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On strong approximation by Cesàro means of negative order

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1. Let $\{\varphi_n(x)\}$ be an orthonormal system on the finite interval (a, b). We shall consider series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{1.1}$$

with real coefficients satisfying

$$\sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.2}$$

By the Riesz—Fischer theorem, series (1.1) converges in the metric L^2 to a square-integrable function $f(x)$. We denote the partial sums and the (C, α)-means of series (1.1) by $s_n(x)$ and $\sigma_n^\alpha(x)$, respectively. Furthermore, T_n will denote a positive regular summation method determined by a triangular matrix (α_{nk}/A_n) $\Big(\alpha_{nk} \geqq 0$ and $A_n := \sum_{k=0}^{n} \alpha_{nk}$, and if s_k tends to s, then

$$T_n := \frac{1}{A_n} \sum_{k=0}^{n} \alpha_{nk} s_k \to s\Big).$$

G. SUNOUCHI [7] proved the following result.

Theorem A. *If* $0<\gamma<1$ *and*

$$\sum_{n=1}^{\infty} c_n^2 n^{2\gamma} < \infty, \tag{1.3}$$

then

$$\left\{\frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} |f(x)-s_\nu(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{1.4}$$

holds almost everywhere (a.e.) for any $\alpha>0$ *and* $0<p<\gamma^{-1}$, *where* $A_n^\alpha := \binom{n+\alpha}{n}$.

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In [3] we generalized this result in such a way that we replaced the partials in (1.4) by Cesàro means of negative order. Our theorem reads as follows, where and in the sequel K will denote positive constant, not necessarily the same one.

Theorem B. *Suppose that* $0<\gamma<1$, $0<p<\gamma^{-1}$ *and* (1.3) *holds, furthermore that there exists a number* $\varrho>1$ *such that*

$$\frac{\varrho p}{\varrho-1} \geqq 2, \tag{1.5}$$

and with this ϱ *for any* $0<\omega<1$ *and* $2^m<n\leqq 2^{m+1}$

$$\sum_{l=0}^{m}\Big\{\sum_{\nu=2^l-1}^{\min(2^{l+1},n)} \alpha_{n\nu}^{\varrho}(\nu+1)^{\varrho(1-\omega)-1}\Big\}^{1/\varrho} \leqq Kn^{-\omega}A_n. \tag{1.6}$$

Then, for arbitrary

$$d>1-\frac{\varrho-1}{\varrho p}, \tag{1.7}$$

$$\left\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^{d-1}(x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{1.8}$$

holds a.e. in (a, b).

It is easy to verify that in the special case $\alpha_{n\nu}=A_{n-\nu}^{\alpha-1}\ (\alpha>0)$ condition (1.6) is satisfied, thus Theorem B with $d=1$ reduces to Theorem A.

But if we set $\alpha_{n\nu}=(\nu+1)^{\beta-1}\ (\beta>0)$, then condition (1.6) will be satisfied only if $\beta\geqq 1$. Consequently, for $0<\beta<1$, we cannot apply Theorem B to get an estimate for the following strong Riesz means

$$h_n(f,d,\beta,p;\ x) := \Big\{(n+1)^{-\beta}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|f(x)-\sigma_\nu^{d-1}(x)|^p\Big\}^{1/p},$$

but the Riesz summability is a frequently used summation method in connection with strong approximation. Nevertheless, if we want to get the estimate

$$h_n(f,d,\beta,p;\ x) = o_x(n^{-\gamma}) \tag{1.9}$$

for some $0<\beta<1$, then, as a possible solution, we can try to weaken the requirement of (1.6).

One of our aims is to give such a generalization of Theorem B.

