On inaccessible and minimal congruence relations. I

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To professor Ladislaus Rédei on his 60th birthday

§ 1. Introduction

If we are given an abstract algebra A, then, partially ordering the set $\Theta(A)$ of all congruence relations of A under the usual rule: $\Theta \leqq \Phi$ if and only if $x \equiv y(\Theta)$ implies $x \equiv y(\Phi)$, $\Theta(A)$ becomes a complete lattice. It is customary to select two special types of congruence relations: the *inaccessible (from below)* congruence relations and the *minimal* ones. A congruence relation Θ is called inaccessible from below, if whenever the set $\{\Theta_\alpha\}$ of congruence relations is closed under finite joins, then $\Theta = \bigvee \Theta_\alpha$ implies $\Theta \in \{\Theta_\alpha\}$. A special type of inaccessible from below congruence relations is the minimal one: a congruence relation Θ is called minimal if there exists a pair of elements a, b of A, such that $a \equiv b(\Theta)$, and Θ is minimal with respect to this property. This Θ will be denoted by Θ_{ab}. (See [2] and [4].)

It is immediate from the definitions that the property of being inaccessible from below depends only on the structure of $\Theta(A)$, while the property of being minimal depends on the structure of A.

The minimal congruence relations are those which may be most easily described within A (see e. g. [3] and [5]). Further, the minimal congruence relations are those, which are closely connected with the elements of A. Therefore, in examining the structural properties of $\Theta(A)$, it seems to be useful to change A to an other abstract algebra $\bar{A}$ such that $\Theta(A) \simeq \Theta(\bar{A})$ and in $\Theta(\bar{A})$ as many congruence relations are minimal as possible. Since the minimal congruence relations are inaccessible from below, further the property of being inaccessible from below is preserved under lattice isomorphisms, we see that at most the inaccessible from below congruence relations of A may become minimal (relative to $\bar{A}$).

The aim of the present note is to prove that this optimal case may always be achieved, that is, we prove the following

Theorem. *To any abstract algebra A there exists an abstract algebra $\bar{A}$ such that $\Theta(A) \cong \Theta(\bar{A})$ and in $\Theta(\bar{A})$ every inaccessible from below congruence relation is minimal (relative to $\bar{A}$).*

In the light of this theorem it seems to have some importance to characterize all abstract algebras $\bar{A}$ with the property stated in the theorem. In other words, given a class of abstract algebras, to determine all algebras of this class in which every inaccessible from below congruence relation is minimal.

This problem is extremely difficult, we do not hope a general solution of it. In the second part of this paper we shall solve it in the very special case of distributive lattices.

For the notions we will make use of we refer to [1].

§ 2. Preliminaries

In the proof of our Theorem we shall need two lemmata which will be proved in this section.

If A is an abstract algebra, then let $M(A)$ denote the set of all operations defined on A.

Lemma 1. *To any abstract algebra A one can find an abstract algebra A' such that $\Theta(A) \cong \Theta(A')$ and $M(A')$ consists of operations of one variable.*

Proof. We define an abstract algebra A' on the same set as A. Let $\varphi(x_1, \ldots, x_n)$ be an arbitrary operation of A. If $n > 1$ we may fix $n-1$ variables of $\varphi(x_1, \ldots, x_n)$, to get an operation $\varphi(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$ of a single variable x_i. We define the operations of A' as follows: they are the operations of one variable of A, further all the operations of one variable that have been made from the operations of more than one variable of A in all possible ways. It is easy to prove that $\Theta(A) \cong \Theta(A')$. Even more is true: if under the natural isomorphism $\Theta \to \bar{\Theta}$, then the congruence classes modulo Θ are indentical with the congruence classes modulo $\bar{\Theta}$.

Lemma 1 makes us possible to restrict ourselves to general algebras in which all the operations are of one variable.

Let us suppose that we are given a set of general algebras $\{B_\alpha\}$ satisfying the following axioms:

(A) the sets $\{B_\alpha\}$ form a chain, that is, given B_α and B_β, either $B_\alpha \subseteq B_\beta$ or $B_\beta \subseteq B_\alpha$, and in case $B_\alpha \subseteq B_\beta$ the operations of B_α are extended[1]) to B_β, which formally may be denoted by $M(B_\alpha) \subseteq M(B_\beta)$;

[1]) That is to any operation $\varphi(x)$ of B_α there exists an operation $\psi(x)$ of B_β such that $\varphi(a) = \psi(a)$ if $a \in B_\alpha$.

