a free monoid closed under the shuffle operation. As noted for example in [9], by considering all boolean combinations of shuffle ideals on a given free monoid, one obtains exactly all piecewise testable languages over that monoid. In fact, the shuffle, the class of piecewise testable languages, the Green's  $\mathcal{J}$ -relation for semigroups and the class of monotonically ordered monoids are all concepts which are strongly connected to each other, and we shall use these connections to investigate the notion of shuffling for general algebras.

The shuffle operation cannot be directly defined for any given  $\Sigma X$ -trees, since even the product of two trees cannot be uniquely defined in a way that would suit

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\*University of Turku, E-mail: ville.piirainen@utu.fi

all applications. While the operation itself does not generalize directly, the shuffle ideals, as languages, have direct counterparts in the tree language setting, as we shall see.

After this first section of introduction and preliminaries, in the second section, we introduce the shuffle relation  $\mathcal{SH}$  and the shuffle ideals, and investigate their basic properties. In the third section, we establish a connection between so-called monotonically ordered algebras and the  $\mathcal{SH}$ -relation. Finally, we discuss some connections between the relation  $\mathcal{SH}$  and piecewise testable tree languages.

As a general reference on algebraic tree language theory, we recommend [11]. It contains most of the basic theory on which this paper is built, and also some discussion on the points one has to take into account when moving from word languages to tree languages. However, we recall here a few of the most important definitions and notions that we need in this paper, since some of them have various different versions in the literature.

We are mainly interested in trees and their languages, and we follow the theoretical framework of [11] which depends heavily on universal algebra. The tree recognizers, general algebras, have a finite number of named operations, from which all other operations of the algebra are composed. Moreover, the number of arguments of each operation is fixed. Hence, trees considered here are terms over suitable alphabets, in which each node of a tree labeled with a given symbol always has a fixed number of children. We use the following notation.

**Definition 1.1.** *A ranked alphabet  $\Sigma$  is a finite set of function symbols, and for all  $m \geq 0$ ,  $\Sigma_m \subseteq \Sigma$  denotes the subset of symbols of rank  $m$ . A  $\Sigma$ -algebra  $\mathcal{A} = (A, \Sigma)$  consists of a non-empty set  $A$  equipped with operations  $f^{\mathcal{A}} : A^m \rightarrow A$ , for all  $m \geq 0$ ,  $f \in \Sigma_m$ .*

For the rest of the paper,  $\mathcal{A} = (A, \Sigma)$  is an arbitrary given  $\Sigma$ -algebra.

In the framework we use, the inner nodes and leaves of a tree have different labelings. In addition to ranked alphabets, we use leaf-alphabets, finite sets of symbols that are disjoint from the ranked alphabets. We identify trees with terms defined in the following definition.

**Definition 1.2.** *For a set  $X$ , called the leaf alphabet, the set of all  $\Sigma X$ -terms  $T_{\Sigma}(X)$  is the smallest set such that  $X \cup \Sigma_0 \in T_{\Sigma}(X)$ , and for every  $m > 0$ ,  $t_1, \dots, t_m \in T_{\Sigma}(X)$  and  $f \in \Sigma_m$ ,  $f(t_1, \dots, t_m) \in T_{\Sigma}(X)$ .*

For transforming a word concept into a tree concept we need a way to regard words as special trees. As usual, we regard words over an alphabet  $A$  as unary trees equipped with a single special leaf symbol  $\xi$ , and letters of the alphabet  $A$  are regarded as unary symbols of the ranked alphabet  $\Sigma$ . More precisely, let  $A$  be an alphabet, let  $X = \{\xi\}$  and let  $\Sigma = \Sigma_1 = A$ . Let  $\chi : A^* \rightarrow T_{\Sigma}(X)$  be the map such that  $\varepsilon\chi = \xi$  and  $(wa)\chi = a(w\chi)$  for any  $a \in A$  and  $w \in A^*$ . Obviously,  $\chi$  forms a bijective correspondence between  $A^*$  and  $T_{\Sigma}(X)$ .