We mention that in the special case $d=1$, i.e. if we approximate the function $f(x)$ with the partial sums $s_\nu(x)\ (=\sigma_\nu^0(x))$, then already an estimate of type (1.9) is known. Namely, in a joint paper with H. Schwinn [6], we proved among others:

Theorem C. *If* $\gamma>0$ *and* $0<p\gamma<\beta$ *then condition* (1.3) *implies*

$$(1.10)\quad H_n(f,\beta,p,\nu;\,x):=\Big\{(n+1)^{-\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|f(x)-s_{\nu_k}(x)|^p\Big\}^{1/p}=o_x(n^{-\gamma})$$

a.e. in (a,b) *for any increasing sequence* $\nu:=\{\nu_k\}$ *of natural numbers.*

In [5] we investigated the so-called *limit case* of the restriction of the parameters, i.e. if $\beta=p\gamma$. Among others we proved:

Theorem D. *If* p *and* β *are positive numbers then for any increasing sequence* $\nu:=\{\nu_k\}$

$$(1.11)\qquad \sum_{n=1}^{\infty}c_n^2n^{2\beta/p}<\infty\quad (\gamma=\beta/p)$$

implies

$$(1.12)\qquad H_n(f,\beta,p,\nu;\,x)\doteq o_x\big(n^{-\beta/p}(\log n)^{1/p}\big)$$

a.e. in (a,b).

Theorem E. *If* α *and* p *are positive numbers then for any increasing sequence* $\nu:=\{\nu_k\}$

$$(1.13)\qquad \sum_{n=1}^{\infty}c_n^2n^{2/p}<\infty\quad (\gamma=1/p)$$

implies

$$(1.14)\quad C_n(f,\alpha,p,\nu;\,x):=\Big\{\frac{1}{A_n^\alpha}\sum_{k=0}^{n}A_{n-k}^{\alpha-1}|f(x)-s_{\nu_k}(x)|^p\Big\}^{1/p}=o_x\big(n^{-1/p}(\log n)^{1/p}\big)$$

a.e. in (a,b).

We want to point out that Theorems C, D and E contrary to Theorems A and B do not claim the extra restriction $\gamma<1$. This is a great advantage of these theorems, but they do not allow of approximating with Cesàro means of negative order.

The common kernel of the proof of Theorems A and B is a very interesting result of T. M. Flett [1] and a useful lemma of G. Sunouchi [7] (here Lemma 2 and Lemma 3, respectively) and they, unfortunately, require the assumption $0<\gamma<1$. The proofs of Theorems C, D and E run on a perfectly different line, and these proofs do not use the assumption $\gamma<1$.

In the present paper we prove such a general theorem which generalizes Theorem B and includes all of Theorems C, D and E if $\gamma<1$. Unfortunately, we cannot extend the validity of our result for $\gamma\geqq1$. This remains as an interesting open problem, in our view.

Using the notations introduced above we can formulate our results.

Theorem. *Suppose that* $p>0$ *and* $0<\gamma<1$. *Moreover let us suppose that there exists a number* $\varrho>1$ *with property* (1.5) *and that with this* ϱ *and with* $n(l):=$

$:=\min(2^l, n)$, $2^m < n \leqq 2^{m+1}$

$$(1.15) \qquad \sum_{l=0}^{m} \Big\{ \sum_{\nu=n(l)-1}^{n(l+1)} \alpha_{n\nu}^{\varrho} (\nu+1)^{\varrho(1-\gamma p)-1} \Big\}^{1/\varrho} \leqq K g(n) A_n n^{-\gamma p},$$

where $g(t)$ denotes a non-decreasing positive function defined for $0 \leqq t < \infty$.

Then, for any d satisfying (1.7), (1.3) *implies*

$$(1.16) \qquad \Big\{ \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |f(x) - \sigma_\nu^{d-1}(x)|^p \Big\}^{1/p} = O_x\big(g(n)^{1/p} n^{-\gamma}\big)$$

a.e. in (a, b).

If, in addition, for every fixed l,

$$(1.17) \qquad \Big\{ \sum_{\nu=n(l)-1}^{n(l+1)} \alpha_{n\nu}^{\varrho} \Big\}^{1/\varrho} = o\big(g(n) A_n n^{-\gamma p}\big), \quad as \quad n \to \infty,$$

then the O_x in (1.16) *can be replaced by o_x.*

Hence, by a useful lemma (here Lemma 1) we easily get the following result.

