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On additive functions with values in a compact Abelian group

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1. Introduction

Let G be an additively written, metrically compact Abelian topological group, $\mathbf{N}$ be the set of all positive integers. A function $f\colon \mathbf{N}\to G$ is called a completely additive, if

$$f(nm) = f(n)+f(m)$$

holds for all $n, m\in\mathbf{N}$. Let $\mathscr{A}_G^*$ denote the class of all completely additive functions $f\colon \mathbf{N}\to G$.

Let $A>0$ and $B\neq 0$ be fixed integers. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $D[A, B]$ if for any convergent subsequence $\{x_{\nu_n}\}_{n=1}^\infty$ the sequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ has a limit, too. We say that it is of property $E[A, B]$ if for any convergent subsequence $\{x_{A\nu_n+B}\}_{n=1}^\infty$ the sequence $\{x_{\nu_n}\}_{n=1}^\infty$ is convergent. We shall say that an infinite sequence $\{x_\nu\}_{\nu=1}^\infty$ in G is of property $\Delta[A, B]$ if the sequence $\{x_{A\nu+B}-x_\nu\}_{\nu=1}^\infty$ has a limit.

Let $\mathscr{A}_G^*(D[A, B])$, $\mathscr{A}_G^*(E[A, B])$ and $\mathscr{A}_G^*(\Delta[A, B])$ be the classes of those $f\in\mathscr{A}_G^*$ for which $\{x_\nu=f(\nu)\}_{\nu=1}^\infty$ is of property $D[A, B]$, $E[A, B]$ and $\Delta[A, B]$, respectively.

It is obvious that

$$\mathscr{A}_G^*(\Delta[A, B]) \subseteqq \mathscr{A}_G^*(D[A, B]) \quad\text{and}\quad \mathscr{A}_G^*(\Delta[A, B]) \subseteqq \mathscr{A}_G^*(E[A, B]).$$

Z. Daróczy and I. Kátai proved in [1] that

$$\mathscr{A}_G^*(\Delta[1, 1]) = \mathscr{A}_G^*(D[1, 1]),$$

and in [2] they deduced the following assertion: If $f\in\mathscr{A}_G^*(\Delta[1,1)$, then there exists a continuous homomorphism $\Psi\colon \mathbf{R}_x\to G$, $\mathbf{R}_x$ denotes the multiplicative group of the positive reals, such that $f(n)=\Psi(n)$ for all $n\in\mathbf{N}$.

For the case $A=2$ and $B=-1$ the complete characterization of $\mathscr{A}_G^*(D[2, -1])$ and $\mathscr{A}_G^*(\Delta[2, -1])$ has been given by Z. Daróczy and I. Kátai [3], [4].

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In a recent paper [5] we gave a complete characterization of $\mathcal{A}_G^*(E[A, B])$ and $\mathcal{A}_G^*(\Delta[A, B])$. Namely we showed that

$$\mathcal{A}_G^*(E[A, B]) = \mathcal{A}_G^*(\Delta[A, B])$$

and

$$\mathcal{A}_G^*(\Delta[A, B]) = \mathcal{A}_G^*(\Delta[1, 1]).$$

In the other words, if $f \in \mathcal{A}_G^*(E[A, B]) = \mathcal{A}_G^*(\Delta[A, B])$, then there is a continuous homomorphism $\Psi\colon \mathbf{R}_x \to G$ such that $f(n) = \Psi(n)$ for all $n \in \mathbf{N}$.

Our main purpose in this paper is to give a complete determination of $\mathcal{A}_G^*(D[A, B])$. We note that it is enough to characterize those classes $\mathcal{A}_G^*(D[A, B])$ for which $(A, B) = 1$, since

$$\mathcal{A}_G^*(D[Ad, Bd]) = \mathcal{A}_G^*(D[A, B])$$

holds for each $d \in \mathbf{N}$.

