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DOUBLE-LAYERED EQUATION OF MOTION: PLATONIC-ARCHIMEDEAN SPHERICAL CELLULAR AUTOMATA IN THE SOLUTION OF THE INDIRECT VON-NEUMANN PROBLEM ON SPHERE FOR TRANSFORMATIONS OF REGULAR TESSELLATIONS

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ABSTRACT

This paper shows a method of the parallel description of transformations of a structure which involves two levels of hierarchy and its transformations concern both hierarchy levels. The method reworks the cellular automaton model, but uses it in the direction of the indirect von Neumann problem (BÉRCZI 1991).

Tabular order of the transformational relations between Platonic and Archimedean (without distinguished rotational axis) solids was given on the basis of the truncation operation 13 years ago (BÉRCZI 1980). In this paper the reformulation of the system by a cellular automata in the form of indirect von-Neumann problem (BÉRCZI 1985) solution is given.

THE TRANSFORMATIONS OF A WHOLE BY ITS PHASES

If we are not present during the transformations of an object: the WHOLE, then we need to introduce a principle to be able to describe these transformations. Step by step in time, we may observe the transformations, so we can describe the process by the sequence of the moment-observations. These moment observations are the phases of the transformational process. So the description of the process will be given in the form of the sequence of these phases.

THE TWO HIERARCHY LEVELS OF SYMMETRY

The concept of symmetry inherently involves two levels of hierarchy of the structure. Symmetry is the invariance of a WHOLE over transformations, which may exchange some of the ELEMENTS, which build up the WHOLE. So the ELEMENTs and the WHOLE form two different levels of hierarchy. If symmetry is manifested in the form of a geometrical object, then the elements, which build up the geometrical object, form a pattern. This pattern is a regular arrangement on a surface. Both the regular LOCAL arrangement of the elements, and the regular GLOBAL structure of the WHOLE are important constituents of the SYMMETRY OF THE WHOLE. Both regularities of the two hierarchy levels are used up in construc-

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tion and specification of the cellular automaton model. The coherency between the two regularities makes it possible to decipher local operations of the elements from the global transformations of the whole. This problem of deciphering will be the indirect von-Neumann problem.

THE CELLULAR AUTOMATON MODEL: A TENTATIVE AXIOMATIC APPROACH

The unification of the former two structural descriptions: the TRANSFORMATIONS OF A WHOLE BY ITS PHASES (*Fig. 1*) and the SYMMETRY OF A WHOLE results in a constructive approach to build up the cellular automaton model. The

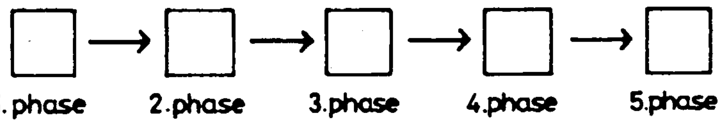


Fig. 1. The TRANSFORMATIONS BY ITS PHASES principle.

principle: TRANSFORMATIONS BY ITS PHASES gives a step by step discrete description of the transformations of the WHOLE and the ELEMENTS. The principle: SYMMETRY OF THE WHOLE gives the structural constraint of the description of transformations on two hierarchy levels: on the level of the ELEMENTS, and on the level of the WHOLE. (*Fig.* 2). So in the cellular automaton model the description of the transformations is given on both hierarchy levels in parallel form. (*Fig. 3* and *4*)

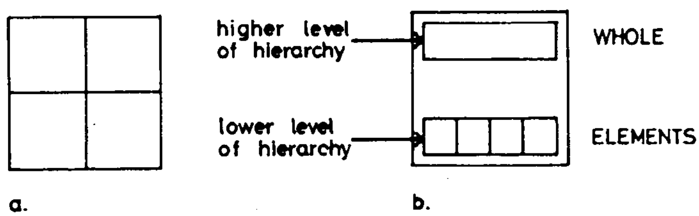


Fig. 2. The decomposition of the WHOLE, which has symmetries. **a.** The symbol shows, that elements build up the whole. **b.** The symbol shows the two hierarchy levels of the whole; used later.

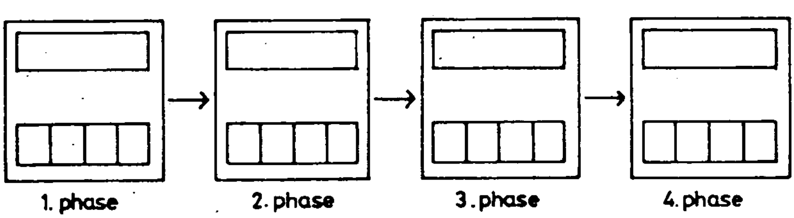


Fig. 3. The WHOLE with symmetry and so with two hierarchy levels has been substituted into the graph of the TRANSFORMATIONS BY ITS PHASES principle (*Fig. 1*) in the form of **b.** of *Fig. 2.*

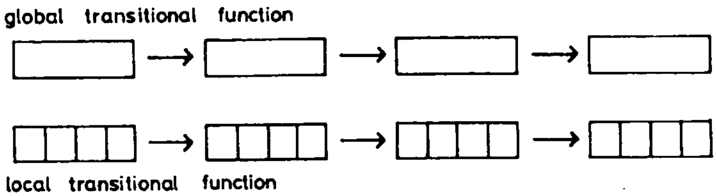


Fig. 4. Separation of phase blocks to its two parts gives the most characteristical framework of the cellular automatic description: the two parallel "equations of motion": that on the higher level of hierarchy — i.e. the GLOBAL TRANSITIONAL FUNCTION, and that on the lower level of hierarchy — i.e. the LOCAL TRANSITIONAL FUNCTION. Both functions are discrete functions in time. Detailed specifications of the model are given in *Fig. 5.*

FRAMEWORK OF THE CELLULAR AUTOMATIC SYSTEM'S DESCRIPTION

On the basis of the three introductory principles and other earlier works (i.e. VOLLMAR 1978, BÉRCZI 1991) we may summarize the sturctural characteristics and specifications of the description style of cellular automata as a framework. It is composed from two parts of the description on two hierarchy levels. The two parts are: the CELLULAR BACKGROUND, which is the structural one, and the TRANSITIONAL FUNCTION, which is the kinematical one. The two levels of hierarchy are: the LOCAL one, that of the cells, and the GLOBAL one, that of the WHOLE, the SYSTEM of the cell-mosaic itself.

The two X two (2x2) system of condiditons of description in the framework of cellular automata system suggests a matrix-summary of conditions. So first we give the tabular form (from BÉRCZI 1991) and then the Relation-table of the conditions (*Fig. 5*).

