Acta Sci. Math., 56 (1992), 311—321

Approximation theorems for modified Szász operators*)

ZHU-RUI GUO and DING-XUAN ZHOU

1. Introduction

The Bernstein operators on $C[0, 1]$ are given by

(1.1) $$B_n(f, x) = \sum_{k=0}^{n} f(k/n)\binom{n}{k} x^k(1-x)^{n-k}.$$

In 1972, H. BERENS and G. G. LORENTZ [3] gave the pioneering theorem on Bernstein operators in the form

(1.2) $$|B_n(f, x)-f(x)| \leqq M\big(x(1-x)/n\big)^{\alpha/2} \Leftrightarrow \omega_2(f, t) = O(t^\alpha),$$

where $0<\alpha<2$, and

(1.3) $$\omega_2(f, t) = \sup_{0\leqq h\leqq t} \sup_{h\leqq x\leqq 1-h} |f(x-h)-2f(x)+f(x+h)|.$$

In 1978, M. BECKER [1], R. J. NESSEL [2] gave similar results for Szász and Baskakov operators, Meyer—König and Zeller operators.

Berens—Lorentz type theorems for all exponential-type operators were given by K. SATO [9] in 1982. All these exponential-type operators reproduce linear functions, however if this is not the case, then similar Berens—Lorentz type results for non-Feller modified exponential-type operators have not been obtained till now.

In this paper we shall give such a result for modified Szász operators defined in 1985 by S. M. MAZHAR and V. TOTIK [8]:

(1.4) $$L_n(f, x) = \sum_{k=0}^{\infty} \Big(n \int_0^{\infty} f(t)\, p_{n,k}(t)\, dt\Big)\, p_{n,k}(x),$$

where

(1.5) $$p_{n,k}(x) = e^{-nx}(nx)^k/k!.$$

*) Supported by ZPSF and NSF of China.

Received November 19, 1990 and in revised form January 29, 1991.

Concerning these operators, Mazhar and Totik stated: "However it is far less obvious how the analogues of Theorem 2—3 look like in the case of L_n." Our interest stems from this problem. In fact, we will show a result, which, together with some known theorems, yields for $0<\alpha<1$

$$|L_n(f,x)-f(x)| \leqq M\big(1/n+(x/n)^{1/2}\big)^\alpha \Leftrightarrow |S_n(f,x)-f(x)| \leqq M_1\big(1/n+(x/n)^{1/2}\big)^\alpha$$

$$|S_n(f,x)-f(x)| \leqq M_2(x/n)^{\alpha/2} \Leftrightarrow \omega_1(f,t) = O(t^\alpha),$$

where $S_n(f,x)$ are the Szász operators given by

$$S_n(f,x) = \sum_{k=0}^{\infty} f(k/n)\, p_{n,k}(x). \tag{1.6}$$

We shall also give an equivalence theorem involving the smoothness of functions and the derivatives of the modified Szász operators.

2. A Berens—Lorentz type theorem

First let us give some identities.

Lemma 1 [8]. *For $L_n(f(t),x)$ given by* (1.4), *we have*

$$L_n(t,x) = x+1/n;$$

$$L_n\big((t-x)^2,x\big) = 2x/n+2n^{-2}. \tag{2.1}$$

Mazhar and Totik [8] gave the following direct theorem for modified Szász operators:

$$|L_n(f,x)-f(x)| \leqq K\omega_1\big(((x+1/n)/n)^{1/2}\big), \tag{2.2}$$

here

$$\omega(f,t) = \sup_{0\leqq h\leqq t}\ \sup_{x\geqq h/2} |f(x+h/2)-f(x-h/2)|$$

is the usual modulus of smoothness of f.