We shall prove the following

Theorem. *Let $A > 0$ and $B \neq 0$ be fixed integers for which $(A, B) = 1$ and let G be a metrically compact Abelian topological group. If $f \in \mathcal{A}_G^*(D[A, B])$, then there are $U \in \mathcal{A}_G^*$ and a continuous homomorphims $\Phi\colon \mathbf{R}_x \to G$, $\mathbf{R}_x$ denotes the multiplicative group of positive reals, such that*

$$\text{(I)} \qquad f(n) = \Phi(n) + U(n) \quad \forall n \in \mathbf{N},$$

$$\text{(II)} \qquad U(n+A) = U(n) \quad \forall n \in \mathbf{N},\ (n, A) = 1,$$

(III) *If X_1, Γ denote the set of all limit points of $\{\Phi(n) | n \in \mathbf{N}\}$ and $\{U(n) | n \in \mathbf{N}\}$, respectively, then*

$$X_1 \cap \Gamma = \{0\}$$

and Γ is the smallest closed group generated by

$$\{U(m) | 1 \leqq m < A, (m, A) = 1\} \quad \text{and} \quad \{U(p) | p \text{ is prime, } p | A\}.$$

Conversely, let $\Phi\colon \mathbf{R}_x \to G$ be an arbitrary continuous homomorphism, X_1 be the smallest compact supgroup generated by $\{\Phi(n) | n \in \mathbf{N}\}$. Let $U \in \mathcal{A}_G^*$ be so chosen that $U(n+A) = U(n)$ for all $n \in \mathbf{N}, (n, A) = 1$ and the smallest closed group Γ generated by $U(\mathbf{N})$ has the property $X_1 \cap \Gamma = \{0\}$. Then the function

$$f(n) := \Phi(n) + U(n)$$

belongs to $\mathcal{A}_G^*(D[A, B])$.

2. Preliminary lemmas

In this section we shall prove some results which will be used in the proof of our theorem.

Lemma 1. *We have*

$$\mathscr{A}_G^*(D[A, B]) \subseteqq \mathscr{A}_G^*(D[A, 1])$$

for all fixed integers $A>0$ *and* $B\neq 0$.

Proof. Let $A>0$, $B\neq 0$ be fixed integers. Assume that

$$f\in\mathscr{A}_G^*(D[A, B]).$$

Let

$$n_1<\ldots<n_\nu<\ldots \quad (n_\nu\in\mathbf{N})$$

be an infinite sequence for which the sequence $\{f(n_\nu)\}_{\nu=1}^{\infty}$ is convergent. Then, it is obvious that the sequence $\{f(|B|n_\nu)\}_{\nu=1}^{\infty}$ has also a limit, consequently we get from the definition of $\mathscr{A}_G^*(D[A, B])$ that

$$\lim_{\nu\to\infty} f\left[An_\nu+\frac{B}{|B|}\right] = \lim_{\nu\to\infty} f[A|B|n_\nu+B]-f(|B|)$$

exists as well. This implies in the case $B>0$ that $f\in\mathscr{A}_G^*(D[A, 1])$.

We now assume that $B<0$. In this case we have $f\in\mathscr{A}_G^*(D[A, -1])$. Since $\{f(n_\nu)\}_{\nu=1}^{\infty}$ is convergent, therefore the sequence $\{f(An_\nu^2)\}_{\nu=1}^{\infty}$ is convergent, too. Thus, by using the fact $f\in\mathscr{A}_G^*(D[A, -1])$, it follows that the following limit exists:

$$\lim_{\nu\to\infty} f(An_\nu+1) = \lim_{\nu\to\infty} f[(An_\nu)^2-1]-\lim_{\nu\to\infty} f[An_\nu-1].$$

This shows that $f\in\mathscr{A}_G^*(D[A, 1])$.

So we have proved Lemma 1.

In the following we assume that $A>0, B\neq 0$ are fixed integers and G is a metrically compact Abelian topological group. Let

$$f\in\mathscr{A}_G^*(D[A, B]).$$

We shall denote by X the set of limit points of $\{f(n)|n\in\mathbf{N}\}$, i.e. $g\in X$ if there exists a sequence

$$n_1<\ldots<n_\nu<\ldots \quad (n_\nu\in\mathbf{N})$$

for which $f(n_\nu)\to g$. Let X_1 ($\subseteqq X$) be the set of limit points of $\{f(An+1)|n\in\mathbf{N}\}$. Since $\mathbf{N}$ and the positive integers $m\equiv 1 \pmod A$ form semingroups, therefore $\{f(n)|n\in\mathbf{N}\}$ and $\{f(An+1)|n\in\mathbf{N}\}$ are semigroups as well. Thus, X and X_1 are closed semigroups in the compact group G, so by a known theorem (see [6], Theorem

(9.16)) they are compact subgroups in G. Since $0\in X_1\subseteq X$ we have $f(n)\in X$ and $f(An+1)\in X_1$ for each $n\in\mathbf{N}$.