A. CELLULAR BACKGROUND

Aa. Local characteristics of the cell-mosaic system give the form of cells, their connections and neighbourhood relations (with initial condiditons)

Ab. Global characteristics of the cell-mosaic system give the enclosure of the local relations to form a whole, a surface of the cell-mosaic system

B. TRANSITIONAL FUNCTIONS

Ba. Local transitional function for cell-mosaic elements which are individual automata (this is a discrete funciton of steps in space and time)

Bb. Global transitional function for the whole surface built up by the cell-mosaic system as a whole (this function is also a discrete one consisting of the sequence of the stages of discrete transformations summarized from cellular steps)

Cellular automatic framework about description of transformations	L O C A L	G L O B A L
B A C K G R O U N D	Aa the form, the elementary neighbourhood, the connections and the initial states of cells	Ab the cell mosaic system built together from cells; a surface or spatial region with its initial parameters
T R A N S I T I O N	Ba transition of cellular states, / local transitional function/: depends on the states of neighbour cells, on the earlier state of the cell itself, and on the program written into the cell	Bb transition of the cell mosaic system /surface or spatial region/ composed of the cells: this global transitional function is summarized from the local transitions of the cells, step by step in time
H I E R A R C H Y	First, lower level of hierarchy	Second, higher level of hierarchy

Direction of construction of operation — direct von Neumann problem →

Deciphering of state changes and their description — indirect von Neumann problem ←

Fig. 5. Relation table of the specifications of the cellular automata models. Directions of the direct and indirect VON NEUMANN problems are also shown below.

DEFINITION OF THE INDIRECT VON-NEUMANN PROBLEM

If we consider the cellular automatic description of a deformational motion of a cell-mosaic-system as a new motion-description on a flexible or plastic background, then the first step is the formation and definition of the background and the second one is the formulation of transitional function. But in the cellular automaton modelling there is a double-level description, so the direction of problem formulation is open both for Ba→Bb and Bb→Ba cases. The classical way of construction and development of the cellular automaton modelling was: /1/ construction of Aa and Ab background, /2/ construction of the Ba local transitional function, /3/ deduction of the Bb global transitional function. Although iteration could happen between Ba and Bb function formulation, in the given sequence of formulation of problem solving the last step was the summary of the model. So the Ba→Bb direction of construction is characteristical to such modelling. We call this direction of construction to the direct von NEUMANN problem. The principal aim of the construction of von Nemann's cellular automata model was to build a self-reproducing structure on the level of the global background by a global transitional function.

The direction of our efforts in problem solutions in this paper is the opposite direction in the level of transitional functions, if compared to that of von NEUMANN's problem solution. Our aim is to read out local transitional function (uniform for all cells) from the given or reconstructed global transitional function. Therefore we call our program and formulation to indirect von NEUMANN problem (Bb→Ba). *(Fig. 5)*

FEED-BACKS AND COHERENCY OF LOCAL AND GLOBAL STRUCTURE

Symmetry of the cell-mosaic background means a coherency of regularity between its local and global structure. Therefore symmetry results in simplicity in the formulation of the transitional functions in the cellular automaton model (Ba and Bb). But symmetry of the cell-mosaic structure has another benefit, too. Symmetry may make it possible to formulate easily the direct (Ba→Bb) or the indirect (Bb→Ba) program in model-construction; these transcriptions make complete the cellular automaton model. In our formulations of the indirect programs (transcriptions of the global transitional function into the local one) we shall use up these benefits of symmetry.

Global structure (background, *Fig. 6*) is a kind of feed-back of local regularity (transitional function, *Fig. 6*) into itself, if the surface is at least partly closed. Movements, which are important cellular-automatic local operations in our formulations, partly rearrange this feed-back structure, too. Movements between cells are allowed, because a degree of freedom remains to carry out it: /1/ if the surface is only partly closed, or /2/ if a regular separation-operation make cells partly and temporaly free in the temporaly and partly loosed structure. After rearranging cell-movements cells fix their new contacts. These kinds of operations are the main benefits of our modelling symmetry by cellular automata (BÉRCZI 1985).

In our paper we refer to other ones and show one such kind of model where partial and local deformations of the cells form the local operations.

THE MAPPING OF PHENOMENA IN ORDER TO SELECT THOSE WHICH NEED CELLULAR AUTOMATIC DESCRIPTION

We may construct a compositional diagram for three basic components. The components are: basic characteristics of phenomena in everyday experiences. These were the following basic characteristics: RIGID, GRANULAR-COARSE, and CRUMPLED-SOFT. Different mixing of these components in phenomena occurs in those ones which take place inside the triangle (according to the compositional regulas of VIVIANI's theorem). The mostly viable phenomena which are combined from the three basic characteristics can be found in the centre of the triangle (*Fig. 7.*): they are weighted with almost equal weights from the three components. These phenomena are: crops of plants and some parts of plants themselves, embryos, cell systems and nets, and crystal-structure rearrangements, etc.

Cellular automatic description of cell-mosaic system's deformational transformations does not fix the time-scale of the phenomena. So the same transformational process may represent a short and a very long time process. That is the case

		SPECIFICATIONS IN THE CELLULAR AUTOMATON MODEL			
		BACKGROUND		TRANSITIONAL FUNCTIONS	
		Aa LOCAL	Ab GLOBAL	Ba LOCAL	Bb GLOBAL
CLASSICAL FORM OF DESCRIPTION OF TRANSFORMATIONS IN CELL-SYSTEM	SYSTEM OF REFERENCE	THE NETWORK (MOSAIC) AND SHAPE OF THE CELLS	INVARIANT PROPERTIES OF SURFACE		
	INITIAL CONDITIONS	SPECIFICATIONS OF THE INITIAL VALUES OF CELL STATES	SPECIFICATIONS OF THE INITIAL PARAMETERS OF THE NETWORK		
	BORDER CONDITIONS		SPECIFICATIONS OF THE TYPE OF THE SURFACE (TOPOLOGY)		
	FEEDBACK INVOLVED IN STRUCTURE		DETERMINED BY RETURNING BANDS OF CELLS ON THE SURFACE	DETERMINED BY NEIGHBOURHOOD EFFECTS OF CELLS	
	EQUATIONS OF MOTION			TRANSITIONS OF STATES OF CELLS DEPEND ON CELL INTERACTIONS	TRANSITIONS OF THE WHOLE SURFACE SUMMED UP FROM CELL-TRANS.

Fig. 6. Feedbacks appear on two hierarchy levels and in two types of specifications in the cellular automaton model. (with black frame). Feedback in the global background is resulted in by the enclosure of the surface into itself (or with a BORN-KRÁMÁN boundary condition).

with our crystal structures during crystallization, or recrystallization under high pressure. Different final products of the process can be found in different truncated stages but the reconstruction of the transformational sequence can be considered as a short term transformation during the development of an individual, (an entity of a mineral,) and at the same time it can be considered as a stage of transformational process which resulted in reaching of different final stages of the process modelled by the cellular automatic description. So our first example will show the cellular automatic description of truncation of Platonic and Archimedean solids and tesselations: it is partly a summary of earlier papers (BÉRCZI 1979, 1980, 1991).