Corollary 1. *Under the assumptions of Theorem we have the estimate*

$$(1.18) \qquad \Big\{ \frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |f(x) - \sigma_\nu^{d-1}(\mu;\, x)|^p \Big\}^{1/p} = O_x\big(g(n)^{1/p} n^{-\gamma}\big)$$

a.e. in (a, b), where $\mu := \{\mu_k\}$ is an increasing sequence of natural numbers and

$$\sigma_n^{\beta}(\mu;\, x) := \frac{1}{A_n^{\beta}} \sum_{k=0}^{n} A_{n-k}^{\beta-1} s_{\mu_k}(x).$$

If (1.17) *also holds, then the O_x in* (1.18) *can be replaced by o_x.*

From Corollary 1, by an easy consideration to be detailed later on, we get the following results:

Corollary 1.1. *If $0 < \gamma < 1$, $d > \max\big(1/2, (p-1)/p\big)$ and $0 < p\gamma < \beta$, then* (1.3) *implies*

$$\Big\{ (n+1)^{-\beta} \sum_{\nu=0}^{n} (\nu+1)^{\beta-1} |f(x) - \sigma_\nu^{d-1}(\mu;\, x)|^p \Big\}^{1/p} = o_x(n^{-\gamma})$$

a.e. in (a, b) for any increasing sequence $\mu := \{\mu_k\}$.

Corollary 1.2. *If* $\alpha>0$, $0<\gamma<1$, $0<p<\gamma^{-1}$ *and* $d>\max(1/2, (p-1)/p, (p-\alpha)/p)$, *then* (1.3) *implies*

$$\left\{\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|f(x)-\sigma_\nu^{d-1}(\mu;\ x)|^p\right\}^{1/p} = o_x(x^{-\gamma}) \tag{1.19}$$

a.e. in (a, b) *for any increasing sequence* $\mu:=\{\mu_k\}$.

Corollary 1.3. *If* $0<\gamma<1$, $d>\max(1/2, (p-1)/p)$ *and* $\beta=p\gamma$, *then* (1.3) *implies*

$$\left\{(n+1)^{-\beta}\sum_{\nu=0}^{n}(\nu+1)^{\beta-1}|f(x)-\sigma_\nu^{d-1}(\mu;\ x)|^p\right\}^{1/p} = o_x((\log n)^{1/p}n^{-\gamma}) \tag{1.20}$$

a.e. in (a, b) *for any increasing sequence* $\mu:=\{\mu_k\}$.

Corollary 1.4. *If* $0<\gamma<1, p\gamma=1, \alpha>0$ *and* $d>\max(1/2, (p-1)/p, (p-\alpha)/p)$, *then* (1.3) *implies*

$$\left\{\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}|f(x)-\sigma_\nu^{d-1}(\mu;\ x)|^p\right\}^{1/p} = o_x((\log n)^{1/p}n^{-\gamma}) \tag{1.21}$$

a.e. in (a, b) *for any increasing sequence* $\mu:=\{\mu_k\}$.

First we remark that Corollary 1.2 is a slight improvement of Theorem 1 proved in [2].

Furthermore we mention that since $\sigma_k^0(\mu; x)=s_{\mu_k}(x)$, thus the special case $d=1$ of Corollary 1.1 coincides with Theorem C if $0<\gamma<1$. But if $\gamma\geqq 1$ then Corollary 1.1 does not work, consequently, we cannot say that Corollary 1.1 is a generalization of Theorem C. So, we can say that Corollary 1.1 is a generalization of Theorem C if and only if the range of parameter γ is restricted to $0<\gamma<1$.

The same assertion can be made regarding Corollary 1.3 and Theorem D, moreover in connection with Corollary 1.4 and Theorem E.

Finally we deduce one more statement from Corollary 1.

Corollary 1.5. *If* $p>0$, $0<\gamma<1$ *and* $d>\max(1/2, (p-1)/p)$, *then* (1.3) *implies*

$$\left\{\frac{1}{n}\sum_{\nu=n+1}^{2n}|f(x)-\sigma_\nu^{d-1}(\mu;\ x)|^p\right\}^{1/p} = o_x(n^{-\gamma}) \tag{1.22}$$

a.e. in (a, b) *for any increasing sequence* $\mu:=\{\mu_k\}$.

We point out that Corollary 1.5 sharpens and generalizes Corollary 1 proved in [3]. It can be used for any positive p, not only if $p<\gamma^{-1}$ as in [3].