We have the Berens—Lorentz type inverse result as follows:

Theorem 1. *Let $f\in C[0,\infty)$ be bounded. Then with $0<\alpha<1$,*

$$|L_n(f,x)-f(x)| \leqq M(x/n+n^{-2})^{\alpha/2} \quad (x \geqq 0,\ n\in \mathbf{N}) \tag{2.3}$$

holds if and only if

$$\omega_1(f,t) = O(t^\alpha) \quad (t>0). \tag{2.4}$$

Remark 1. The assumption that f is bounded is necessary, which can be seen from the following example: Let $f(x)=(x+1)\ln(x+1)-(x+1)$. Then $\omega_1(f,t)\neq$

$\neq O(t^\alpha)$ for $\alpha=1/2$. However (2.3) is satisfied: For $x \geqq 1/n$, we have

$$|L_n(f,x)-f(x)| = \left|f'(x)\,L_n(t-x,x)+L_n\Big(\int_x^t (t-u)f''(u)\,du, x\Big)\right| \leqq$$

$$\leqq \ln(x+1)/n+\|uf''(u)\|_\infty L_n\big((t-x)^2/x, x\big) \leqq$$

$$\leqq Mx^{1/4}/n+2/n+2/(n^2x) \leqq (4+M)(x/n+n^{-2})^{1/4}.$$

For $x<1/n$ we have

$$|L_n(f,x)-f(x)| = \left|L_n\Big(\int_x^t \ln(u+1)\,du, x\Big)\right| \leqq$$

$$\leqq L_n\Big(\int_x^t u\,du, x\Big) \leqq L_n(t^2+x^2, x) =$$

$$= 2x^2+4x/n+2n^{-2} \leqq 8(x/n+n^{-2})^{1/4}.$$

Thus we have proved that the boundedness cannot be dropped.

Proof of Theorem 1. By (2.2) we shall only prove the necessity. For $d>0$, let

$$f_d(x) = d^{-1}\int_0^d f(x+s)\,ds. \tag{2.5}$$

Then we have for $f\in C[0,\infty)\cap L_\infty[0,\infty)$

$$\|f_d-f\|_\infty \leqq \omega_1(f,d);$$

$$\|f_d'\|_\infty \leqq d^{-1}\omega_1(f,d). \tag{2.6}$$

Note that since

$$L_n'(f,x) = nx^{-1}\sum_{k=0}^\infty n\int_0^\infty f(t)\,p_{n,k}(t)\,dt\,(k/n-x)\,p_{n,k}(x), \tag{2.7}$$

$$= n\sum_{k=0}^\infty n\int_0^\infty f(t)\big(p_{n,k+1}(t)-p_{n,k}(t)\big)\,dt\,p_{n,k}(x), \tag{2.8}$$

we have

$$|L_n'(f_d-f,x)| \leqq nx^{-1}\|f-f_d\|_\infty S_n(|t-x|,x) \leqq (n/x)^{1/2}\omega_1(f,d);$$

$$|L_n'(f_d-f,x)| \leqq 2n\|f-f_d\|_\infty,$$

where we used that

$$S_n(|t-x|,x) \leqq \big(S_n((t-x)^2,x)\big)^{1/2} = (x/n)^{1/2} \text{ (see, e.g. [1, 12])}.$$

Hence

$$|L_n'(f_d-f,x)| \leqq 2\omega_1(f,d)\min\{(n/x)^{1/2}, n\}. \tag{2.9}$$

From (2.8) and [7] we can also derive

$$|L_n'(f_d, x)| = \left|n \sum_{k=0}^{\infty} \int_0^{\infty} p_{n,k+1}(t) f_d'(t)\, dt\, p_{n,k}(x)\right| \leqq d^{-1}\omega_1(f, d).$$

Now for any $t>0$ and $0<h\leqq t$, $x\in(0,\infty)$, we get from (2.3) for any $n\in\mathbf{N}$

$$(2.10)\qquad |f(x+h)-f(x)| \leqq |f(x+h)-L_n(f, x+h)| + |f(x)-L_n(f, x)| +$$

$$+\left|\int_0^h L_n'(f_d, x+u)\, du\right| + \left|\int_0^h L_n'(f-f_d, x+u)\, du\right| \leqq$$

$$\leqq 2M\big((x+h)/n+n^{-2}\big)^{\alpha/2} + d^{-1}\omega_1(f, d)\,h + 2\omega_1(f, d)\int_0^h \min\big\{(n/(x+u))^{1/2}, n\big\}\, du \leqq$$

$$\leqq 2M\big(d(n, x, h)\big)^{\alpha} + 8h\omega_1(f, d)\big(d^{-1}+1/d(n, x, h)\big),$$