EVOLUTIONARY SERIES FROM MULTITUDE.

Where can be found a community of the same type of structures in order to select a representative series for the development and evolution of the structure of

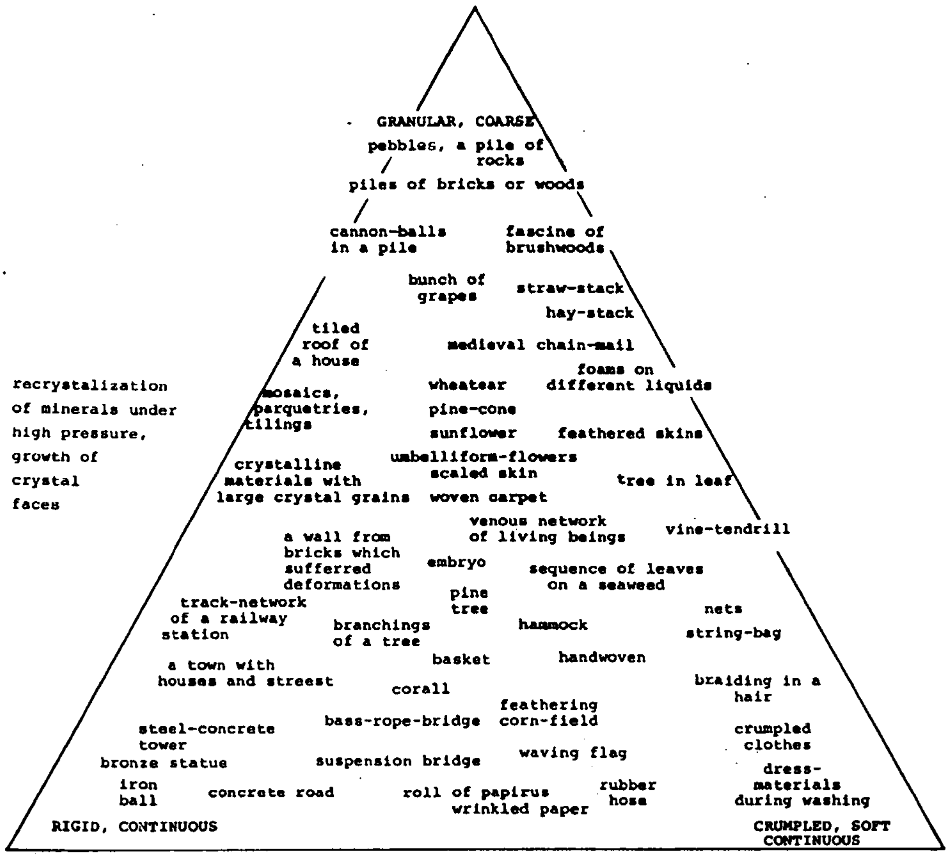


Fig. 7. Mapping of phenomena (different structures) according to three basic characteristics: RIGID, GRANULAR-COARSE and CRUMPLED-SOFT. The problem shown in this paper can be found in the center of this map, signed by a dot. Other related problems can be found in frame.

such type? Because the description: CHANGES BY THE PHASES needs such a community as a background of the phenomena to model the evolutionary events. Individuals in the community grow with different speeds, so in a community (or multitude) we can find representatives from different periods of the life-stage of individual structures. At one moment these individuals of such multitude represent a stage of an evolutionary or developmental sequence (depending on the time scale to be considered). Such multitude for example can be an open or globular cluster of stars, or a forest of trees. Observing the characteristics of the elements (the wholes, the individuals) of the multitude, a sequence of representative individuals can be selected from the multitude, they can be arranged into an evolutioary series according to these time dependent characteristics. This sequence (or chain) of the whole (individuals) gives a discrete description about the stages of transformations of the structure type, so in this form the principle of CHANGES BY THE PHASES of transformations was used on the multitude. The principle of selection of EVOLUTIONARY SERIES FROM MULTITUDE is an important tool, when we intend to formalize the abstract way of the foundation of cellular automatic description of wholes. (*Fig. 8.*) A further principle is necessary, however, if we want a next step to the double-layered description of transformations. This principle has been detailed earlier: the SYMMETRY OF THE WHOLE, which implies two layered structure of the whole.

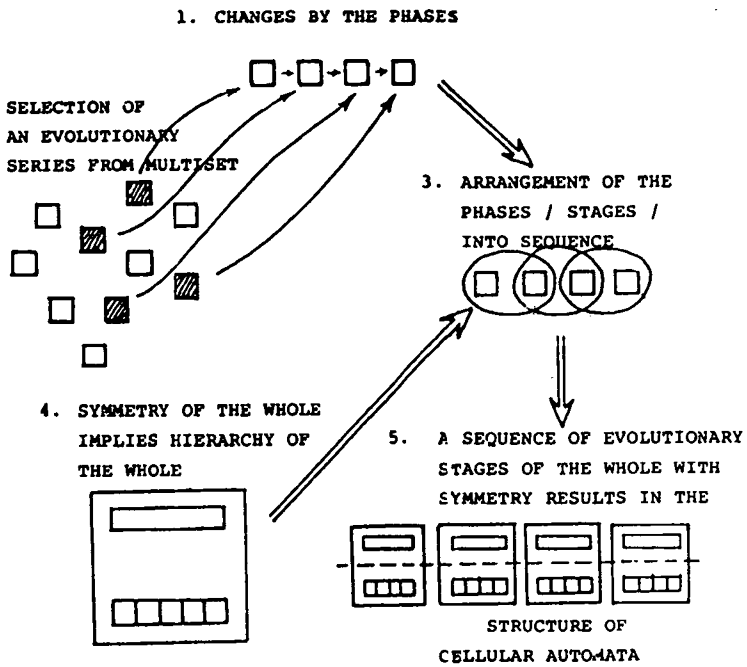


Fig. 8. The summary of principles which were necessary to the foundation of cellular automata description of the state changes of a whole with symmetry. The steps of the using of these principles in solving a problem can be given as follows: /1/ I can describe kinematics by CHANGES BY THE PHASES of a whole. /2/ Because of evolutionary events on individual wholes I can select a series of wholes from a multiset to describe the evolutionary steps. /3/ I can arrange them into a sequence. /4/ If the wholes are with symmetry, than they have two layers of hierarchy in their structure. /5/ Then cellular automata decomposition of the evolutionary sequence is possible.