For the purpose of generalizing the semigroup concept shuffle for  $\Sigma$ -algebras, we have chosen to follow the convention that the root of a  $\Sigma X$ -term corresponds

to the right end, and the leaf symbols to the left end of a word. This follows the usual tradition on how words and terms (trees) are read by their respective ordinary automata, from left to right and from leaf to root. This convention has the following consequences. The right translations of semigroups correspond to the algebraic translations of the term algebra  $\mathcal{T}_\Sigma(X)$  of  $\Sigma X$ -trees, while the left translations correspond to the endomorphisms of the same term algebra. We use the translations in our effort to generalize the ideas of insertion and the shuffle ideal for trees in Section 3.

**Definition 1.3.** *A unary map  $p : A \rightarrow A$  is an elementary translation of an algebra  $\mathcal{A}$ , if there exist  $m > 0$ ,  $f \in \Sigma_m$ ,  $i = 1, \dots, m$ , and  $a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m$  such that*

$$p(a) = f^{\mathcal{A}}(a_1, \dots, a_{i-1}, a, a_{i+1}, \dots, a_m),$$

*for all  $a \in A$ . The set of all elementary translations of  $\mathcal{A}$  is denoted  $\text{ETr}(\mathcal{A})$ . The set of translations of  $\mathcal{A}$ , denoted  $\text{Tr}(\mathcal{A})$ , is the smallest set which includes the identity map and the elementary translations, and is closed under functional composition.*

The translations of a term algebra  $\mathcal{T}_\Sigma(X)$  are induced by the  $\Sigma X$ -contexts, that is, the trees  $p \in T_\Sigma(X \cup \{\xi\})$ , where the symbol  $\xi$  appears exactly once. To simplify notation, a context  $p \in T_\Sigma(X \cup \{\xi\})$  and the map  $\hat{p} : T_\Sigma(X) \rightarrow T_\Sigma(X)$ ,  $t \mapsto p(t)$  it induces are identified.

The concept of an ideal is common in algebra, and we introduce here a certain type of an ideal. We note that since we consider here general algebras with no additional requirements, the ideals presented here might differ from ideals defined for different purposes. The theory investigated here is closely related to that of ordered algebras, and as a reference concerning notation and points of view, we offer [6]. From this paper we adopt the following definition.

**Definition 1.4.** *An ideal of an algebra  $\mathcal{A}$  is a non-empty set  $I \subseteq A$  such that for any  $p \in \text{Tr}(\mathcal{A})$ , and  $a \in I$ ,  $p(a) \in I$ . The ideal generated by an element  $a$  is denoted  $I(a)$ .*

In essence, this definition states that if we choose any element from the ideal, any  $n - 1$  elements of the algebra ( $n > 0$ ), and apply to them any  $n$ -ary function of the algebra, the resulting element is still in this ideal. Hence, the notion resembles that of a semigroup ideal, though not that of a Dedekind ideal. Namely, in ring theory there is such a distinction between the two operations that cannot be required in any given arbitrary  $\Sigma$ -algebra in a meaningful way.

In our effort to generalize the idea of shuffling for general algebras, and for non-linear trees, we have taken as a starting point the following definition from [9].

**Definition 1.5.** *For an alphabet  $X$ , a shuffle ideal of the free monoid  $X^*$  is a non-empty set  $I \subseteq X^*$  such that for any words  $u \in I$  and  $v \in X^*$ , their shuffle is included in the set  $I$ .*

For any word  $u \in X^*$ , if  $u = x_1 \cdots x_n$  where  $x_1, \dots, x_n \in X$ , then the shuffle ideal generated by  $u$  is the language  $X^*x_1X^* \cdots X^*x_nX^*$ .

We connect the shuffle ideal to the homeomorphic embedding relation used in term rewriting theory. When words are interpreted as unary trees, it turns out that these notions are very naturally related to one another (see Example 2.2).

**Definition 1.6.** *The homeomorphic embedding relation  $\leq_{emb}$  on  $T_\Sigma(X)$  is defined as follows. For any  $s, t \in T_\Sigma(X)$ ,  $s \leq_{emb} t$  if and only if,*

- (1)  $t \in X \cup \Sigma$  and  $s = t$ , or
- (2)  $t = f(t_1, \dots, t_m)$ ,  $s = f(s_1, \dots, s_m)$  and  $s_i \leq_{emb} t_i$  for  $i = 1, \dots, m$ , or
- (3)  $t = f(t_1, \dots, t_m)$  and  $s \leq_{emb} t_i$  for some  $i = 1, \dots, m$ .