Let $g\in X$ and $f(n_\nu)\to g$ as $\nu\to\infty$. Then, by using Lemma 1, it follows that the sequence $\{f(An_\nu+1)\}_{\nu=1}^{\infty}$ is convergent. Let $f(An_\nu+1)\to g'(\in X_1)$. It is easily seen that g' is determined by g, and so the correspondence

$$H: g \to g' \quad (g\in X,\ g'\in X_1)$$

is a function.

Lemma 2. *The function H: $X\to X_1$ is continuous and*

$$H(X) = X_1.$$

Proof. We can prove Lemma 2 by the same method as was used in [1] (see Lemma 4 and Lemma 5), so we omit the proof.

Lemma 3. *We have*

$$H\big(g+h+f(A)\big)+H(g) = H\big(g+H(h+H(g))\big) \tag{2.1}$$

for all $g\in X$ and $h\in X$.

Proof. Let $g\in X$ and $h\in X$ be arbitrary elements. Let

$$n_1<\ldots<n_\nu<\ldots \quad \text{and} \quad m_1<\ldots<m_\nu<\ldots \quad (n_\nu, m_\nu\in\mathbf{N})$$

be such sequences for which $f(n_\nu)\to g$ and $f(m_\nu)\to h$. By using the following relation

$$(A^2n_\nu m_\nu+1)(An_\nu+1) = An_\nu[Am_\nu(An_\nu+1)+1]+1$$

and using the definition of H, we get immediately that (2.1) holds. So, we have proved Lemma 3.

Lemma 4. *Let*

$$E(f) := \{\varrho\in X \mid H(\varrho) = 0\}.$$

Then $E(f)\neq\emptyset$. Furthermore, if $\varrho\in E(f)$, then

$$H\big(k\varrho+(k-1)f(A)\big) = 0 \tag{2.2}$$

for every integer k. In particular, we have

$$H\big(-f(A)\big) = 0. \tag{2.3}$$

Proof. Since X_1 is a group, therefore $0\in X_1$. Thus, it follows from $H(X)=X_1$ that there is at least one $\varrho\in X$ for which $H(\varrho)=0$. Then $E(f)\neq\emptyset$. Furthermore, it is easily seen from (2.1) that

$$H\big(\varrho_1+\varrho_2+f(A)\big) = 0 \quad \text{if} \quad H(\varrho_1) = H(\varrho_2) = 0. \tag{2.4}$$

Assume that $\varrho \in E(f)$, i.e. $H(\varrho)=0$. By using (2.4) and induction on k we get immediately that (2.2) holds for every $k \in \mathbf{N}$. Let

$$V_\varrho = \{k(\varrho + f(A)) \mid k \in \mathbf{N}\}.$$

Since (2.2) holds for every $k \in \mathbf{N}$, therefore we have

$$H(\delta - f(A)) = 0 \quad \text{for all} \quad \delta \in V_\varrho. \tag{2.5}$$

Let $\bar{V}_\varrho$ be the smallest closed set containing V_ϱ. It is clear that V_ϱ is a semigroup, therefore $\bar{V}_\varrho$ is a closed semigroup in G. Thus, by using a known theorem of [6], we get that $\bar{V}_\varrho$ is a compact group. Since H is continuous function and $\bar{V}_\varrho$ is the smallest closed set containing V_ϱ, it follows that (2.5) holds for all $\delta \in \bar{V}_\varrho$, consequently (2.2) holds for every integer k. So (2.2) is proved.

Finally, by applying (2.2) with $k=0$, we obtain (2.3).

The proof of Lemma 4 is finished.