where $d(n, x, h)=((x+h)/n+n^{-2})^{1/2}$. Note that $d(n, x, h)\geqq d(n+1, x, h)\geqq$ $\geqq d(n, x, h)/2$ for any $n\in\mathbf{N}$, hence for any $1/2>\delta>0$ we can choose $n\in\mathbf{N}$ such that

$$2d(n, x, h) \geqq \delta > d(n, x, h).$$

With this choice we get from (2.10) the estimate

$$|f(x+h)-f(x)| \leqq 2M\delta^{\alpha} + 36h\omega_1(f, \delta)/\delta,$$

hence

$$\omega_1(f, t) \leqq 2M\delta^{\alpha} + 36t\omega_1(f, \delta)/\delta \leqq (2M+36)\big(\delta^{\alpha} + t\omega_1(f, \delta)/\delta\big), \quad (t, \delta > 0)$$

which implies $\omega_1(f, t)=O(t^{\alpha})$ (see [3, 6]). Our proof is complete.

Remark 2. The same statement is true for Szász operators, however we shall omit the proof since it is just the same.

Remark 3. In [14], we have proved for Bernstein—Durrmeyer operators

$$(2.11)\qquad D_n(f, x) = \sum_{k=0}^{n}\left((n+1)\int_0^1 f(t)\binom{n}{k} t^k(1-t)^{n-k}\, dt\right)\binom{n}{k} x^k(1-x)^{n-k},$$

that for $1<\alpha<2$ there exist no functions $\{\psi_{n,\alpha}(x)\}_{n\in\mathbf{N}}$ such that the following equivalence holds for $f\in C[0, 1]$

$$(2.12)\qquad \omega_2(f, t) = O(t^{\alpha}) \Leftrightarrow |D_n(f, x)-f(x)| \leqq M\psi_{n,\alpha}(x).$$

In view of this result we cannot expect a similar characterization theorem by the modified Szász operators for functions satisfying

$$\omega_2(f, t) = O(t^{\alpha}) \quad \text{with} \quad 1 < \alpha < 2.$$

3. Derivatives and smoothness

Some results on the relation between the order of derivatives and smoothness have been obtained in [5, 7], most of which characterize the Ditzian—Totik modulus of smoothness. Z. Ditzian gave a result on the characterization of the usual modulus of smoothness by the derivatives of Bernstein polynomials [4]. Recently one of the authors gave similar results for higher order of smoothness [14].

Let

$$\omega_2(f, t) = \sup_{0<h\leqq t}\ \sup_{x\geqq 0} |f(x)-2f(x+h)+f(x+2h)|.$$

For the modified Szász operators, we can prove

Theorem 2. *For* $f\in C[0,\infty)\cap L_\infty[0,\infty)$, $0<\alpha<2$, *we have*

$$\omega_2(f, t) = O(t^\alpha) \Leftrightarrow |L_n''(f, x)| \leqq M(\min\{n^2, n/x\})^{(2-\alpha)/2}. \tag{3.1}$$

Theorem 3. *For* $f\in C[0,\infty)\cap L_\infty$, $0<\alpha<1$, *we have*

$$\omega_1(f, t) = O(t^\alpha) \Leftrightarrow |L_n'(f, x)| \leqq M(\min\{n^2, n/x\})^{(1-\alpha)/2}. \tag{3.2}$$

Proof of Theorem 2. Proof of the direction "$\Rightarrow$": Suppose $\omega_2(f, t)\leqq Mt^\alpha$. By simple calculation one can get

$$L_n''(g, x) = n^2 \sum_{k=0}^{\infty}\Big(n\int_0^\infty g(t)\big(p_{n,k}(t)-2p_{n,k+1}(t)+p_{n,k+2}(t)\big)\,dt\Big)\,p_{n,k}(x) \tag{3.3}$$

$$= n^2x^{-2}\sum_{k=0}^{\infty}\Big(n\int_0^\infty g(t)\,p_{n,k}(t)\,dt\Big)\big((k/n-x)^2-kn^{-2}\big)\,p_{n,k}(x), \tag{3.4}$$

hence

$$|L_n''(g, x)| \leqq 4n^2\,\|g\|_\infty, \tag{3.5}$$

and

$$|L_n''(g, x)| \leqq 2n/x\,\|g\|_\infty. \tag{3.6}$$

Now for $f\in C[0,\infty)\cap L_\infty(0,\infty)$, let us define the Steklov function as

$$f_d(x) = 4d^{-2}\iint_0^{d/2}\big(2f(x+u+v)-f(x+2u+2v)\big)\,du\,dv. \tag{3.7}$$