(*B*) if $B_\alpha \subseteq B_\beta$ and $\Theta \in \Theta(B_\alpha)$, then there is one and only one congruence relation $\bar{\Theta}$ of B_β such that $a \equiv b(\bar{\Theta})$ $(a, b \in B_\alpha)$ is equivalent to $a \equiv b(\Theta)$.

$\bar{\Theta}$ is called the congruence relation *induced* by Θ.

We define an abstract algebra $\bar{B}$ as follows: $x \in \bar{B}$ if $x \in B_\alpha$ for some α and $M(\bar{B}) = \bigvee_\alpha M(B_\alpha)$. From (*A*) it follows that $\bar{B}$ is an abstract algebra. Now we state

Lemma 2. *If the abstract algebra $\bar{B}$ is added to the set $\{B_\alpha\}$ then the arising set also satisfies the conditions* (*A*) *and* (*B*).

Proof. Condition (*A*) follows directly from the definition of $\bar{B}$. To prove (*B*) we take a B_α and a $\Theta \in \Theta(B_\alpha)$. Denote by Φ_β that congruence relation of $B_\beta \supseteq B_\alpha$ which is induced by Θ. We define $x \equiv y(\bar{\Theta})$ if and only if $x \equiv y(\Phi_\beta)$ for some β. It is easy to check that $\bar{\Theta}$ is a congruence relation of $\bar{B}$ and the only one having the property: $x \equiv y(\Theta)$ $(x, y \in B_\alpha)$ if and only if $x \equiv y(\bar{\Theta})$. Thus property (*B*) and so Lemma 2 is proved.

Corollary. *The isomorphism $\Theta(B_\alpha) \cong \Theta(\bar{B})$ holds for every α.*

§ 3. The construction of A_1

Let an abstract algebra A be given. We may suppose — owing to Lemma 1 — that all the operations of A are of one variable. We fix two elements a, b of A and construct an extension A_1 of A such that in A_1 every congruence relation of the form $\Theta_{ax} \cup \Theta_{by}$ is minimal and of course $\Theta(A) \cong \Theta(A_1)$.

To do this we define formally to every element $x \in A$ a symbol x^* subject to the sole rule: $a = b^*$. The set of all x^* is denoted by A^*. Thus $x \to x^*$ is a one-to-one correspondence between A and A^* and the only common element of A and A^* is a.

We consider the set $A_1 = A \cup A^*$ and define operations of one variable on this set.

1. The definition of $f(x)$: if $x \in A$, $f(x) = x^*$ and $f(x^*) = a^*$.
2. The definition of $g(x)$: if $x \in A$ then $g(x) = b$, and $g(x^*) = x$.
3. We extend all the operations $\omega(x)$ of A to A_1 by setting $\omega(x^*) = \omega(a)$.

$M(A_1)$ consists of the operations $f(x), g(x)$ and the $\omega(x)$ defined under 1—3. A_1 with the operations $M(A_1)$ is an abstract algebra.

We first prove $\Theta(A) \cong \Theta(A_1)$.

Let $\Theta \in \Theta(A)$ and define $u \equiv v(\bar{\Theta})$ if and only if one of the following conditions hold:

I. $u, v \in A$ and $u \equiv v(\Theta)$;

II. $u = x^*, v = y^*$ and $x \equiv y(\Theta)$;

III. $x \equiv a(\Theta)$, $b \equiv z(\Theta)$ and $u = x$ or $u = z^*$ and $v = x$ or $v = z^*$.

We show that the relation $\bar{\Theta}$ of A_1 is a congruence relation.

It is clear that $\bar{\Theta}$ is reflexive and symmetric. To show the transitivity of $\bar{\Theta}$ suppose that $u \equiv v(\bar{\Theta})$ and $v \equiv w(\bar{\Theta})$. If $u, v, w \in A$ or $u, v, w \in A^*$, then the transitivity (i. e. $u \equiv w(\bar{\Theta})$) is obvious. Consider the case $u, v \in A$ and $w \in A^*$ (the case $u \in A$ and $v, w \in A^*$ is quite similar). Then we have $u \equiv v(\Theta), v \equiv a(\Theta), x \equiv b(\Theta)$ where x is defined by $w = x^*$ (we get these by III and I), thus by the transitivity of Θ we get $u \equiv a(\Theta)$ and hence by III $u \equiv v(\bar{\Theta})$. It remained to consider the case $u, w \in A$ and $v \in A^*$ (the case $v \in A, u, w \in A^*$ may be discussed in the same way). From the assumptions, owing to III, we get $u \equiv a(\Theta), b \equiv x(\Theta), v = x^*, w \equiv a(\Theta), b \equiv x(\Theta)$ and the first and third of these congruences imply $u \equiv w(\Theta)$ what is (by I) the same as $u \equiv w(\bar{\Theta})$.