Lemma 5. *We have*

$$H(g+\tau) = H(g) + \tau \tag{2.6}$$

for all $g \in X$ *and* $\tau \in X_1$.

Proof. We first prove that

$$H(\tau - f(A)) = \tau \quad \text{for all} \quad \tau \in X_1 \tag{2.7}$$

and

$$H(g - H(g)) = 0 \quad \text{for all} \quad g \in X. \tag{2.8}$$

Let $\tau \in X_1$. Then, it follows from $H(X)=X_1$ that there is one $h \in X$ such that $H(h)=\tau$. We apply (2.1) with $g=-f(A)$ and using (2.3), we have

$$H(H(h) - f(A)) = H(h),$$

which with $H(h)=\tau$ proves (2.7). It is clear that (2.8) is a consequence of (2.1) and (2.3) in the case $h+H(g)=-f(A)$.

We now prove Lemma 5.

Let $g \in X$ and $\tau \in X_1$ be arbitrary elements. By using (2.8), we have

$$H[(g+\tau) - H(g+\tau)] = 0$$

and

$$H[g - H(g)] = 0.$$

Applying Lemma 4 with $\varrho = g - H(g)$ and $k=-1$, we get that

$$H[-g + H(g) - 2f(A)] = 0.$$

Let

$$\varrho_1 := g + \tau - H(g+\tau) \quad \text{and} \quad \varrho_2 := -g + H(g) - 2f(A).$$

Then $H(\varrho_1)=H(\varrho_2)=0$, and so by (2.4) we have

$$H[(g+\tau-H(g+\tau))+(-g+H(g)-2f(A))+f(A)] = 0,$$

i.e.

$$H[(\tau-H(g+\tau)+H(g))-f(A)] = 0. \tag{2.9}$$

Since $\tau\in X_1$, $H(g+\tau)\in X_1$, $H(g)\in X_1$ and X_1 is a group, therefore

$$\tau-H(g+\tau)+H(g)\in X_1. \tag{2.10}$$

Finally, from (2.7), (2.9) and (2.10) we get that

$$\tau-H(g+\tau)+H(g) = 0,$$

which proves (2.6).

So we have proved Lemma 5.

Lemma 6. *We have*

$$H(g+h+f(A)) = H(g+h)+H(0) = H(g)+H(h) \tag{2.11}$$

for all $g\in X$ *and* $h\in X$.

Proof. Let $g\in X$ and $h\in X$. Since $H(h+H(g))\in X_1$ and $H(g)\in X_1$, by using Lemma 5, we have

$$H(g+H(h+H(g))) = H(g)+H(h+H(g)) = H(g)+H(h)+H(g).$$

This with (2.1) implies that

$$H(g+h+f(A)) = H(g)+H(h). \tag{2.12}$$

Thus, (2.12) holds for all $g\in X$ and $h\in X$.

On the other hand, we get from (2.12) that

$$H(g+h+f(A)) = H(g+h)+H(0).$$

This with (2.12) shows that (2.11) holds for all $g\in X$ and $h\in X$. The proof of Lemma 6 is finished.

3. Proof of the theorem

Assume that $A>0$ and $B\neq 0$ are fixed integers for which $(A,B)=1$ and G is a metrically compact Abelian topological group. Let

$$f\in\mathcal{A}_G^*(D[A,B]).$$

As in the Section 2, we denote by X and X_1 the set of limit points of $\{f(n)|\in\mathbf{N}\}$ and $\{f(An+1)|n\in\mathbf{N}\}$, respectively. Let $H\colon X\to X_1$ be a continuous function which is defined in Section 2, i.e., if $f(n_\nu)\to g$, then $f(An_\nu+1)\to H(g)$.

For an arbitrary $n \in \mathbf{N}$, let $S(n)$ be the product of all prime factors of n composed from the prime divisors of A, $R(n)$ be defined by $n=S(n)\cdot R(n)$, i.e. $(A, R(n))=1$ and every prime divisor of $S(n)$ is a divisor of A. Let $\bar{R}(n)$ be the smallest positive integer for which

$$\bar{R}(n) \equiv R(n) \pmod{A}.$$

It is obvious that $(\bar{R}(n), A)=1$ and $1 \leqq \bar{R}(n) < A$.