Then

$$\|f-f_d\| \leqq \omega_2(f, d), \tag{3.8}$$

and

$$\|f_d''\| \leqq 9d^{-2}\omega_2(f, d). \tag{3.9}$$

For f_d one can verify (see [7])

$$|L_n''(f_d, x)| \leqq \Big|\sum_{k=0}^{\infty} n\int_0^\infty p_{n,k+2}(t)f_d''(t)\,dt\,p_{n,k}(x)\Big| \leqq 9d^{-2}\omega_2(f, d). \tag{3.10}$$

Thus, we have for $d=(\min\{n^2, n/x\})^{-1/2}$

$$\begin{aligned}|L_n''(f,x) &\leqq |L_n''(f_d,x)|+|L_n''(f-f_d,x)| \leqq \\ &\leqq 9d^{-2}\omega_2(f,d)+4\min\{n^2, n/x)\,\omega_2(f,d) \leqq \\ &\leqq 13M(\min\{n^2, n/x\})^{1-\alpha/2}.\end{aligned}$$

Proof of the direction "$\Leftarrow$": To prove this part, we need the combination of $\{L_n\}$ defined as

(3.11) $$L_{n,1}(f;x) = a_0(n)L_{n_0}(f,x)+a_1(n)L_{n_1}(f,x),$$

where $|a_0(n)|+|a_1(n)|\leqq B$, $n=n_0<n_1\leqq An$, with A, B absolute constants, having the property

(3.12) $$L_{n,1}(t^i,x) = x^i, \quad i=0,1 \text{ (see e.g. [7])}.$$

Then we have for $f\in C[0,\infty)\cap L_\infty[0,\infty)$ by the method from [1, 3]

(3.13) $$|L_{n,1}(f,x)-f(x)| \leqq M\omega_2(f, 1/n+(x/n)^{1/2}).$$

Now we can give our proof, where the commutativity of $\{L_n\}$ is crucial (see [7]). For $n,m\in\mathbf{N}$, $x\in(0,\infty)$, $0<h\leqq t$, we have

(3.14)
$$|L_m(f,x)-2L_m(f,x+h)+L_m(f,x+2h)| \leqq 4M\omega_2\big(L_mf, 1/n+((x+2h)/n)^{1/2}\big)+$$
$$+\iint_0^h \big|L_n''(L_{n,1}(f), x+u+v)\big|\,du\,dv$$

Now we shall estimate the second term. First we have

$$\big|L_m''(L_{n,1}(f), x+u+v)\big| \leqq \|L_{n,1}''(f)\|_\infty \leqq 2B(An)^{2-\alpha}.$$

On the other hand, note that by

$$|x^{1-\alpha/2}L_{n,1}''(f,x)| \leqq MABn^{1-\alpha/2}$$

we have

$$\big|x^{1-\alpha/2}L_m''(L_{n,1}(f),x)\big| = \Big|x^{1-\alpha/2}\sum_{k=0}^\infty m\int_0^\infty p_{m,k+2}(t)\,L_{n,1}''(f,t)\,dt\,p_{m,k}(x)\Big| \leqq$$

$$\leqq MABn^{1-\alpha/2}x^{1-\alpha/2}\sum_{k=0}^\infty m\Big(\int_0^\infty p_{m,k+2}(t)\,t^{-1}\,dt\Big)^{1-\alpha/2}\Big(\int_0^\infty p_{m,k+2}(t)\,dt\Big)^{\alpha/2}p_{m,k}(x) \leqq$$

$$\leqq MABn^{1-\alpha/2}x^{1-\alpha/2}\Big(\sum_{k=0}^\infty mp_{m,k}(x)/(k+2)\Big)^{1-\alpha/2} \leqq MABn^{1-\alpha/2},$$

hence

$$\iint_0^h \big|L_m''(L_{n,1}(f), x+u+v)\big|\,du\,dv \leqq MABn^{1-\alpha/2}\iint_0^h (x+u+v)^{\alpha/2-1}\,du\,dv \leqq$$

$$\leqq MABn^{1-\alpha/2}h^\alpha\big(M_1h^2/(x+2h)\big)^{1-\alpha/2} \leqq M_2h^2\big(n/(x+2h)\big)^{1-\alpha/2},$$

here we have used the fact that

$$\iint_0^h 1/(x+u+v)\,du\,dv \leqq M_1 h^2/(x+2h), \quad 0 < h \leqq 1, \quad x \geqq 0$$