If  $s \leq_{emb} t$  ( $s, t \in T_\Sigma(X)$ ), then essentially this means that all the nodes of the term  $s$  are embedded in the structure of  $t$ , in such a way that they retain their rank (arity) and relative position. For example, if  $X = \{x, y\}$ , and  $\Sigma = \{g/1, f/2, h/2\}$ , then

$$x \leq_{emb} f(x, y) \leq_{emb} f(g(x), h(y, x)) \leq_{emb} h(f(g(x), h(g(y), x)), h(x, y)).$$

## 2 Shuffle ideal

What we call a shuffle ideal borrows ideas from the shuffle operation and ideal defined for word languages (see [9]) as well as the embedding relation from rewriting theory (see [1]). These notions share a common idea: starting from a single element of a language, using suitable insertions, obtain the elements which contain the original element embedded in their structure. We begin by defining a relation that specifies the types of insertions in which we are interested here.

**Definition 2.1.** *Let  $\Rightarrow_{\mathcal{SH}}$  be the relation on  $A$  such that for any  $a, b \in A$*

$$a \Rightarrow_{\mathcal{SH}} b,$$

*if and only if there exist an element  $c \in A$  and translations  $q, r \in \text{Tr}(\mathcal{A})$  such that  $a = q(c)$  and  $b = q(r(c))$ .*

In essence, we decompose the element  $a$  into a product of an element  $c$  and a translation  $q$ , and then insert an another translation  $r$  into the middle of the product.

In the next example we show concretely how such insertions work in a term algebra  $\mathcal{T}_\Sigma(X)$ . The original term, which is embedded in the derived terms, is printed in boldface.

**Example 2.1.** Let  $\Sigma = \{f/2, g/1\}$  and  $X = \{x, y\}$ . Then, for example

$$\mathbf{f(x, y)} \Rightarrow_{\mathcal{SH}} \mathbf{f(f(y, x), y)} \Rightarrow_{\mathcal{SH}} \mathbf{f(f(y, x), g(y))} \Rightarrow_{\mathcal{SH}} \mathbf{f(f(f(y, y), x), g(y)).}$$

Consider for example the second step of the derivation. We can write  $f(f(y, x), y) = f(f(y, x), \xi)(y)$ , and by applying the context  $g(\xi)$  we obtain  $f(f(y, x), \xi)(g(\xi)(y)) = f(f(y, x), g(y))$ .

In the following example we show how derivations can be made in the free monoid generated by the alphabet  $\{a, b\}$ . We denote by  $e$  the empty word, and by  $u\xi v \in \text{Tr}(X^*)$ , for any  $u, v \in X^*$ , the (two-sided) translation such that  $u\xi v(w) = uvw$ .

**Example 2.2.** Let  $X = \{a, b\}$ , and let  $w, w', w'' \in X^*$ . We have for example the following derivation.

$$ab \Rightarrow_{\mathcal{SH}} aw'b \Rightarrow_{\mathcal{SH}} waw'bw''.$$

In the first step we can write that  $ab = a\xi b(e)$ , and further apply the translation  $w'\xi e$  to obtain  $a\xi b(w'\xi e(e)) = aw'b$ . In the second step, we write first  $aw'b = \xi(aw'b)$ , and by using the translation  $w\xi w''$  we obtain  $\xi(w\xi w''(aw'b)) = \xi(waw'bw'') = waw'bw''$ . In general, it is easy to see, that  $ab \Rightarrow_{\mathcal{SH}}^* w$  if and only if  $w \in X^*aX^*bX^*$ .

The following lemmas are direct consequences of the Definition 2.1.

**Lemma 2.1.** For all  $a \in A$  and  $p \in \text{Tr}(\mathcal{A})$ ,  $a \Rightarrow_{\mathcal{SH}} p(a)$ .

*Proof.* Let  $a \in A$ . Then,  $a = \text{id}(a)$ , and  $\text{id}(p(a)) = p(a)$ , for any  $p \in \text{Tr}(\mathcal{A})$ .  $\square$

**Lemma 2.2.** If  $a \Rightarrow_{\mathcal{SH}} b$ , then  $p(a) \Rightarrow_{\mathcal{SH}} p(b)$ , for any  $p \in \text{Tr}(\mathcal{A})$  and  $a, b \in A$ .