THE "ACTORS" OF OUR MODEL: THE PLATONIC AND ARCHIMEDEAN SOLIDS

The basic process to be studied here by cellular automata description is the hypothetical changes of faces on crystal. But the same process advances when recrystallization happens under high pressure: there the coordination numbers change because of different compressibility of different ion-balls. We shall see that this second type of transformation is a higher dimensional equivalent "truncation" problem, as the first one. Both cases can be used in formulating the most simple cellular automata model, in crystallography.

Transformation of a spherical cellular system is a common phenomenon in development of embryonal structures too. These transformations may be formulated according to different languages of descriptions. Among the most simple cases of such transformations there are those which preserve some properties of the initial cellular arrangement. One form of initial simplicity of cellular arrangements is the symmetry. Now we study those spherical cell-arrangements which have symmetries that of the Platonic solids. We do not distinguish solids (which are covered by regular faces) and spherical tessellations (which are the corresponding cellular mosaics on the circumscribed spheres by central projection of the former solids).

Platonic solids are covered by congruent regular polygons of the same kind, while Archimedean solids are covered by 2 or 3 types of such regular polygons. Symmetry of these solids means regularity of not only the covering polygons, but the uniformity of the vertex configurations, too. This uniformity of the vertices allows a simple naming of these solids according to the polygons meeting at a vertex (listed in a given circulating order around a vertex). These are the STEINER symbols: the cube is named (4,4,4), the octahedron is (3,3,3,3) according to Steiner's terminology, (where numbers mean the sides of a regular polygon meeting at a vertex). The solids involved in our problem-solving are given in *Fig. 9*. Those solids, which traditionally are also Archimedean solids, and can be

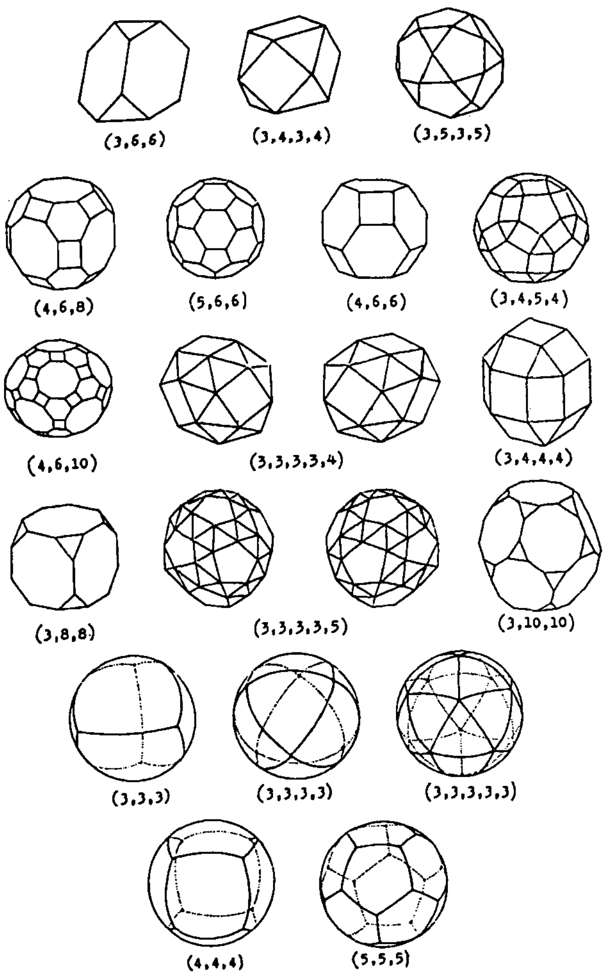


Fig. 9. The complete set of Archimedean solids (upper four rows) and Platonic solids (in their spherical tessellation form, lower two rows) which were arranged in a periodic table according to a cellular automatic operation: by truncation.

given by Steiner symbols, but has distinguished rotational axis, do not take part in our transformational system (*Fig. 10*). These prisms and antiprisms can not involved into a sequence where solids are listed according to a coherent transformational operation on their faces (on sphere on cells): this operation is the truncation.

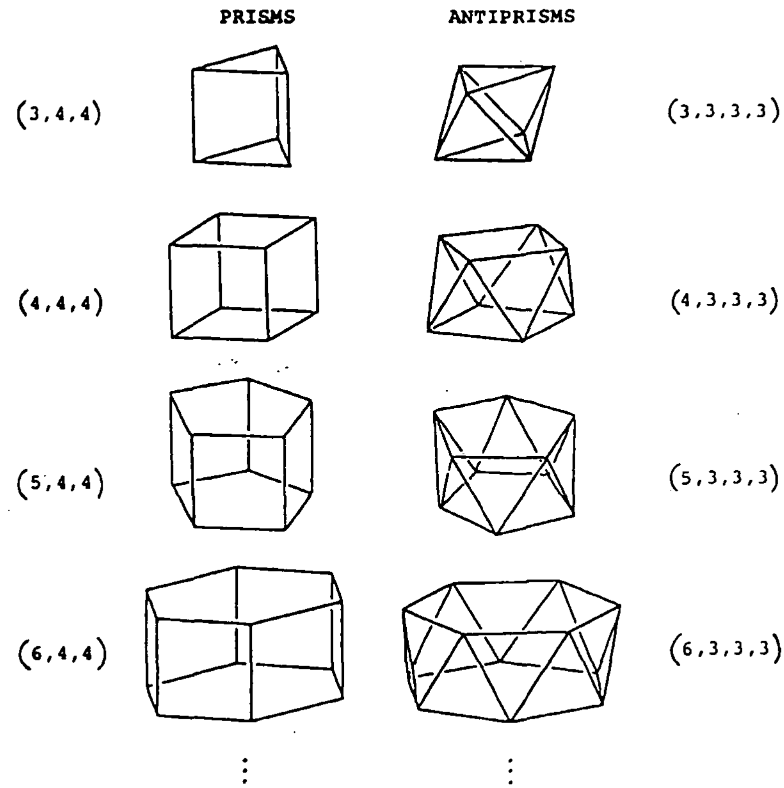


Fig. 10. The complete set of those Archimedean solids, which were not involved in the periodic table, because they have distinguished rotational axis (except (3,3,3,3) and (4,4,4), which are regular solids).

