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A note on the strong de la Vallée Poussin approximation

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1. Let $\{\varphi_n(x)\}$ be an orthonormal system on the finite interval (a, b). We shall consider series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \tag{1.1}$$

with real coefficients satisfying

$$\sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.2}$$

By the Riesz—Fischer theorem, series (1.1) converges in the metric L^2 to a square-integrable function $f(x)$. We denote the n-th partial sum of series (1.1) by $s_n(x)$.

It is well known that the notion of *strong summability* is due to HARDY and LITTLEWOOD [3], and the notion of *strong approximation* is due to ALEXITS [2].

Since the strong approximation investigations have started in the 1960s it has become more and more clear that most of the results concerning any property of ordinary approximation have an analogue in strong sense. In other words, we have the same rate of approximation for strong means as for ordinary ones if we consider any one of the most frequently used means. This is true in spite of the facts that, in general, neither strong summability nor strong approximation do not follow from the suitable general ordinary summability and approximation (see MÓRICZ [11] and LEINDLER [9]). Some sample theorems showing the great analogy between the ordinary and strong approximation results can be found e.g. in the works [1], [4], [5], [6], [8], [10].

Recently we have discovered that even in the case of the classical de la Vallée Poussin approximation there exists a result which has only a weaker analogue in strong sense. One of the aims of this note is to fill up this gap.

In order to formulate our statements precisely we recall some definitions and theorems.

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Now we define the ordinary, furthermore the strong and very strong de la Vallée Poussin means with exponent p $(p>0)$:

$$V_n(x) := \frac{1}{n}\sum_{\nu=n}^{2n-1} s_\nu(x), \quad (n \geqq 1),$$

$$V_n|p;\, x| := \left\{\frac{1}{n}\sum_{\nu=n}^{2n-1} |s_\nu(x)-f(x)|^p\right\}^{1/p}$$

and

$$V_n|p, \nu;\, x| := \left\{\frac{1}{n}\sum_{k=n}^{2n-1} |s_{\nu_k}(x)-f(x)|^p\right\}^{1/p},$$

where $\nu := \{\nu_k\}$ denotes an arbitrary increasing sequence of positive integers.

In [4] we proved

Theorem A. *Let $\{\lambda_n\}$ be a monotonic sequence of positive numbers such that*

$$\sum_{k=0}^{m} \lambda_{2^k}^2 \leqq K\lambda_{2^m}^2. \text{ *)} \tag{1.3}$$

If

$$\sum_{n=0}^{\infty} c_n^2 \lambda_n^2 < \infty, \tag{1.4}$$

then we have that

$$V_n(x)-f(x) = o_x(\lambda_n^{-1})$$

holds almost everywhere (a.e.) in (a, b).

A similar, but weaker result in connection with the strong approximation $(p=1)$ was proved in [6] which reads as follows.

Theorem B. *Let $\{\lambda_n\}$ be a monotonic sequence of positive numbers with* (1.3). *If*

$$\sum_{n=0}^{\infty} c_n^2 \lambda_{2n}^2 < \infty, \tag{1.5}$$

then

$$V_n|p, x| = o_x(\lambda_n^{-1}) \tag{1.6}$$

holds a.e. in (a, b) for any $0<p\leqq 2$.

It is easy to see that if $\lambda_n = n^\gamma$ with $\gamma>0$, then conditions (1.4) and (1.5) are equivalent; but if e.g. $\lambda_n = q^n$ with $q>1$, then (1.5) requires much more that (1.4) does in order to have the same order of approximation.

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*) K will denote positive constant not necessarily the same one at each occurrence.

First we show that (1.4) also always implies (1.6), but this proof will be longer than that of the implication (1.5)$\Rightarrow$(1.6) in [6]. Thereafter, using a very general lemma proved only in 1982, we shall extend our result to the very strong approximation; i.e. we shall prove the following result.