2. In order to prove our theorem and corollaries we need three known lemmas and one to be proved now.

Lemma 1 ([4]). *Let* $\delta>0$ *and* $\{\delta_n\}$ *an arbitrary sequence of non-negative numbers. Suppose that for any orthonormal system* $\{\varphi_n(x)\}$ *the condition*

$$\sum_{n=0}^{\infty}\delta_n\Big(\sum_{m=n}^{\infty}c_m^2\Big)^{\delta}<\infty \tag{2.1}$$

implies that the sequence $\{s_n(x)\}$ *of the partial sums of* (1.1) *possesses a property P, then any subsequence* $\{s_{\nu_n}(x)\}$ *also possesses property P.*

Lemma 2 ([1]). *Set*

$$\tau_n^{\alpha}:=\tau_n^{\alpha}(x):=n\{\sigma_n^{\alpha}(x)-\sigma_{n-1}^{\alpha}(x)\}\quad (=\alpha\{\sigma_n^{\alpha-1}(x)-\sigma_n^{\alpha}(x)\}\ \textit{if}\ \alpha>0).$$

Let $\bar p\geqq\bar q>1$, $\bar\varrho>0$, $\bar\alpha>\bar\varrho-1$ *and* $\bar\beta\geqq\bar\alpha+(\bar q)^{-1}-(\bar p)^{-1}$. *Then*

$$\Big\{\sum_{n=0}^{\infty}(n+1)^{\bar p\bar\varrho-1}|\tau_n^{\bar\beta}|^{\bar p}\Big\}^{1/\bar p}\leqq K\Big\{\sum_{n=0}^{\infty}(n+1)^{\bar q\bar\varrho-1}|\tau_n^{\bar\alpha}|^{\bar q}\Big\}^{1/\bar q}. \tag{2.2}$$

Lemma 3 ([7]). *If* $0<\gamma<1$ *and* (1.3) *holds, then*

$$\int_a^b\Big\{\sum_{n=0}^{\infty}(n+1)^{2\gamma-1}|\sigma_n^{\alpha-1}(x)-\sigma_n^{\alpha}(x)|^2\Big\}\,dx\leqq K\sum_{n=1}^{\infty}c_n^2n^{2\gamma}$$

for any $\alpha>1/2$.

Lemma 4. *Under the assumptions of Theorem we have the inequality*

$$\int_a^b\Big\{\sup_{0\leqq n<\infty}\Big(\frac{n^{p\gamma}}{g(n)A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^{d-1}(x)-\sigma_\nu^{d}(x)|^p\Big)^{1/p}\Big\}^2dx\leqq K\sum_{n=1}^{\infty}c_n^2n^{2\gamma}. \tag{2.3}$$

Proof. Set $\varrho':=\varrho/(\varrho-1)$, then by (1.5) and (1.7), we have

$$\varrho'p\geqq 2\quad\text{and}\quad d>1-(\varrho'p)^{-1}. \tag{2.4}$$

Applying Hölder's inequality, by (1.15) and $\varrho>1$, we obtain that

$$\sum_{\nu=0}^{n}\alpha_{n\nu}|\tau_\nu^d(x)|^p\leqq\Big\{\sum_{\nu=0}^{n}\alpha_{n\nu}^{\varrho}(\nu+1)^{(\varrho/\varrho')-\gamma p\varrho}\Big\}^{1/\varrho}\Big\{\sum_{\nu=0}^{n}(\nu+1)^{\gamma p\varrho'-1}|\tau_\nu^d(x)|^{p\varrho'}\Big\}^{1/\varrho'}\leqq \tag{2.5}$$

$$\leqq Kg(n)n^{-\gamma p}A_n\Big\{\sum_{\nu=0}^{n}(\nu+1)^{\gamma p\varrho'-1}|\tau_\nu^d(x)|^{p\varrho'}\Big\}^{1/\varrho'}.$$

By the second statement of (2.4) we can choose α^* such that

$$d-\frac{1}{2}+\frac{1}{\varrho'p}>\alpha^*>\frac{1}{2} \tag{2.6}$$

holds. By (2.6), $0<\gamma<1$ and $\varrho'\varrho\geqq2$ the conditions of Lemma 2 are fulfilled with

$\bar{p}=\varrho' p$, $\bar{q}=2$, $\bar{\varrho}=\gamma$, $\bar{\alpha}=\alpha^*$ and $\bar{\beta}=d$. Using Lemma 2 we get

$$\Big\{\sum_{\nu=0}^{\infty}(\nu+1)^{\gamma p\varrho'-1}|\tau_\nu^d(x)|^{p\varrho'}\Big\}^{1/p\varrho'} \leqq K\Big\{\sum_{\nu=0}^{\infty}(\nu+1)^{2\gamma-1}|\tau_\nu^{\alpha^*}(x)|^2\Big\}^{1/2}. \tag{2.7}$$