Now we prove for $\bar{\Theta}$ the substitution law. We have to show that $\psi(x) \in M(A_1)$ and $u \equiv v(\bar{\Theta})$ imply $\psi(u) \equiv \psi(v)(\bar{\Theta})$. We distinguish three cases:

a) $u, v \in A$. Then $f(u) \equiv f(v)(\bar{\Theta})$ by II. Concerning the operation $g(x)$ it results the trivial $g(u) = b \equiv b = g(v)(\bar{\Theta})$. If $\omega(x)$ is an operation of A which is extended to A_1, then $\omega(u) \equiv \omega(v)(\Theta)$, thus by I $\omega(u) \equiv \omega(v)(\bar{\Theta})$.

b) $u, v \in A^*$. Then $u = x^*, v = y^*$ and we have $x \equiv y(\Theta)$. Thus $x^* \equiv y^*(\bar{\Theta})$ and concerning the operation $f(x)$ we get $a^* \equiv a^*(\bar{\Theta})$. The operation $g(x)$ yields $x \equiv y(\bar{\Theta})$ which is by I also true. Finally, with an $\omega(x)$ we get $\omega(a) \equiv \omega(a)(\bar{\Theta})$ which is also trivial.

c) $u \in A$, $v \in A^*$. Then $v = x^*$ and the relations $u \equiv a(\Theta), b \equiv x(\Theta)$ are valid. From $u \equiv v(\bar{\Theta})$ we get concerning the operations $f(x), g(x)$ and $\omega(x)$ the following relations: $f(u) \equiv f(a)(\bar{\Theta}), b \equiv x(\bar{\Theta}), \omega(a) \equiv \omega(u)(\bar{\Theta})$ and all these relations are easy consequences of $u \equiv a(\Theta)$ and $b \equiv x(\Theta)$.

Hence we have proved that $\bar{\Theta}$ is a congruence relation. The following step is to show that the correspondence $\Theta \to \bar{\Theta}$ is an isomorphism between $\Theta(A)$ and $\Theta(A_1)$.

The congruence relation $\bar{\Theta}$ of A_1, induces in the natural way an equivalence relation on A which is just Θ. Since Θ completely determines $\bar{\Theta}$ the mapping $\Theta \to \bar{\Theta}$ is one-to-one from $\Theta(A)$ *into* $\Theta(A_1)$. It remains only to show that it is *onto*.

To do this suppose $\Phi \in \Theta(A_1)$. We define a relation Θ of A by $x \equiv y(\Theta)$ $(x, y \in A)$ if and only if $x \equiv y(\Phi)$. This Θ is a congruence relation of A and we prove $\bar{\Theta} = \Phi$. Since the laws I—III are consequences of the transitivity of Θ alone and that of the substitution law, therefore $\bar{\Theta} \leqq \Phi$ is trivial. Thus we have to show only that every relation $u \equiv v(\Phi)$ follows from the relations of type $x \equiv y(\Theta)(x, y \in A)$, using the laws I—III. Let $u \equiv v(\Phi)$. If $u, v \in A$, then the assertion is trivial. It is also clear in case $u, v \in A^*$, for the validity of $x \equiv y(\Theta)$ is equivalent to that of $x^* \equiv y^*(\bar{\Theta})$. If $u \in A$ and $v \in A^*(v = x^*)$, then from $u \equiv v(\Phi)$ we get $u^* = f(u) \equiv f(v) = a^*(\Phi)$ and then $u = g(u^*) \equiv g(a^*) = a(\Phi)$. Thus $a \equiv v(\Phi)$ that is $b = g(a) \equiv g(v) = g(x^*) = x(\Phi)$. Thus we have under Θ the congruences $u \equiv a(\Theta)$ and $b \equiv x(\Theta)$ from which using the law III we get the required $u \equiv v(\Phi)$, as we wished to prove.