FORMULATION OF TRUNCATION AS CELLULAR AUTOMATIC OPERATION

The members of the complete set of regular and semiregular cellular arrangements on sphere (*Fig. 11*) were considered as stages of tranformational sequences, where the transformation was generated by an operation: the truncation (BÉRCZI 1980.). This truncation changes the cellular surface of a Platonic or Archimedean solid (or spherical mosaic) but does not change the symmetry group of the solid. The concept of truncation has a visually imaginable meaning for solids: the pyramids at vertices of regular solids are cut leaving a face on the place of the vertex: the base of pyramid. The cutting plane is perpendicular to the radius vector coming from the center of the regular solid to the vertex. Advanced truncation cuts truncated pyramids. In the spherical case truncation means: blowing up of initial vertex "points" of a regular spherical tessellation. In both variants of coordinate

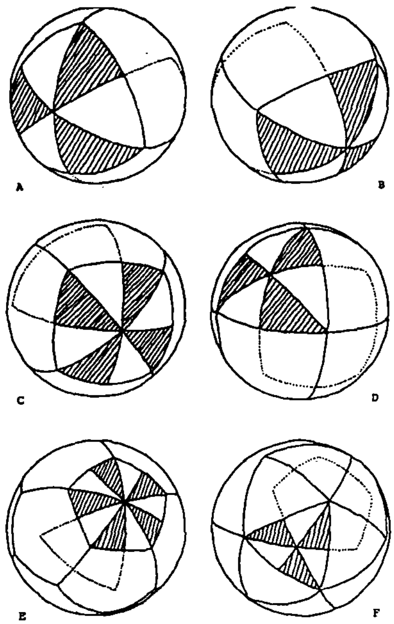


Fig. 11. Faces of dual regular solids are different regular-face-forming blocks of fundamental regions in the spherical tessellation of fundamental regions of tetrahedral- (A,B,), octahedral-(C,D,), and icosahedral- (E,F,) groups. A third kind of block-forming should result in rombohedral faces: these solids are also missing from the periodic table.

systems (solid-representation or spherical representation) three different stages with equal edge-lengths for all faces appear, when trucation started from a regular solid and advanced till the reaching of the dual solid of the initial one. The two closing regular (Platonic) solids in the truncational sequence flanks three Archimedean solids as stages of truncation operation. It is shown for the case of cube-octahedron sequence in *Fig. 12 and 13.*

The definition of the indirect von Neumann problem (implicit formulation: BÉRCZI 1980, 1985, explicit formulation: BÉRCZI 1991.) allows an easy formulation of the truncational transformations (*Fig. 14*). Let us consider the simple truncational sequences as global transitional functions with 5 stages (steps) for solids. *(Fig. 15)* Then the local transitional functions are the blowing up sequences for vertices (or complementary equivalents: the truncational sequences of initial faces or polygons (or cells)). These formulations — the global and local transitional functions — were given parallel for the higher dimensional case of the transformations for spatial cube-tessellation (BÉRCZI 1980.). This was the first implicit formulation of the Platonic-Archimedean Spherical Cellular Automata (PASCA). *(Fig. 16)*

EXTENSIONS OF PASCA: THE PERIODIC TABLE AND HIGHER DIMENSIONAL DEVELOPMENTS

Fitting together the corresponding PASCA sequences of the three 3D spherical symmetry groups (terahedral, octahedral, icosahedral) and also to planar and hyperbolic tessellations, a periodic table was given, as a summary of the deduction system built by truncation. (*Fig. 17*, BÉRCZI 1980) .This Periodic Table of Platonic

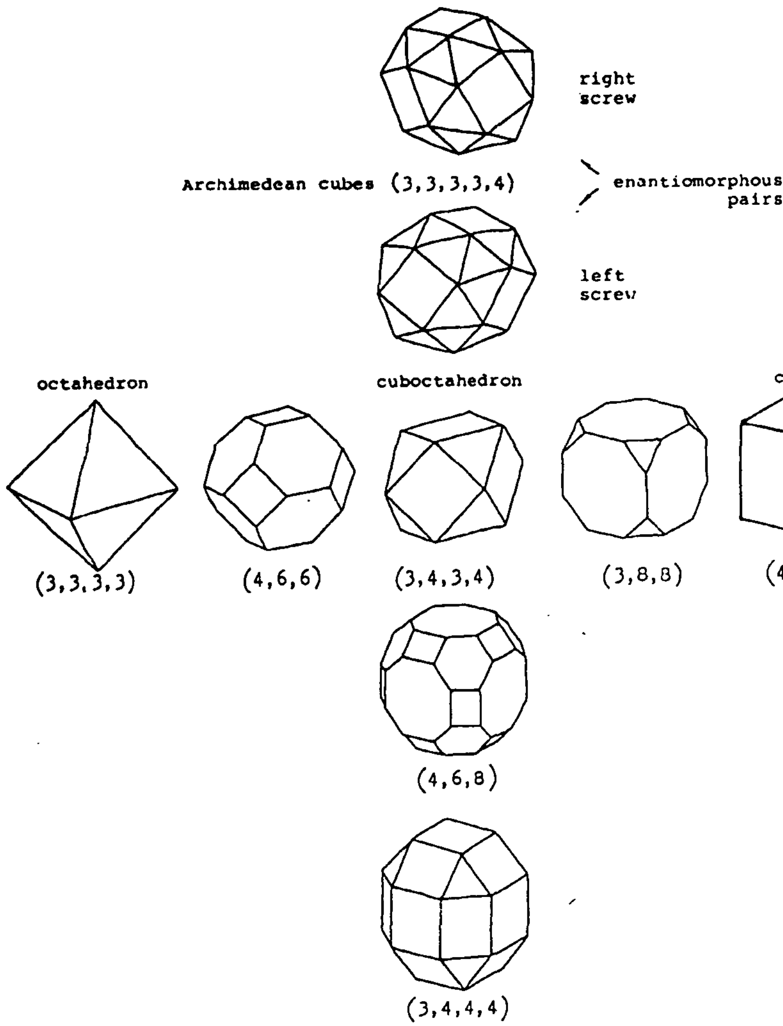
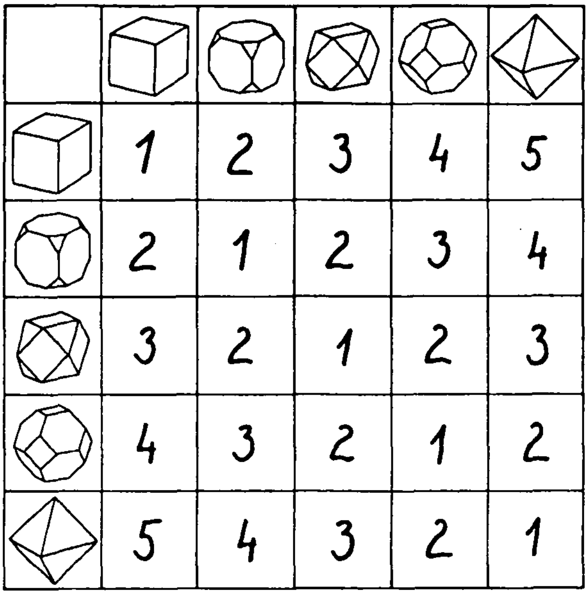


Fig. 12. Truncations in the octahedron-cube (hexahedron) system. The two Platonic solids close the simple truncation sequence. Halfway between them, the cuboctahedron is the generator of the complex truncation sequence (below (3,4,3,4)), and the snub-truncated enantiomorphous pairs (above (3,4,3,4)). The simple truncation sequence is considered to be the global transitional function in *Fig. 14.*