Thus, by (2.5), (2.6), (2.7) and Lemma 3, we have

$$\int_a^b\Big\{\sup_{0\leqq n<\infty}\Big(\frac{n^{\gamma p}}{g(n)A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\tau_\nu^d(x)|^p\Big)^{1/p}\Big\}^2 dx \leqq$$

$$\leqq K\int_a^b\Big\{\sum_{\nu=0}^{\infty}(\nu+1)^{2\gamma-1}|\tau_\nu^{\alpha^*}(x)|^2\Big\}dx \leqq K\sum_{n=1}^{\infty}c_n^2 n^{2\gamma}<\infty,$$

which gives statement (2.3).

3. Proof of Theorem. It is clear that

$$\Big\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^{d-1}(x)|^p\Big\}^{1/p} \leqq \tag{3.1}$$

$$\leqq K\Big(\Big\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^{d}(x)|^p\Big\}^{1/p}+\Big\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^d(x)-\sigma_\nu^{d-1}(x)|^p\Big\}^{1/p}\Big).$$

First we show that the first term has the required order.

Since $d>1/2$, so, e.g. by Theorem A with $p=1$, we get that

$$f(x)-\sigma_n^d(x) = o_x(n^{-\gamma}) \tag{3.2}$$

a.e. in (a, b).

Let now $\varepsilon>0$ be given. If x is a point where (3.2) holds, then let $M(x)$ be a positive integer such that for $n>M(x)$ the inequality

$$|f(x)-\sigma_n^d(x)| < \varepsilon n^{-\gamma} \tag{3.3}$$

is valid. For such x we get that

$$\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^d(x)|^p \leqq \Big(\sum_{n(l)\leqq M(x)}+\sum_{\substack{n(l)>M(x)\\ l\leqq m}}\Big)\pi_l, \tag{3.4}$$

where

$$\pi_l:=\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}\alpha_{n\nu}^{\varrho}(\nu+1)^{\varrho(1-\gamma p)-1}\Big\}^{1/\varrho}\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}\frac{1}{\nu+1}\big((\nu+1)^{\gamma p}|f(x)-\sigma_\nu^d(x)|^p\big)^{\varrho'}\Big\}^{1/\varrho'}.$$

By (1.15) it is easy to see that the first sum remains $O_x(g(n)A_n n^{-\gamma p})$, but if (1.17) also holds, then its order $o_x(g(n)A_n n^{-\gamma p})$.

On the other hand, by (1.15) and (3.3), the second sum in (3.4) is always less than $O_x(1)\varepsilon^p g(n)A_n n^{-\gamma p}$, that is, its order is always $o_x(g(n)A_n n^{-\gamma p})$.

Consequently we have

$$(3.5)\qquad \left\{\frac{1}{A_n}\sum_{\nu=0}^{n}\alpha_{n\nu}|f(x)-\sigma_\nu^d(x)|^p\right\}^{1/p}=\begin{cases}O_x(g(n)^{1/p}n^{-\gamma}), & \text{always,}\\ o_x(g(n)^{1/p}n^{-\gamma}), & \text{if (1.17) holds,}\end{cases}$$

a.e. in (a, b).

Next we show that the second term in (3.1) also has the suitable orders of (3.5), according as (1.17) is not or is satisfied.

Now let ε be any positive number. Let us choose N so large that

$$(3.6)\qquad \sum_{n=N+1}^{\infty} c_n^2 n^{2\gamma} < \varepsilon^3.$$

By means of N we split series (1.3) into

$$\sum_{n=1}^{N} c_n^2 n^{2\gamma} < \infty \quad\text{and}\quad \sum_{n=N+1}^{\infty} c_n^2 n^{2\gamma} < \varepsilon^3,$$

and consider the corresponding orthogonal series. More exactly, let

$$(3.7)\qquad \sum_{n=0}^{\infty} a_n\varphi_n(x) \quad\text{with}\quad a_n=\begin{cases}c_n & \text{for}\quad n\le N,\\ 0 & \text{for}\quad n>N;\end{cases}$$

and

$$(3.8)\qquad \sum_{n=0}^{\infty} b_n\varphi_n(x) \quad\text{with}\quad b_n=\begin{cases}0 & \text{for}\quad n\le N,\\ c_n & \text{for}\quad n>N.\end{cases}$$