Theorem. *Let* $\{\lambda_n\}$ *be a monotonic sequence of positive numbers with* (1.3). *If* (1.4) *holds, then*

$$V_n|p, \nu; x| = o_x(\lambda_n^{-1}) \tag{1.7}$$

a.e. in (a, b) *for any* $0<p\leqq 2$ *and for any increasing sequence* $\nu:=\{\nu_k\}$ *of positive integers.*

2. In order to prove our theorem we need two known lemmas.

Lemma 1 (Kronecker lemma, see e.g. [1] p. 68). *If* $s_n(a)$ *denotes the n-th partial sum of the numerical series* $\sum_{m=0}^{\infty} a_m$ *and* $\{\lambda_n\}$ *is an increasing sequence of positive numbers such that* $\lambda_n\to\infty$ *and the series* $\sum_{m=0}^{\infty} a_m\lambda_m^{-1}$ *converges, then* $s_n(a)=o(\lambda_n)$ *holds.*

Lemma 2 ([7]). *Let* $\delta>0$ *and* $\{\delta_n\}$ *an arbitrary sequence of non-negative numbers. Suppose that for any orthonormal system* $\{\varphi_n(x)\}$ *the condition*

$$\sum_{n=0}^{\infty} \delta_n\Big(\sum_{m=n}^{\infty} c_m^2\Big)^{\delta} < \infty \tag{2.1}$$

implies that the sequence $\{s_n(x)\}$ *of the partial sums of* (1.1) *possesses a property P, then any subsequence* $\{s_{\nu_n}(x)\}$ *also possesses property P.*

3. First we carry out the proof of (1.7) when $p=2$ and $\nu_k=k$. An elementary consideration shows that

$$V_n|2, x|^2 \leqq K\Big\{\frac{1}{n}\sum_{\nu=n}^{2n-1} |s_\nu(x)-V_\nu(x)|^2+\frac{1}{n}\sum_{\nu=n}^{2n-1} |V_\nu(x)-f(x)|^2\Big\}. \tag{3.1}$$

The second term on the right hand side of (3.1) is $o_x(\lambda_n^{-2})$ a.e. in (a, b) on account of Theorem A regarding the monotonicity of the sequence $\{\lambda_n\}$. Thus we have only to estimate the first term. For that purpose we first show that

$$\sum\nolimits_1 := \sum_{n=1}^{\infty} \lambda_n^2 n^{-1} \int_a^b (s_n(x)-V_n(x))^2\,dx < \infty. \tag{3.2}$$

Namely, an easy calculation gives that

$$\sum\nolimits_1 = \sum_{n=1}^{\infty} \lambda_n^2 n^{-1} \frac{1}{n^2} \sum_{\nu=n+1}^{2n-1} (2n-\nu)^2 c_\nu^2 \leqq$$

$$\leqq \sum_{n=1}^{\infty} \lambda_n^2 \frac{1}{n} \sum_{\nu=n+1}^{2n-1} c_\nu^2 \leqq \sum_{\nu=1}^{\infty} c_\nu^2 \sum_{\nu/2<n<\nu} \lambda_n^2 n^{-1} \leqq \sum_{\nu=1}^{\infty} c_\nu^2 \lambda_\nu^2,$$

whence, by (1.4), (3.2) obviously follows. From (3.2), using B. Levi's theorem, we get that

$$\sum_{n=1}^{\infty} \lambda_n^2 n^{-1}(s_n(x)-V_n(x))^2 < \infty$$

a.e. in (a, b). Hence, by Lemma 1, the estimation

$$\sum_{n=1}^{2m} \lambda_n^2 (s_n(x)-V_n(x))^2 = o_x(2m) \tag{3.3}$$

holds a.e. in (a, b). But (3.3) clearly implies that

$$\frac{1}{m} \sum_{n=m}^{2m-1} |s_n(x)-V_n(x)|^2 = o_x(\lambda_m^{-2})$$

also holds a.e. in (a, b). Summing up our partial results we get that

$$V_n|2, x| = o_x(\lambda_n^{-1}). \tag{3.4}$$

On account of the Hölder's inequality, we get, for any $0<p\leqq 2$, that

$$V_n|p, x| \leqq V_n|2, x|$$

whence, by (3.4),

$$V_n|p, x| = o_x(\lambda_n^{-1}) \tag{3.5}$$

also holds a.e. in (a, b). This completes the proof when $\nu_k = k$.

Finally, the statement of Theorem in its generality, i.e. for arbitrary $\nu := \{\nu_k\}$, follows from (3.5) using Lemma 2 with $\delta=1$ and $\delta_n := \lambda_n^2 - \lambda_{n-1}^2$ $(\lambda_{-1}=0)$; furthermore the property P in this case will be just the estimation given by (3.5).

Theorem is hereby proved completely.

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