1	2	3	4	5
2	1	2	3	4
3	2	1	2	3
4	3	2	1	2
5	4	3	2	1

Fig. 13. Truncational distance matrix of the cube-octahedron sequence. Numbers mean the number of steps between the solids of a column and a row. Identical step is 1, without further truncation.

and Archimedean Solids and Tessellations shows the price payed for the ordered arrangement of solids given in *Fig. 9.* Some of the structures occurs more than once. But, on the other hand, the functional aspect of the relations between solids arises the source of simplicity: it is both in symmetry of initial conditions (Sphere, symmetric mosaic), and in formulation of the operation. Indirect von Neumann problems can be formulated and solved for the first cases when symmetry reduces the number of states of cells to small numbers. (*Fig. 18*)

SUMMARY

A classical problem of the functional arranging of regular and semi-regular (Platonic and Archimedean) solids (and tessellations) was solved and tabularly formulated by a cellular automatic operation using up the framework of the indirect von Neumann problem. (*Fig. 19*) The cellular transformation were double-terminated by regular dual-solids; therefore these arrangements of cells seemed more like a wave-motion of a global state of the spherical surface between opposite wave-formations. But the method and the approach to the problem may be useful, when more complex cellular states and arrangement are to be discerned and described.

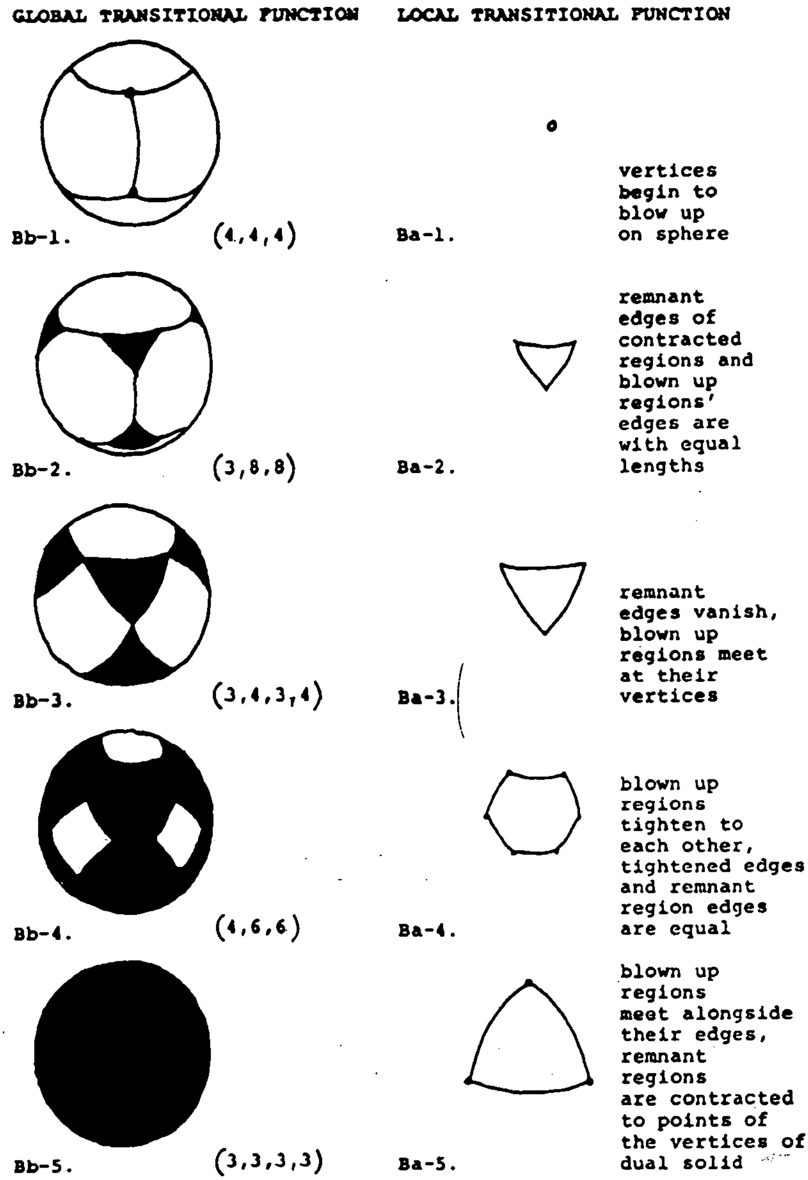


Fig. 14. The global (left column) and the local (right column) transitional function in the cellular automatic formulation of the truncational transformation between duals of regular solids.