*Proof.* If  $a = q(c)$ , and  $b = q(r(c))$ , for some  $c \in A$  and  $r, q \in \text{Tr}(\mathcal{A})$ , then  $p(a) = p(q(c))$ , and  $p(b) = p(q(r(c)))$ , for any  $p \in \text{Tr}(\mathcal{A})$ , which proves the claim.  $\square$

As usual, we denote

$$\Rightarrow_{\mathcal{SH}}^* = \bigcup_{n \geq 0} \Rightarrow_{\mathcal{SH}}^n.$$

**Definition 2.2.** We call a non-empty subset  $I \subseteq A$  a shuffle ideal of  $\mathcal{A} = (A, \Sigma)$ , if for all  $a, b \in A$ ,

(SI)  $a \in I$  and  $a \Rightarrow_{\mathcal{SH}} b$  imply  $b \in I$ .

The following lemma is easy to prove.

**Lemma 2.3.** The intersection of a set of shuffle ideals is either empty or a shuffle ideal.

By the previous lemma, for a given element  $a \in A$ , we can define the *shuffle ideal generated by  $a$*  as the intersection of the shuffle ideals containing  $a$ . We denote this by  $SH(a)$ .

**Lemma 2.4.** For any  $a \in A$ ,  $SH(a) = \{b \in A \mid a \Rightarrow_{\mathcal{SH}}^* b\}$ .

The following lemma is a direct consequence of Lemma 2.1.

**Lemma 2.5.** For all  $a \in A$  and  $p \in \text{Tr}(\mathcal{A})$ ,  $SH(p(a)) \subseteq SH(a)$ .

Note that a shuffle ideal is always an ideal. The shuffle ideal generated by an element contains the ideal generated by the same element, but in general these sets are not the same, as demonstrated by the following example.

**Example 2.3.** Let  $\mathcal{A} = (\{1, 2, 3\}, \{f/1, g/1\})$  be the algebra described in Figure 1, originally presented in [7]. A direct calculation shows that  $I(3) = \{3\}$  but  $SH(3) = \{2, 3\}$ .

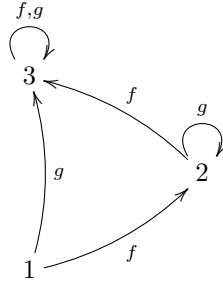


Figure 1: The algebra  $\mathcal{A}$

Note that when interpreted for the free monoid  $X^*$ , the shuffle ideal generated by a word  $w \in X^*$  corresponds exactly to the original notion. Indeed, if  $X$  is an alphabet,  $w = x_1 \cdots x_n \in X^*$  and

$$u = u_1 x_1 u_2 \cdots u_n x_n u_{n+1} \in X^* x_1 X^* \cdots X^* x_n X^*,$$

then

$$x_n(\cdots x_1(\xi) \cdots) \Rightarrow_{\mathcal{SH}}^* u_{n+1}(x_n(u_n(\cdots (u_2(x_1(u_1(\xi)))) \cdots))),$$

by Lemma 2.4. The converse is analogous.

Lemma 2.4 also gives us a naive algorithm to calculate the shuffle ideals  $SH(a)$  of a finite algebra. The algorithm works in two parts. First we calculate  $a \Rightarrow_{\mathcal{SH}}$  for each element  $a \in A$ , and then the equivalence closure  $a \Rightarrow_{\mathcal{SH}}^*$ .

1. Compute the table of translations for the algebra.
2. For each element  $a \in A$  find all possible decompositions  $a = p(b)$  ( $p \in \text{Tr}(\mathcal{A}), b \in A$ ) from the table of translations.
3. For each decomposition  $a = p(b)$ , form all elements  $p(r(b))$ , where  $r \in \text{Tr}(\mathcal{A})$ . These elements form the sets  $a \Rightarrow_{\mathcal{SH}}$ .
4. Compute the reflexive transitive closure  $\Rightarrow_{\mathcal{SH}}^*$  of the relation  $\Rightarrow_{\mathcal{SH}}$ .

Since the algorithm follows exactly the steps of the definitions of the shuffle ideal and the shuffle relation, it is obvious that this algorithm produces exactly the desired sets  $SH(a)$  for all  $a \in A$ .