(see [1]). Thus, combining the above estimates with (3.14), we have

$$\text{(3.15)} \qquad |L_m(f, x) - 2L_m(f, x+h) + L_m(f, x+2h)| \leqq$$
$$\leqq 4M\omega_2\big(L_m f, 1/n + ((x+2h)/n)^{1/2}\big) + M_3 h^2\big(1/n + ((x+2h)/n)^{1/2}\big)^{\alpha-2},$$

where M_3 is a constant independent of n, x, h and m.

Let C be a constant which will be determined later. Since

$$1/n + \big((x+2h)/n\big)^{1/2} < 1/(n-1) + \big((x+2h)/(n-1)\big)^{1/2} \leqq 2\big(1/n + ((x+2h)/n)^{1/2}\big),$$

we can choose $n \in \mathrm{N}$, such that

$$t/(2C) \leqq 1/n + \big((x+2h)/n\big)^{1/2} \leqq t/C.$$

Then we get from (3.15) by induction

$$\omega_2(L_m f, t) \leqq 4M\omega_2(L_m f, t/C) + (2C)^{(2-\alpha)} M_3 t^\alpha \leqq$$
$$\leqq \dots$$
$$\leqq (4M)^k \omega_2(L_m f, tC^{-k}) + (2C)^{2-\alpha} M_3 t^\alpha \sum_{l=0}^{k-1} (4MC^{-\alpha})^l \leqq$$
$$\leqq t^2 (4M)^k C^{-2k} \|(L_m f)''\|_\infty + (2C)^{2-\alpha} M_3 t^\alpha C^\alpha/(C^\alpha - 4M).$$

Now if we take here $C = (1+4M)^{1/\alpha}$, and let $k \to \infty$, we obtain

$$\omega_2(L_m f, t) \leqq 4C^2 M_3 t^\alpha/(C^\alpha - 4M),$$

which implies $\omega_2(f, t) = O(t^\alpha)$, since the constant $4C^2 M_3/(C^\alpha - 4M)$ is independent of $m \in \mathrm{N}$.

Our proof is complete. We shall omit the proof of Theorem 3, since it is almost the same as the proof of Theorem 2.

4. A direct theorem for uniform approximation

When treating uniform approximation we shall always assume the boundedness of the functions. Let C_B be the set of bounded and continuous functions on $[0, \infty)$.

Theorem 4. *For $f \in C_B$, $L_n(f, x)$ given by* (1.4), *we have*

$$\text{(4.1)} \qquad \|L_n f - f\|_\infty \leqq C\big(\omega_\varphi^2(f, n^{-1/2})_\infty + \omega_1(f, n^{-1}) + n^{-1}\|f\|_\infty\big),$$

where C is a constant independent of n, and $\omega_\varphi^2(f, n^{-1/2})_\infty$ is the so-called Ditzian—

Totik modulus of smoothness defined as

$$\omega_\varphi^2(f,t)_\infty = \sup_{0<h\leqq t} \|\Delta_{h\varphi}^2 f\|_\infty, \tag{4.2}$$

$$\varphi(x) = x^{1/2},$$

$$\Delta_h^2 f(x) = f(x-h) - 2f(x) + f(x+h), \quad x \geqq h;$$

$$\Delta_h^2 f(x) = 0, \quad \text{otherwise}.$$

Remark 4. In view of the characterization theorems of MAZHAR and TOTIK [8] on the saturation and non-optimal approximation, we can see that our result is of some value.