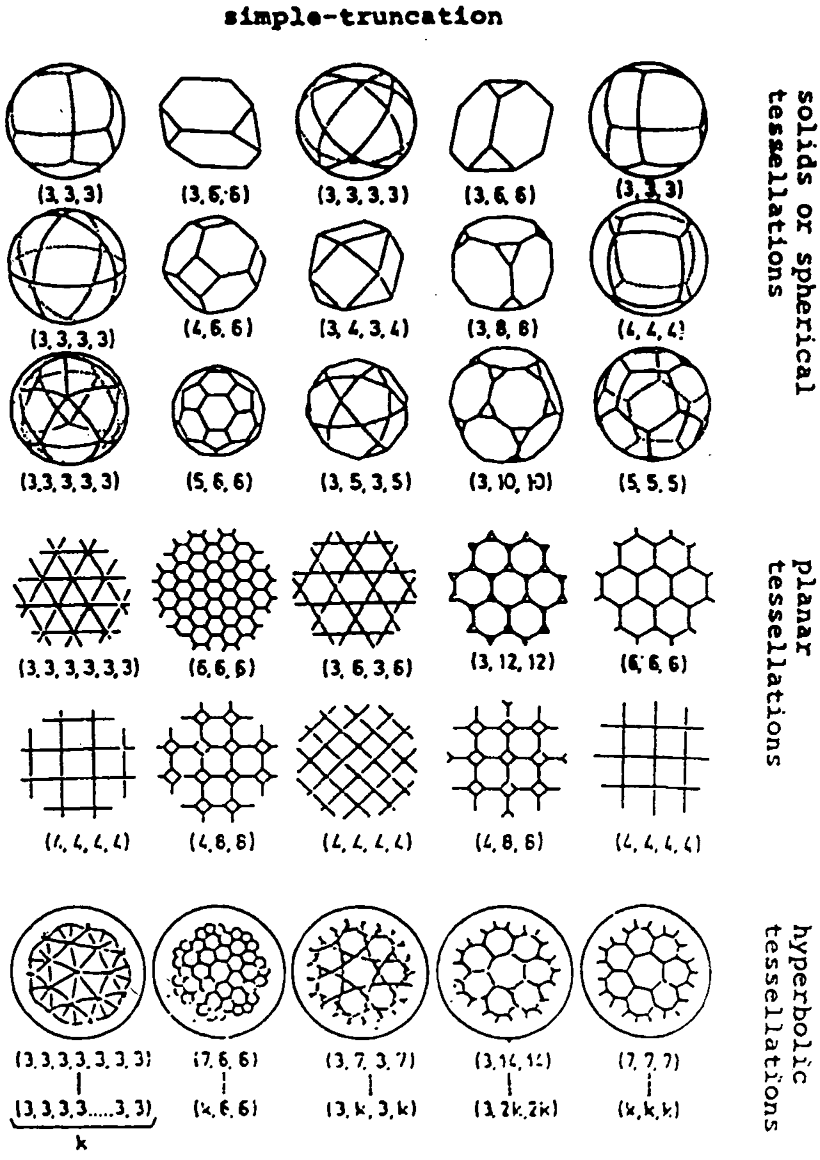


Fig. 15. One direction of extension of the ordering benefits for simplicity and comprehensive role of cellular automatic representation of the simple truncation sequence is to all regular solids (or spherical tessellations) and planar and hyperbolic tessellations. (Bérczi 1979.) The other direction of extension of the principle is to higher dimensional cases of regular solids and tessellations. *Fig. 14.* and *Fig. 16.* show, that the local transitional function in D dimension is the global transitional function in D-1 dimensions. Both extensions shown here prove that truncation operation in such a cellular automatic formulation is a comprehensive principle in deduction of regular solids and tessellations in any dimensions.

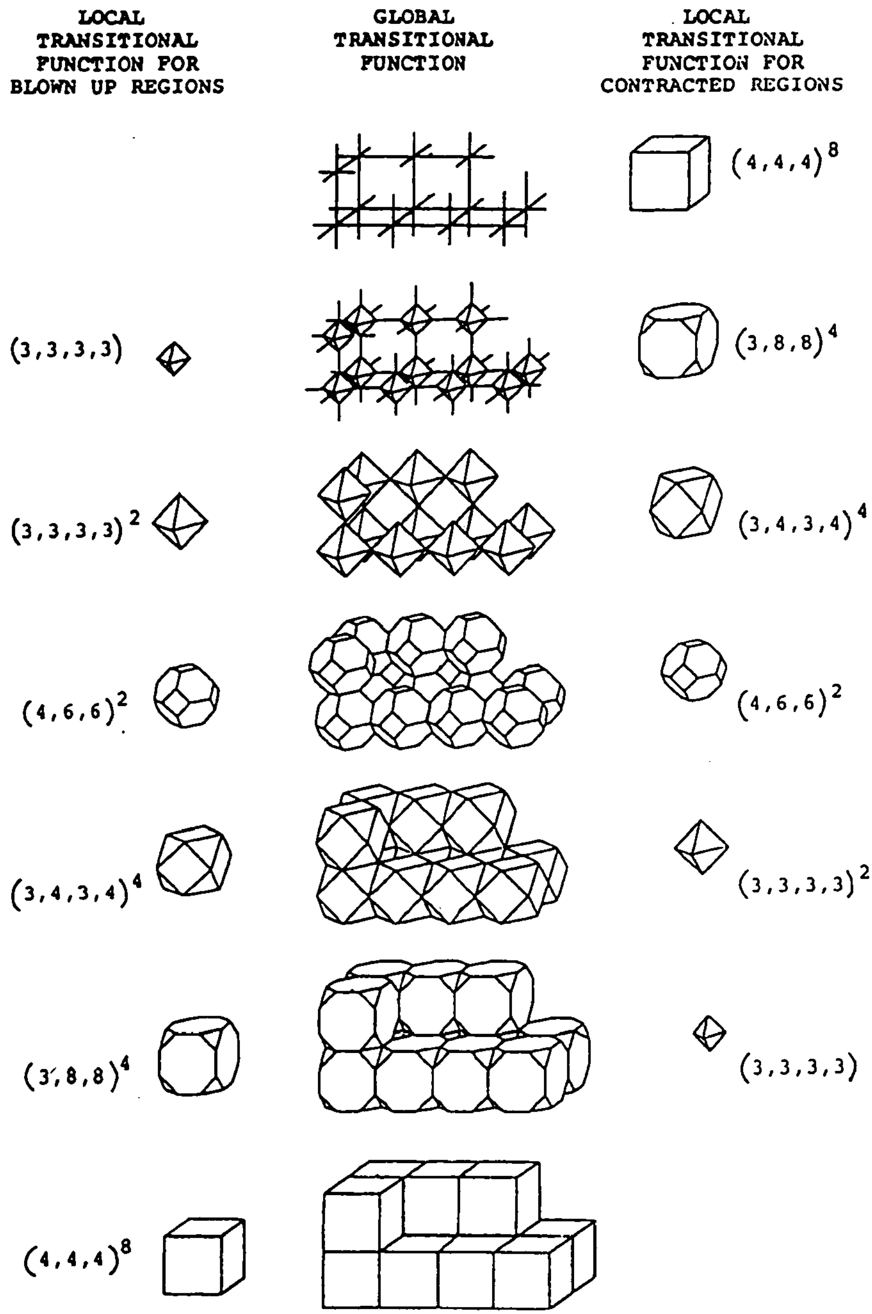


Fig. 16. Simple truncational sequence of spatial tessellation $(4,4,4)^8$ in cellular automatic formulation (BÉRCZI 1979, 1980).

