Acta Sci. Math., 56 (1992), 365—367

On the joint Weyl spectrum. III

MUNEO CHŌ

Dedicated to Professor Tsuyoshi Ando on his 60th birthday

1. Introduction. In [3], we proved that the Weyl theorem holds for a commuting pair of normal operators on a Hilbert space. In this paper we show, by a simple proof, that the Weyl theorem holds for a commuting n-tuple of normal operators and, moreover, its Weyl spectrum coincides with the essential spectrum.

Let $\mathfrak{H}$ be a complex Hilbert space. Let $\mathcal{B}(\mathfrak{H})$ be the algebra of all bounded linear operators on $\mathfrak{H}$ and $\mathcal{K}(\mathfrak{H})$ be the ideal of all compact operators on $\mathfrak{H}$. Let $\mathcal{C}(\mathfrak{H})$ denote the Calkin algebra $\mathcal{B}(\mathfrak{H})/\mathcal{K}(\mathfrak{H})$, with corresponding Calkin map $\pi\colon \mathcal{B}(\mathfrak{H})\to\mathcal{C}(\mathfrak{H})$. Let $\mathbf{T}=(T_1, \ldots, T_n)$ be a commuting n-tuple of operators on $\mathfrak{H}$. Let $\sigma(\mathbf{T})$ be the (Taylor) joint spectrum of $\mathbf{T}$. We refer the reader to [9] for the definition of $\sigma(\mathbf{T})$.

The joint Weyl spectrum $\omega(\mathbf{T})$ of $\mathbf{T}=(T_1, \ldots, T_n)$ is defined as the set

$$\omega(\mathbf{T}) = \bigcap\{\sigma(\mathbf{T}+\mathbf{K})\colon \mathbf{T}+\mathbf{K} = (T_1+K_1, \ldots, T_n+K_n)$$
$$\text{is a commuting } n\text{-tuple for } K_1, \ldots, K_n\in\mathcal{K}(\mathfrak{H})\}.$$

The joint essential spectrum $\sigma_e(\mathbf{T})$ of $\mathbf{T}=(T_1, \ldots, T_n)$ is defined as the set

$$\sigma_e(\mathbf{T}) = \sigma(\pi(\mathbf{T})),$$

where $\pi(\mathbf{T})=(\pi(T_1), \ldots, \pi(T_n))$.

For a commuting n-tuple $\mathbf{T}=(T_1, \ldots, T_n)$, $\pi_{00}(\mathbf{T})$ is the set of all isolated points in $\sigma(\mathbf{T})$ which are joint eigenvalues of finite multiplicity.

2. Theorem. From Corollary 3.8 in [6] and Theorem 2.6 in [7], we have the following

Received May 22, 1991.

Theorem 1. *Let* $\mathbf{T}=(T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of hyponormal operators on H. Then* $\mathbf{z}=(z_1, \ldots, z_n)\in\sigma_e(\mathbf{T})$ *if and only if there exists a sequence* $\{x_k\}$ *of unit vectors in* $\mathfrak{H}$ *with* $x_k\to 0$ *weakly such that* $(T_i-z_i)^* x_k\to 0$ *as* $k\to\infty$.

Immediately, we have the following result.

Theorem 2. *Let* $\mathbf{T}=(T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of hyponormal operators on* $\mathfrak{H}$. *Then* $\sigma_e(\mathbf{T})\subset\omega(\mathbf{T})$.

Lemma 3. *Let* $\mathbf{T}=(T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of hyponormal operators on* $\mathfrak{H}$. *If* $\boldsymbol{\alpha}=(\alpha_1, \ldots, \alpha_n)$ *is an isolated point of* $\sigma(\mathbf{T})$, *then* $\boldsymbol{\alpha}$ *is a joint eigenvalue of* $\mathbf{T}$.

Proof. Let Γ be a surface $|\mathbf{z}-\boldsymbol{\alpha}|=\varepsilon$ $(\varepsilon>0)$, whose interior has no point of $\sigma(\mathbf{T})$ except $\boldsymbol{\alpha}$. Define

$$P=\frac{1}{(2\pi i)^n}\int_{\Gamma} R_{\mathbf{z}-\mathbf{T}}\wedge dz_1\wedge\ldots\wedge dz_n.$$

Then P is a nonzero projection which commutes with every T_i $(i=1, \ldots, T_n)$ (see [10]). Let $\mathbf{T}_{|P}=(PT_1P, \ldots, PT_nP)$. Then $\mathbf{T}_{|P}$ is a doubly commuting n-tuple of hyponormal operators and $\sigma(\mathbf{T}_{|P})=\{\boldsymbol{\alpha}\}$. By Theorem 3.4 in [5], $\boldsymbol{\alpha}$ is a joint eigenvalue of $\mathbf{T}$.

Theorem 4. *Let* $\mathbf{T}=(T_1, \ldots, T_n)$ *be a doubly commuting n-tuple of hyponormal operators on* $\mathfrak{H}$. *Then* $\omega(\mathbf{T})\subset\sigma(\mathbf{T})-\pi_{00}(\mathbf{T})$.