Summing up, $\Theta \rightarrow \bar{\Theta}$ is a one-to-one correspondence between $\Theta(A)$ and $\Theta(A_1)$, further, from the definition of $\bar{\Theta}$ it is clear that $\Theta > \Phi$ if and only if $\bar{\Theta} > \bar{\Phi}$. This implies that the correspondence in question is an isomorphism, thus

$$\Theta(A) \simeq \Theta(A_1).$$

Secondly, we prove

$$\bar{\Theta}_{ax} \cup \bar{\Theta}_{by} = \bar{\Theta}_{xy^*}.$$

Indeed, $b \equiv y(\bar{\Theta}_{ax} \cup \bar{\Theta}_{by})$ and from this we obtain $a = b^* = f(b) \equiv f(y) = y^*(\bar{\Theta}_{ax} \cup \bar{\Theta}_{by})$ and comparing this with $x \equiv a(\bar{\Theta}_{ax} \cup \bar{\Theta}_{by})$ it results that $x \equiv y^*(\bar{\Theta}_{ax} \cup \bar{\Theta}_{by})$, that is, $\bar{\Theta}_{xy^*} \leqq \bar{\Theta}_{xa} \cup \bar{\Theta}_{by}$. Conversely, starting from $x \equiv y^*(\bar{\Theta}_{xy^*})$ we get $f(x) \equiv f(y^*) = a^*(\bar{\Theta}_{xy^*})$, that is, $x = g(f(x)) \equiv g(a^*) = a(\bar{\Theta}_{xy^*})$ and the transitivity implies $a = f(b) \equiv f(y)(\bar{\Theta}_{xy^*})$ and $b \equiv y(\bar{\Theta}_{xy^*})$, that is, $\bar{\Theta}_{ax} \cup \bar{\Theta}_{by} \leqq \bar{\Theta}_{xy^*}$, finishing the proof of the equality.

Thus A_1 has all the properties stated at the beginning of this section.

§ 4. The proof of the Theorem

Consider the abstract algebra A and define the set H as the set of all (unordered) pairs of the elements of A. We fix a well-ordering of H and to each $q_\alpha \in H$ we define an abstract algebra A_α in the following way: $A_0 = A$ where q_0 is the first element of H; if A_β is defined for all $\beta < \alpha$ then A_α is that general algebra which is constructed with the method of § 3 from $\bigcup_{\beta<\alpha} A_\beta$ if the fixed pair a, b of elements is just q_α.

We define the abstract algebra A^1 by

$$A^1 = \bigcup_\alpha A_\alpha.$$

We construct from A^1 an abstract algebra A^2 in the same way as A^1 was constructed from A, etc.

$$A \subseteq A^1 \subseteq \cdots \subseteq A^n \subseteq \cdots \qquad (n = 1, 2, \ldots)$$

Let $\bar{A}$ be the union of this chain of abstract algebras. We assert that the abstract algebra $\bar{A}$ fulfill the requirements of the Theorem.

First we prove $\Theta(A) \cong \Theta(A_\alpha)$ by transfinite induction on α. If this is true for all $\beta < \alpha$ and if $\alpha = \gamma + 1$, then the results of § 3 ensure $\Theta(A_\gamma) \cong \Theta(A_\alpha)$, the hypothesis implies $\Theta(A) \cong \Theta(A_\gamma)$ and thus $\Theta(A) \cong \Theta(A_\alpha)$. If α is a limit ordinal, then the Corollary of Lemma 2 together with the results of § 3 prove $\Theta(A) \cong \Theta(A_\alpha)$. Again, owing to Corollary of Lemma 2 we get $\Theta(A) \cong \Theta(A^1)$. In the same way we conclude that $\Theta(A) \cong \Theta(A^i)$ for all i, and again Corollary of Lemma 2 guarantees $\Theta(A) \cong \Theta(\bar{A})$.

Secondly, we prove that every inaccessible from below congruence relation of $\bar{A}$ is minimal. If this were not true, then there would exist an inaccessible from below congruence relation Φ of $\bar{A}$ which is not minimal; it may be supposed that Φ is of the form $\Phi = \Theta_{a_1 a_2} \cup \Theta_{a_3 a_4}$. The a_i-s are elements of $\bar{A}$ thus there exists an A^j containing all the a_i—s. From the construction of A^{j+1} it follows that Φ is minimal in A^{j+1}, thus in $\bar{A}$ too, a contradiction. Thus our Theorem is completely proved.

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