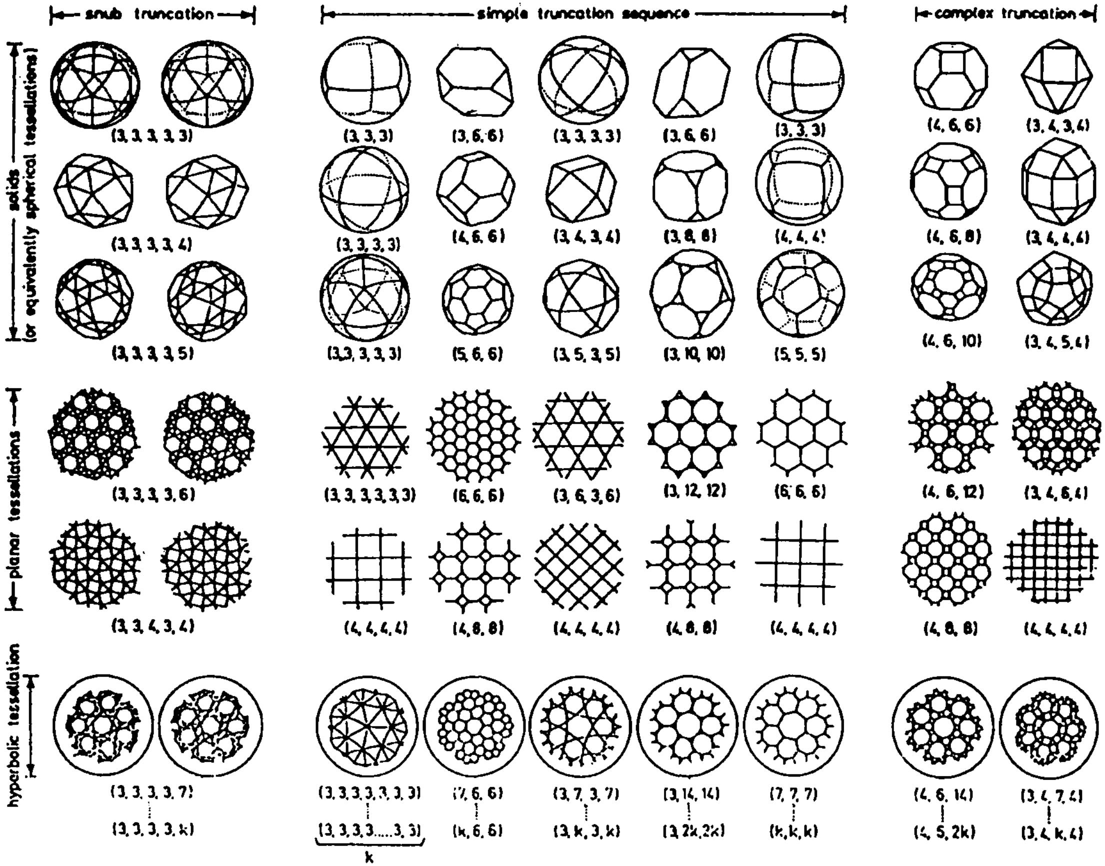


Fig. 17. The periodic table of Platonic and Archimedean solids and tessellations, or the spherical-, planar-, and hyperbolic tessellations. Regular solids are represented by their spherical tessellations for the sake of emphasis: they are the generators for semi-regular ones. (BÉRCZI 1979, 1980)

Two simple cellular automata mosaic systems with global symmetry	Type of operation	Number of cell-types / i.e. that of the local transitional functions/	Number of different "steps" /i.e. the state-ABC of local trans.func./	Type of the global symmetry of the surface
/Bérczi, 1985./ FIBONACCI	one unit sliding of cells	1	2	cy- lind- rical
/this paper/ PLATONIC-ARCHIMEDEAN	blowing up or shrinking of cells	2	1	sphe- rical

Fig. 18. Comparison of the characteristical features of the two cellular mosaic automata system models worked out in the form of indirect von Neumann problem by the author. Global symmetry not only reflects the global boundary condition and the form of the surface but means the form of feed-back-directions by the local transitional function.

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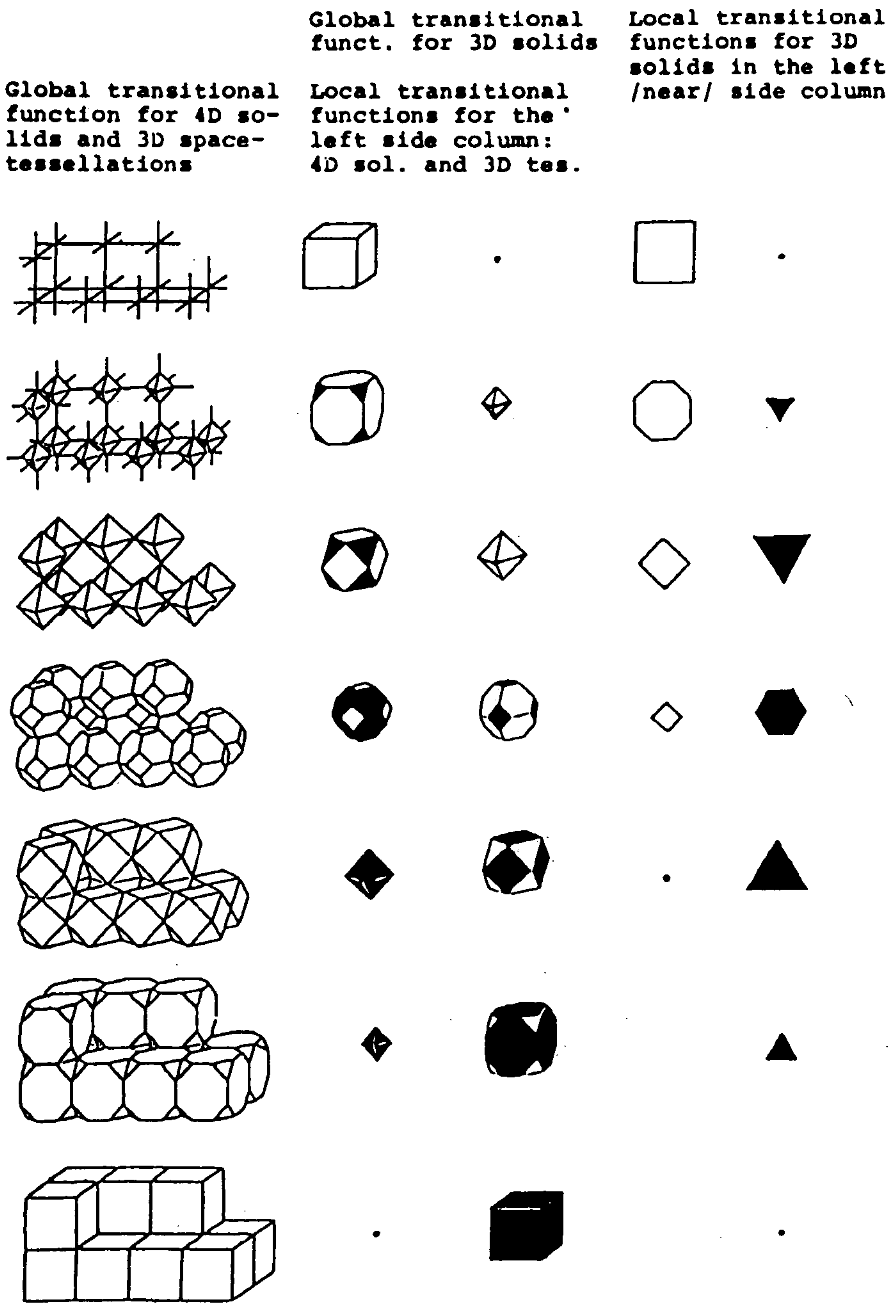

Fig. 19. Depending on the dimensions of the space where the problem is composed the truncation sequences may serve both as local and as global transitional functions, as shown here for three layers of construction.