If $\sigma_n^\alpha(a; x)$ and $\sigma_n^\alpha(b; x)$ denote the (C, α)-means of series (3.7) and (3.8), respectively, then

$$(3.9)\qquad \sigma_n^\alpha(x)=\sigma_n^\alpha(a;\, x)+\sigma_n^\alpha(b;\, x).$$

Since the number of the coefficients $a_n\ne 0$ is finite,

$$\sigma_\nu^{d-1}(a;\, x)-\sigma_\nu^d(a;\, x)=\frac{1}{A_\nu^d}\sum_{k=0}^{N} kA_{\nu-k}^{d-1}c_k\varphi_k(x)$$

if $\nu>N$; and for any $k\le N$ $A_{\nu-k}^{d-1}/A_\nu^d=O(1/\nu)$, so

$$(3.10)\qquad \sum_{\nu=0}^{n}\alpha_{n\nu}|\sigma_\nu^d(a;\, x)-\sigma_\nu^{d-1}(a;\, x)|^p\le$$

$$\le\sum_{l=0}^{m}\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}\alpha_{n\nu}^\varrho(\nu+1)^{\varrho(1-\gamma p)-1}\Big\}^{1/\varrho}\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}(\nu+1)^{-1+\gamma p\varrho'}|\sigma_\nu^d-\sigma_\nu^{d-1}|^{p\varrho'}\Big\}^{1/\varrho'}\le$$

$$\le O_x(1)\sum_{l=0}^{m}\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}\alpha_{n\nu}^\varrho(\nu+1)^{\varrho(1-\gamma p)-1}\Big\}^{1/\varrho}\Big\{\sum_{\nu=n(l)-1}^{n(l+1)}(\nu+1)^{p\varrho'(\gamma-1)-1}\Big\}^{1/\varrho'}:=$$

$$:=O_x(1)\sum_{l=0}^{m}A_l\cdot B_l.$$

By $\gamma<1$ and $B_l \leqq 2\cdot 2^{(l-1)p(\gamma-1)} (\leqq 2)$, for any $\varepsilon>0$, there exists a positive integer l_0 such that if $l>l_0$ then $B_l<\varepsilon$. Thus

$$\sum_{l=0}^{m} A_l B_l = \Big(\sum_{l=0}^{l_0} + \sum_{l=l_0+1}^{m}\Big) A_l B_l \leqq 2 \sum_{l=0}^{l_0} A_l + \varepsilon \sum_{l=l_0+1}^{m} A_l,$$

whence, by (1.15), we get that

$$\sum_{l=0}^{m} A_l B_l \leqq K g(n) A_n n^{-\gamma p}; \tag{3.11}$$

and if (1.17) also holds then we have the estimate

$$\sum_{l=0}^{m} A_l B_l = o\big(g(n) A_n n^{-\gamma p}\big). \tag{3.12}$$

Summing up estimates (3.10), (3.11) and (3.12) we obtain that

$$\left\{\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^d(a;\, x) - \sigma_\nu^{d-1}(a;\, x)|^p\right\}^{1/p} = \begin{cases} O_x\big(g(n)^{1/p} n^{-\gamma}\big), & \text{always,} \\ o_x\big(g(n)^{1/p} n^{-\gamma}\big), & \text{if (1.17) holds,} \end{cases} \tag{3.13}$$

a.e. in (a, b).