The complexity of the algorithm depends heavily on the structure of the algebra and its translation monoid  $\text{Tr}(\mathcal{A})$ . In most of any meaningful examples  $\Sigma$  is fixed, so we measure complexity based only on  $|A|$ . It is worth mentioning though, that by choosing a suitable ranked alphabet  $\Sigma$ , one can easily devise exotic algebras such that the complexity of computing the elementary translations of the algebra exceeds any given bound which is dependent only on the size  $|A|$  of the algebra, and hence the following analysis is not applicable universally. However, even in such exotic cases the number of different elementary translations has an upper bound which depends only on the size of  $|A|$ . Hence, we assume that we are given elementary translations induced by the algebra as the input for the algorithm.

If  $|A| = n$ , then the size of the translation monoid may equal  $n^n$  (the full transformation monoid on  $A$ ), and its calculation that starts from the elementary translations may have a complexity of as high as  $\mathcal{O}(n^{3n+1})$  depending on the size and structure of  $\text{ETr}(\mathcal{A})$ . The size of table of translations may in the worst case equal  $n^{n+1}$ . Hence, the number of calculations generated by the third step of the algorithm may equal  $n^{2n+1}$ . The transitive closure can be calculated in  $\mathcal{O}(n^3)$  time.

Next we show a concrete example of how the algorithm works.

**Example 2.4.** Let  $\Sigma = \{f/1, g/1\}$ ,  $A = \{1, 2, 3, 4, 5\}$ , and let the operations be defined as in Figure 2.

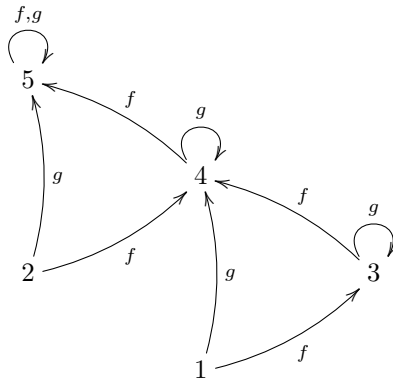


Figure 2: The algebra  $\mathcal{A}$ .

A direct calculation gives the table of translations for the algebra shown in Table 1. Note that for simplicity we have identified unary function symbols with the translations they define, and denoted  $fg$  the operation such that  $(fg)(a) = f(g(a))$  for all  $a \in A$ .

Table 1: Table of translations for  $\mathcal{A}$ .

$\text{Tr}(\mathcal{A})$	1	2	3	4	5
$id$	1	2	3	4	5
$f$	3	4	4	5	5
$g$	4	5	3	4	5
$ff$	4	5	5	5	5
$fg$	5	5	4	5	5
$fff$	5	5	5	5	5

Consider for example  $SH(5)$ . We have that  $5 = g(2)$ , which implies that  $g(f(2)) = 4 \in SH(5)$ , and continuing similarly  $g(f(1)) = 3 \in SH(5)$ . By performing the steps of our algorithm for all such decompositions we obtain the sets

$$\begin{aligned} SH(1) &= \{1, 3, 4, 5\} \\ SH(2) &= \{2, 3, 4, 5\} \\ SH(3) = SH(4) = SH(5) &= \{3, 4, 5\} \end{aligned}$$

We can form a quasi-order on a given algebra based on the inclusion of the ideals  $SH(a)$ . We denote this relation by  $\leq_{SH}$ , and we define it so that for all  $a, b \in A$ ,

$$a \leq_{SH} b \iff SH(a) \supseteq SH(b).$$

In fact,  $\leq_{SH} = \Rightarrow_{SH}^*$ .

In the spirit of Green's relations, we define  $\mathcal{SH} \subseteq A^2$  as the relation such that

$$a \mathcal{SH} b \iff SH(a) = SH(b).$$

By Lemma 2.2 it is a congruence. We say that  $\mathcal{A}$  is  $\mathcal{SH}$ -trivial if  $a \mathcal{SH} b$  implies  $a = b$ . It is clear, that the algebra is  $\mathcal{SH}$ -trivial, if and only if  $\leq_{SH}$  is an order. In the next section we investigate the properties of this order further.

As we saw in Example 2.4, the shuffle ideals  $SH(a)$  of a given finite algebra can be calculated using the algorithm presented earlier in this section. We can then calculate the quasi-order  $\leq_{SH}$ , and also determine whether the algebra is  $\mathcal{SH}$ -trivial or not.