Proof of Theorem 4. For Szász operators given by (1.6), we have for $f \in C_B$ (see [6])

$$\|S_n(f) - f\|_\infty \leqq M\big(\omega_\varphi^2(f, n^{-1/2})_\infty + n^{-1}\|f\|_\infty\big). \tag{4.3}$$

We now need the Szász—Kantorovich operators given by

$$S_n^*(f,x) = \sum_{k=0}^{\infty} n \int_{k/n}^{(k+1)/n} f(t)\,dt\, p_{n,k}(x). \tag{4.4}$$

For these operators, we can easily deduce from (4.3) the direct result as

$$\|S_n^*(f) - f\|_\infty \leqq M\big(\omega_\varphi^2(f, n^{-1/2})_\infty + \omega_1(f, 1/n) + n^{-1}\|f\|_\infty\big). \tag{4.5}$$

We shall use the identities

$$S_n^*(t,x) = x + 1/(2n), \quad S_n^*\big((t-x)^2, x\big) = x/n + 3^{-1}n^{-2}. \tag{4.6}$$

By (4.5), we only need to prove

$$\big\|2(f - S_n^*(f)) - (f - L_n f)\big\|_\infty \leqq C\big(\omega_\varphi^2(f, n^{-1/2})_\infty + n^{-1}\|f\|_\infty\big).$$

Using the Ditzian—Totik K-functional

$$K_{\varphi,2}(f,t)_\infty = \inf_{g\in D}\{\|f-g\|_\infty + t(\|g\|_\infty + \|\varphi^2 g''\|_\infty) + t^2\|g''\|_\infty\},$$

and its equivalence to ω_φ^2, it is sufficient for us to prove for $g \in D = \{g \in C_B\colon g' \in A.C._{\text{loc}},$ $g'' \in L_\infty[0,\infty)\}$ the estimate

$$\big\|2(g - S_n^*(g)) - (g - L_n(g))\big\|_\infty \leqq C\big((\|g\|_\infty + \|\varphi^2 g''\|_\infty)/n + n^{-2}\|g''\|_\infty\big). \tag{4.7}$$

From the above identities, we have for $g\in D$, $x\geqq n^{-1}$

$$|2(g(x)-S_n^*(g,x))-(g(x)-L_n(g,x))| =$$

$$= \left|L_n\Big(\int_x^t (t-u)g''(u)\,du, x\Big)-2S_n^*\Big(\int_x^t (t-u)g''(u)\,du, x\Big)\right| \leqq$$

$$\leqq \|\varphi^2 g''\|_\infty \big(L_n((t-x)^2/x, x)+2S_n^*((t-x)^2/x, x)\big) \leqq 12\,\|\varphi^2 g''\|_\infty/n.$$

For $0<x<1/n$, we have

$$|2(g(x)-S_n^*(g,x))-(g(x)-L_n(g,x))| \leqq$$

$$\leqq \|g''\|_\infty \big(L_n((t-x)^2, x)+2S_n^*((t-x)^2, x)\big) \leqq 12n^{-2}\,\|g''\|_\infty.$$

Thus we have obtained (4.7) and our proof of Theorem 4 is complete.

Remark 5. The second term on the rihgt of (4.1) is necessary, which can be seen from the following example: Let

$$f(x)=\begin{cases} x\ln x - x^2/2+1/2, & \text{for } x\in[0,1);\\ 0, & \text{otherwise.}\end{cases} \tag{4.8}$$

Then we have

$$f'(x)=\begin{cases} \ln x - x+1, & \text{for } x\in(0,1);\\ 0, & \text{otherwise.}\end{cases} \tag{4.9}$$

$$f''(x)=\begin{cases} 1/x-1, & \text{for } x\in(0,1);\\ 0, & \text{otherwise.}\end{cases} \tag{4.10}$$

Therefore we obtain a function $f\in C_B$ which satisfies

$$\omega_\varphi^2(f, n^{-1/2})_\infty = O(1/n) \quad n^{-1}\,\|f\|_\infty = O(1/n),$$

and

$$f'\notin L_\infty.$$

On the other hand, from the saturation class of the modified Szász operators $\{L_n\}$ [8] we have

$$\|L_n f - f\|_\infty \neq O(1/n).$$

Thus we have proved that the second term $\omega_1(f, 1/n)$ is necessary.
We can also give the weak-type inverse estimates for the moduli.