In order to estimate the suitable terms of series (3.8) we use Lemma 4 and (3.6). Then

$$\int_a^b \left\{\sup_{0\leqq n<\infty} \left(\frac{n^{p\gamma}}{g(n)A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^d(b;\, x) - \sigma_\nu^{d-1}(b;\, x)|^p\right)^{1/p}\right\}^2 dx \leqq K\varepsilon^3.$$

Hence

$$\operatorname{meas}\left\{x \,|\, \limsup \left(\frac{n^{p\gamma}}{g(n)A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^d(b;\, x) - \sigma_\nu^{d-1}(b;\, x)|^p\right)^{1/p} > \varepsilon\right\} \leqq K\varepsilon.$$

This, (3.9) and (3.13) imply

$$\left\{\frac{1}{A_n} \sum_{\nu=0}^{n} \alpha_{n\nu} |\sigma_\nu^d(x) - \sigma_\nu^{d-1}(x)|^p\right\}^{1/p} = \begin{cases} O_x\big(g(n)^{1/p} n^{-\gamma}\big), & \text{always,} \\ o_x\big(g(n)^{1/p} n^{-\gamma}\big), & \text{if (1.17) holds,} \end{cases} \tag{3.14}$$

a.e. in (a, b).

Finally, (3.5) and (3.14) yield both statements of Theorem, so our proof is completed.

Proof of Corollary 1. The statements of Corollary 1 follow from the statements of Theorem and from (1.3) using Lemma 1 with $\delta=1$ and $\delta_n := n^{2\gamma} - (n-1)^{2\gamma}$. More precisely, now we have to use Lemma 1 twice. First the property P is that the means $\sigma_v^{d-1}(x)$ of the sequence $\{s_n(x)\}$ approximate $f(x)$, in strong sense regarding the matrix (α_{nk}/A_n) and the exponent p, at the order given in Theorem by (1.16) a.e. in (a, b). Secondly, if (1.17) is also satisfied, then the suitable property P is that the order of the approximation by the means mentioned above is $o_x(g(n)^{1/p}n^{-\gamma})$ a.e. in (a, b).

Proof of Corollary 1.1. Set $\alpha_{n\nu}:=(\nu+1)^{\beta-1}$. Then, regarding the condition $\beta>\gamma p$, an elementary calculation shows that both (1.15) and (1.17) with $g(n)\equiv\equiv 1$ hold for any $\varrho>1$.

On the other hand, since $d>\max(1/2, (p-1)/p)$ and $(1-d)<\min(1/2, 1/p)$ are equivalent, we can give a number $\varrho'>1$ such that $(1-d)p<1/\varrho'<\min(1, p/2)$ and if $\varrho:=\varrho'/(\varrho'-1)$, then both (1.5) and (1.7) hold.

Consequently, with this ϱ, all of the assumptions of Corollary 1 can be satisfied, so, applying Corollary 1, we get (1.18) immediately.

Proof of Corollary 1.2. We set $\alpha_{n\nu}:=A_{n-\nu}^{\alpha-1}$ and follow a similar consideration as in the previous proof with the only change that now we choose ϱ' such that $(1-d)p<1/\varrho'<\min(1, \alpha, p/2)$. Using the suitable ϱ and the condition $p\gamma<1$, elementary calculations show that all of the assumptions of Corollary 1 are satisfied; and Corollary 1 yields (1.19).

The proofs of Corollaries 1.3 and 1.4 run parallel, therefore we detail only the proof of Corollary 1.4.

Proof of Corollary 1.4. Set $\alpha_{n\nu}:=A_{n-\nu}^{\alpha-1}$. Using the assumption $p\gamma=1$ and ϱ' chosen by $(1-d)p<1/\varrho'<\min(1, \alpha, p/2)$, we get that conditions (1.15) and (1.17) with $g(n):=\log n$ and $\varrho:=\varrho'/(\varrho'-1)$ hold, furthermore (1.5) and (1.7) are also fulfilled. Therefore, with these quantities, Corollary 1 can be applied, and we get (1.21).

Proof of Corollary 1.5. Now we set

$$\alpha_{2n,\nu}:=\begin{cases}0, & \text{if } \nu\leqq n,\\ 1, & \text{if } n<\nu\leqq 2n,\end{cases}$$

and

$$\alpha_{2n+1,\nu}:=\begin{cases}\alpha_{2n,\nu}, & \text{if } \nu\leqq 2n,\\ 0, & \text{if } \nu=2n+1.\end{cases}$$

An easy calculation shows that both (1.15) and (1.17) with $g(n)\equiv 1$ hold for any $\varrho>1$. Since the assumption on d yields to choose ϱ' such that $(1-d)p<1/\varrho'<<\min(1, p/2)$, therefore conditions (1.5) and (1.7) can be satisfied simultaneously. Consequently we can apply Corollary 1, whence (1.22) obviously follows.

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