### 3 Monotonically ordered algebras

In this section we investigate algebras which are equipped with a certain type of an order, namely a monotone order (see [3]). We show that algebras equipped with an admissible monotone order are bijectively connected to  $\mathcal{SH}$ -triviality.



**Definition 3.1.** An algebra  $\mathcal{A}$  is monotone, if there exists an order  $\leq$  on  $A$  such that for all  $n \geq 1$ ,  $f \in \Sigma_n$  and  $a_1, \dots, a_n \in A$ ,

$$(M) \quad a_1, \dots, a_n \leq f^{\mathcal{A}}(a_1, \dots, a_n).$$

Note that the condition (M) can be replaced with an equivalent condition:  $a \leq p(a)$  for all  $a \in A$  and for all  $p \in \text{ETr}(\mathcal{A})$ .

Let us recall that a relation on a set is called a *pre-order* if it is reflexive and transitive.

**Definition 3.2.** Let  $\theta$  be a pre-order on  $A$ . It is admissible, if  $a_1 \theta b_1, \dots, a_n \theta b_n$  imply  $f^{\mathcal{A}}(a_1, \dots, a_n) \theta f^{\mathcal{A}}(b_1, \dots, b_n)$  for all  $n \geq 0$ ,  $a_1, \dots, a_n, b_1, \dots, b_n \in A$  and  $f \in \Sigma_n$ .

Equivalently, a pre-order  $\theta$  is admissible, if for all  $a, b \in A$  and for all  $p \in \text{ETr}(\mathcal{A})$ ,  $a \theta b$  implies  $p(a) \theta p(b)$ . An *ordered algebra*  $(\mathcal{A}, \leq)$  consists of an algebra, and an admissible order  $\leq$  on  $A$ .

An ordered algebra  $(\mathcal{A}, \leq)$  is *monotone* if (M) is satisfied for the given order  $\leq$ . Following the definition presented in [7] we call an algebra  $\mathcal{A}$  *monotonically ordered* if there exists an ordered algebra  $(\mathcal{A}, \leq)$  which is monotone. Note that in [7] we used the term monotonously ordered.

Before our main result we prove a useful lemma.

**Lemma 3.1.** If  $(\mathcal{A}, \leq)$  is monotone, then  $a \Rightarrow_{\mathcal{SH}} b$  implies  $a \leq b$  for all  $a, b \in A$ .

*Proof.* Let  $a, b \in A$  be such that  $a \Rightarrow_{\mathcal{SH}} b$ . There exist  $q, r \in \text{Tr}(\mathcal{A})$  and  $c \in A$  such that  $a = q(c)$  and  $b = q(r(c))$ . Now, by the properties of the monotone order on  $A$ , we have that  $c \leq r(c)$ , and hence  $a = q(c) \leq q(r(c)) = b$ .  $\square$

**Theorem 3.1.** An algebra  $\mathcal{A}$  is monotonically ordered if and only if it is  $\mathcal{SH}$ -trivial.

*Proof.* Assume that  $\mathcal{A}$  is  $\mathcal{SH}$ -trivial. Then,  $\leq_{\mathcal{SH}}$  is a partial order on  $A$ . Also,  $a \leq_{\mathcal{SH}} p(a)$ , since  $SH(p(a)) \subseteq SH(a)$  by Lemma 2.5.

For proving that the order is admissible, let  $a, b \in A$  be such that  $a \leq_{\mathcal{SH}} b$ . Now,  $b \in SH(a)$ , and hence  $a \Rightarrow_{\mathcal{SH}}^* b$  by Lemma 2.4, which means that for some  $n \geq 0$ ,  $a \Rightarrow_{\mathcal{SH}}^n b$ . By Lemma 2.2, it follows that  $p(a) \Rightarrow_{\mathcal{SH}}^* p(b)$ , which implies that  $p(b) \in SH(p(a))$ , and therefore  $p(a) \leq_{\mathcal{SH}} p(b)$ .

For the other direction, let  $(\mathcal{A}, \leq)$  be monotone. Assume that  $SH(a) = SH(b)$  for some  $a, b \in A$ . Then,  $a \Rightarrow_{\mathcal{SH}}^* b$ . Now, by Lemma 3.1 we get directly that  $a \leq b$ . By a symmetric argument also  $b \leq a$ , which implies  $a = b$ , which proves that  $\mathcal{A}$  is  $\mathcal{SH}$ -trivial.  $\square$

In the next proposition we show that the order  $\leq_{\mathcal{SH}}$  is the least admissible and monotone order on a given monotonically ordered algebra. Before that, we give a simple example which shows that such an order on an algebra need not be unique.