Lemma 2. *For $L_n(f, x)$ given by* (1.4), $n \in \mathbf{N}$, *we have*

(4.11) $$\|(L_n f)''\|_\infty + \|L_n f\|_\infty \leqq M n^2 \|f\|_\infty, \quad f \in C_B;$$

(4.12) $$\|\varphi^2 (L_n f)''\|_\infty + \|L_n f\|_\infty \leqq M n \|f\|_\infty, \quad f \in C_B;$$

(4.13) $$\|(L_n f)''\|_\infty + \|L_n f\|_\infty \leqq M(\|f''\|_\infty + \|f\|_\infty), \quad f'' \in L_\infty;$$

(4.14) $$\|\varphi^2 (L_n f)''\|_\infty + \|L_n f\|_\infty \leqq M(\|\varphi^2 f''\|_\infty + \|f\|_\infty), \quad \varphi^2 f'' \in L_\infty,$$

where M is a constant independent of n and f.

The proof can be easily obtained from the representations of the derivatives of $L_n f$ in [7].

Theorem 5. *For $f \in C_B$, $L_n(f, x)$ given by* (1.4), *we have*

(4.15) $$\omega_\varphi^2(f, n^{-1/2})_\infty \leqq M_1 n^{-1} \sum_{k=1}^{\infty} \|L_k f - f\|_\infty + n^{-1} \|f\|_\infty;$$

(4.16) $$\omega_1(f, n^{-1}) \leqq M_1 n^{-1} \Big(\sum_{k=1}^{n} \|L_k f - f\|_\infty + \|f\|_\infty \Big).$$

where M_1 is independent of f and $n \in \mathbf{N}$.

The proof is the same as given by Ditzian and Totik [6], (see also [10]), so we shall omit it.

References

[1] M. Becker, Global approximation theorems for Szász—Mirakjan and Baskakov operators in polynomial weight space, *Indiana Univ. Math. J.*, **27** (1978), 127—141.

[2] M. Becker and R. J. Nessel, A global approxiamtion theorem for Meyer—König and Zeller operators, *Math. Z.*, **160** (1978), 195—206.

[3] H. Berens and G. G. Lorentz, Inverse theorems for Bernstein polynomials, *Indiana Univ. Math. J.*, **21** (1972), 693—708.

[4] Z. Ditzian, Derivatives of Bernstein polynomials and smoothness, *Proc. Amer. Math. Soc.*, **93** (1985), 25—31.

[5] Z. Ditzian and K. G. Ivanov, On Bernstein-type operators and their derivatives, *J. Approx. Theory*, **56** (1989), 72—90.

[6] Z. Ditzian and V. Totik, *Moduli of Smoothness*, Springer-Verlag (Berlin/New York, 1987).

[7] M. Heilmann, Direct and converse results for operators of Baskakov—Durrmeyer type, *Approx. Theory Appl.*, **5** (1989), 105—127.

[8] S. M. Mazhar and V. Totik, Approximation by modified Szász operators, *Acta Sci. Math.*, **49** (1985), 257—269.

[9] K. Sato, Global approximation theorems for some exponential-type operators, *J. Approx. Theory*, **32** (1982), 22—36.

[10] V. Totik, An interpolation theorem and its applications to positive operators, *Pacific J. Math.*, **111** (1984), 447—481.

[11] V. Totik, Uniform approximation by exponential-type operators, *J. Math. Anal. Appl.*, **132** (1988), 238—246.

[12] V. Totik, Uniform approximation by positive operators on infinite intervals, *Anal. Math.*, **10** (1984), 163—182.

[13] Ding-Xuan Zhou, Uniform approximation by some Durrmeyer operators, *Approx. Theory Appl.*, 6 (1990), 87—100.

[14] Ding-Xuan Zhou, On smoothness characterized by Berntein-type operators, *Math. Z.*, submitted.

[15] Ding-Xuan Zhou, On a conjecture of Z. Ditzian, *J. Approx. Theory*, **69** (1992), 167—172.

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY;
ADVANCED INSTITUTE OF MATHEMATICS,
HANGZHOU
ZHEJIANG
310 027
PEOPLE'S REPUBLIC OF CHINA