Proof. For every $\mathbf{z}=(z_1, \ldots, z_n)\in\mathbf{C}^n$, $\mathbf{T}-\mathbf{z}=(T_1-z_1, \ldots, T_n-z_n)$ is a doubly commuting n-tuple of hyponormal operators. Hence we may only prove that if $0\in\pi_{00}(\mathbf{T})$, then $0\notin\omega(\mathbf{T})$. Let 0 be in $\pi_{00}(\mathbf{T})$. Then $\mathfrak{N}=\operatorname{Ker}(T_1^*T_1+\ldots+T_1^*T_n)$ is a finite dimensional subspace. Let P denote the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{N}$. Since then P is a compact operator and $PT_i=T_iP=0$ $(i=1, \ldots, n)$, $\mathbf{T}+\mathbf{P}=$ $=\left(T_1+\frac{1}{\sqrt{n}}\cdot P, \ldots, T_n+\frac{1}{\sqrt{n}}\cdot P\right)$ is a doubly commuting n-tuple of hyponormal operators. We let $\mathbf{R}=\left(\left(T_1+\frac{1}{\sqrt{n}}\cdot P\right)_{|\mathfrak{N}}, \ldots, \left(T_n+\frac{1}{\sqrt{n}}\cdot P\right)_{|\mathfrak{N}}\right)$ and $\mathbf{S}=\left(\left(T_1+\frac{1}{\sqrt{n}}\cdot P\right)_{|\mathfrak{N}^\perp}, \ldots \right.$ $\left.\ldots, \left(T_n+\frac{1}{\sqrt{n}}\cdot P\right)_{|\mathfrak{N}^\perp}\right)$. Since then $\mathfrak{N}$ is a reducing subspace for every T_i $(i=1, \ldots, n)$, it follows that $\mathbf{R}$ and $\mathbf{S}$ are doubly commuting n-tuples of hyponormal operators on $\mathfrak{N}$ and $\mathfrak{N}^\perp$ respectively and $\sigma(\mathbf{T}+\mathbf{P})=\sigma(\mathbf{R})\cup\sigma(\mathbf{S})$. It is clear that $0\notin\sigma(\mathbf{R})$. If $0\in\sigma(\mathbf{S})$, then 0 is an isolated point of $\sigma(\mathbf{S})$. Hence by Lemma 3, 0 is a joint eigenvalue of $\mathbf{S}$ and so of $\mathbf{T}$. So there exists a nonzero vector x in $\mathfrak{N}^\perp$ such that $T_ix=0$ $(i=1, \ldots, n)$. This is a contradiction. Therefore we have $0\notin\sigma(\mathbf{T}+\mathbf{P})$.

Theorem 5. *Let* $\mathbf{T}=(T_1, \ldots, T_n)$ *be a commuting n-tuple of normal operators on* $\mathfrak{H}$. *Then* $\sigma_e(\mathbf{T})=\omega(\mathbf{T})=\sigma(\mathbf{T})-\pi_{00}(\mathbf{T})$.

Proof. By Theorems 2 and 4, we may only prove that

$$\sigma(\mathbf{T})-\pi_{00}(\mathbf{T})\subset\sigma_e(\mathbf{T}).$$

In [8], FIALKOW proved that if γ is a nonisolated point of $\sigma(\mathbf{T})$, then $\gamma\in\sigma_e(\mathbf{T})$. It is also clear that if γ is a isolated point of $\sigma(\mathbf{T})$ with infinite multiplicity, then $\gamma\in\sigma_e(\mathbf{T})$.

Acknowledgement. We would like to express our thanke to Professor R. Curto for his advice.

References

[1] J. BAXLEY, Some general conditions implying Weyl's theorem, *Rev. Roum. Math. Pures Appl.*, **16** (1971), 1163—1166.

[2] J. BAXLEY, On the Weyl spectrum of a Hilbert space operators, *Proc. Amer. Math. Soc.*, **34** (1972), 447—452.

[3] M. CHŌ, On the joint Weyl spectrum. II, *Acta Sci. Math.*, **53** (1989), 381—384.

[4] M. CHŌ and M. TAKAGUCHI, On the Joint Weyl spectrum, *Sci. Rep. Hirosaki Univ.*, **27** (1980), 47—49.

[5] M. CHŌ and M. TAKAGUCHI, Some classes of commuting *n*-tuple of operators, *Studia Math.*, **80** (1984), 245—259.

[6] R. CURTO, On the connectedness of invertible *n*-tuples, *Indiana Univ. Math. J.*, **29** (1980), 393—406.

[7] A. T. DASH, Joint essential spectra, *Pacific J. Math.*, **64** (1976), 119—128.

[8] L. A. FIALKOW, The index of an elementary operator, *Indiana Univ. Math. J.*, **35** (1986), 73—102.

[9] J. L. TAYLOR, A joint spectrum for several commuting operators, *J. Funct. Anal.*, **6** (1970), 172—191.

[10] J. L. TAYLOR, The analytic functional calculus for several commuting operators, *Acta Math.*, **125** (1970), 1—38.

DEPARTMENT OF MATHEMATICS
JOETSU UNIVERSITY OF EDUCATION
JOETSU 943, JAPAN