**Example 3.1.** Let  $\Sigma = \{f/1\}$  and  $A = \{a, b\}$ . Define the algebra  $\mathcal{A}$  so that  $f^{\mathcal{A}}(a) = a$  and  $f^{\mathcal{A}}(b) = b$ . Now,  $\leq_{\mathcal{SH}} = \Delta_A$ , but the relation  $\{(a, a), (a, b), (b, b)\}$  is also a monotone and admissible ordering for  $\mathcal{A}$ .

**Proposition 3.1.** *If an ordered algebra  $(\mathcal{A}, \leq)$  is monotone, then  $\leq_{\mathcal{SH}} \subseteq \leq$ .*

*Proof.* If  $a \leq_{\mathcal{SH}} b$ , for some  $a, b \in A$ , then  $a \Rightarrow_{\mathcal{SH}}^* b$ , and Lemma 3.1 implies directly that  $a \leq b$ . □

As we shall see, in the term algebra  $\mathcal{T}_\Sigma(X)$ , the relation  $\Rightarrow_{\mathcal{SH}}^*$  equals the homeomorphic embedding relation of terms. Thus,  $\Rightarrow_{\mathcal{SH}}^*$  can be regarded as a generalization of the embedding relation for general algebras. Before the proposition, we note an obvious lemma.

**Lemma 3.2.** *For any leaf alphabet  $X$  and ranked alphabet  $\Sigma$ , the algebra  $\mathcal{T}_\Sigma(X)$  is monotonically ordered by  $\leq_{emb}$ .*

**Proposition 3.2.** *For any  $X$  and  $\Sigma$ , and  $s, t \in \mathcal{T}_\Sigma(X)$ ,*

$$s \leq_{emb} t \text{ if and only if } s \Rightarrow_{\mathcal{SH}}^* t$$

*Proof.* It follows immediately from the previous lemma, and Lemma 3.1, that  $\Rightarrow_{\mathcal{SH}}^* \subseteq \leq_{emb}$ .

For the other direction, we proceed by structural induction following the definition of the relation  $\leq_{emb}$ . Note that by the previous lemma,  $\mathcal{T}_\Sigma(X)$  is monotonically ordered, or equivalently  $\mathcal{SH}$ -trivial (Theorem 3.1), and  $\Rightarrow_{\mathcal{SH}}^*$  is an admissible, monotone order. Assume that  $s \leq_{emb} t$ .

1. If  $s = t$ , then  $s \Rightarrow_{\mathcal{SH}} t$ .
2. Assume that  $s = f(s_1, \dots, s_n)$  and  $t = f(t_1, \dots, t_n)$ , where  $s_i \leq_{emb} t_i$  for  $i = 1, \dots, n$ , and assume that the claim holds for  $s_i$  and  $t_i$  for all  $i = 1, \dots, n$ . Then,  $s_i \Rightarrow_{\mathcal{SH}}^* t_i$  for  $i = 1, \dots, n$ , and by the  $\mathcal{SH}$ -triviality of  $\mathcal{T}_\Sigma(X)$ ,  $f(s_1, \dots, s_n) \Rightarrow_{\mathcal{SH}}^* f(t_1, \dots, t_n)$ .
3. Assume that  $t = f(t_1, \dots, t_n)$  and  $s \leq_{emb} t_i$  for some  $i = 1, \dots, n$ , and assume that the claim holds for  $s$  and  $t_i$ . Then,  $s \leq_{emb} t_i$  implies  $s \Rightarrow_{\mathcal{SH}}^* t_i \Rightarrow_{\mathcal{SH}} t$ .

□

We conclude the section by considering some variety properties of monotonically ordered algebras. The class of  $\mathcal{SH}$ -trivial algebras (i.e. that of monotonically ordered algebras) is closed under forming direct products and subalgebras, but not homomorphic images [7]. Hence, the class is not a variety. However, in the following we show that the class of monotone ordered algebras is closed under order-preserving homomorphisms, which makes it a variety of ordered algebras [2].

In [2] a pre-order on an ordered algebra is said to be *admissible*, if it is an admissible relation, and contains the ordering of the algebra. If  $\preceq$  is an admissible pre-order on  $\mathcal{A}$ , then  $\sim = \preceq \cap \succeq$  is a congruence on  $\mathcal{A}$ , and  $\mathcal{A}/\sim$  is ordered by the relation  $\preceq$  defined so that for all  $a, b \in A$ ,  $a/\sim \preceq b/\sim$  if and only if  $a \preceq b$  (see [2], p. 201).

**Proposition 3.3.** *The class of monotone ordered algebras is closed under order-preserving homomorphisms, i.e. homomorphisms of ordered algebras.*

*Proof.* Let  $(\mathcal{A}, \leq)$  be a monotone ordered algebra. By Proposition 1.3 in [2], it is sufficient to look at the quotient algebras with respect to the admissible pre-orders on  $(\mathcal{A}, \leq)$ . Hence, assume that  $\preceq$  is an admissible pre-order, and consider the order  $\preceq$  on  $\mathcal{A}/\sim$  derived from  $\preceq$ , where  $\sim = \preceq \cap \succ$ .

Now, let  $n \geq 0$ ,  $a_1, \dots, a_n \in A$ , and  $f \in \Sigma_n$ . For every  $i = 1, \dots, n$ , it follows from  $a_i \leq f^A(a_1, \dots, a_n)$  that  $a_i/\sim \preceq f^A(a_1, \dots, a_n)/\sim = f^{A/\sim}(a_1/\sim, \dots, a_n/\sim)$ .  $\square$

Theorem 2.6 in [2] states that every variety of ordered algebras is defined by a set of inequalities. In the case of monotone orders such a set is immediately given by the definition.

**Example 3.2.** If  $\Sigma = \{f/2\}$ , then the class of monotone ordered  $\Sigma$ -algebras is defined by the set  $\{x \leq f(x, y), y \leq f(x, y)\}$ .

The class of languages corresponding to the class of finite monotonically ordered algebras can be characterized as follows. The  $k$ -piecewise testable tree languages for some fixed  $\Sigma$  and  $X$  were defined in [7] as the unions of  $\pi^k$ -classes, for a certain finite congruence  $\pi^k$ . It was also proved that the algebra  $\mathcal{T}_\Sigma(X)/\pi^k$  is monotonically ordered. Hence, each piecewise testable tree language can be recognized by a finite monotonically ordered algebra, and it was shown also in [7], that all languages recognized by finite monotonically ordered algebras are piecewise testable.

It is clear that the languages recognized by finite monotone ordered algebras in the sense of [6] are included in the variety of tree languages corresponding to the variety of finite algebras generated by the finite monotonically ordered algebras, which are exactly the piecewise testable tree languages. Hence, all languages recognized by finite monotone ordered algebras are piecewise testable. However, for example the language  $\{x\} \subseteq \mathcal{T}_\Sigma(X)$ , where  $X = \{x\}$  and  $\Sigma = \{f/1\}$ , cannot be recognized by a monotone ordered algebra in the sense of [6], even if the language is most certainly piecewise testable.

A shuffle ideal of a term algebra is clearly a piecewise testable tree language. Namely,  $SH(t)$  contains exactly all the terms which have  $t$  as a piecewise subtree. In fact, this implies directly that each piecewise testable tree language can be obtained as a boolean combination of suitable shuffle ideals. This generalizes the result that a piecewise testable word language is a boolean combination of shuffle ideals.

### Further remarks

We presented here a natural generalization of the shuffle ideal, and we established connections between the shuffle relation, the homeomorphic embedding relation and monotonically ordered algebras. Monotonically ordered algebras and the embedding relation were very useful in our earlier work on piecewise testability for trees [7], and hence it is not surprising, that the shuffle ideals investigated here have a similar connection to piecewise testability as in the word case.

Our definition of the shuffle ideal suggests also a definition for the shuffle operation, which would be suitable for terms of term algebras and elements of general

algebras. Such a product would be defined not between two elements, but rather between a translation and an element. Each translation can be decomposed (not in a unique way in general) into a product of elementary translations, and each element of an algebra can also be decomposed into a product of elementary translations and a generator of the algebra. By merging these sequences in a similar manner as shuffling two words, one obtains elements which form a set that could be seen as the shuffle of these objects.

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