Lemma 7. *Let*

$$U(n) := f[S(n)\cdot \bar{R}(n)] + H(0) - H\big(f[S(n)\cdot \bar{R}(n)]\big). \tag{3.1}$$

Then, we have

$$H\big(f(n)\big) - f(n) - H(0) + U(n) = 0 \tag{3.2}$$

for all $n \in \mathbf{N}$.

Proof. Let $\bar{H}\colon X \to X_1$ be the function which is defined by the relation $\bar{H}(g) = H(g) - H(0)$. Then, it is easily seen from Lemma 5 and Lemma 6 that

$$\bar{H}(g+h) = \bar{H}(g) + \bar{H}(h) \quad \forall g, h \in X, \tag{3.3}$$

$$\bar{H}(\tau) = \tau \quad \forall \tau \in X_1 \tag{3.4}$$

and

$$\bar{H}(X) = X_1. \tag{3.5}$$

For each $n \in \mathbf{N}$, let $c(n)$ be the smallest positive integer for which $R(n)\cdot c(n) \equiv 1 \pmod{A}$. Then, it is obvious that

$$f[R(n)\cdot c(n)] \in X_1 \quad \text{and} \quad f[\bar{R}(n)\cdot c(n)] \in X_1$$

hold for every $n \in \mathbf{N}$. By using (3.3) and (3.4), we deduce that

$$\bar{H}[f(n)] + \bar{H}[f(c(n))] = \bar{H}[f(n\cdot c(n))] = f[R(n)\cdot c(n)] + \bar{H}[f(S(n))]$$

and

$$\bar{H}[f(\bar{R}(n))] + \bar{H}[f(c(n))] = \bar{H}[f(\bar{R}(n)\cdot c(n))] = f(\bar{R}(n)\cdot c(n)).$$

These imply that

$$\bar{H}[f(n)] - \bar{H}[f(\bar{R}(n))] = f(R(n)) - f(\bar{R}(n)) + \bar{H}[f(S(n))],$$

consequently

$$\bar{H}[f(n)] - f(n) + \{f(S(n)\cdot \bar{R}(n)) - \bar{H}[f(S(n)\cdot \bar{R}(n))]\} = 0.$$

This with (3.1) proves (3.2)

Lemma 8. *We have*

(i) $U \in \mathscr{A}_G^*$,

(ii) $U(n+A) = U(n)$ *for all* $n \in \mathbf{N}$, $(n, A)=1$,

(iii) *If* $\{a_1, \ldots, a_{\varphi(A)}\}$ *is a reduced residue system moduls* A, *then* $U(a_1), \ldots, U(a_{\varphi(A)})$ *form a group in* G.

(iv) *Let* Γ *denote the set of all limit points of* $\{U(n)|n\in\mathbf{N}\}$. *Then* Γ *is the smallest closed group generated by* $U(a_1), \ldots, U(a_{\varphi(A)}), U(p_1), \ldots, U(p_{\omega(A)})$, *where* $\{a_1, \ldots, a_{\varphi(A)}\}$ *is a reduced residue system moduls* A *and* $p_1, \ldots, p_{\omega(A)}$ *are all distinct prime factors of* A. *Furthermore, we have*

$$X_1\cap\Gamma = \{0\}.$$

Proof. Parts (i) and (ii) follow at once from the definition of U and Lemma 7. The part (iii) is a consequence of (i) and (ii). To prove (iv) we first note that Γ is a closed semigroup in G, and so Γ is a group by Theorem (9.16) of [6]. Hence by (ii) it follows that Γ is the smallest closed group generated by $U(a_1), \ldots, U(a_{\varphi(A)}), U(p_1), \ldots$ $\ldots, U(p_{\omega(A)})$.

Since X_1, Γ are subgroups in G, therefore $0\in X_1\cap\Gamma$. Let us assume that $\delta\in$ $\in X_1\cap\Gamma$. Then there is a sequence $\{n_\nu\}_{\nu=1}^\infty$ for which $U(n_\nu)\to\delta$. Applying (3.2) with $n=n_\nu$, we have

$$H[f(n_\nu)]-f(n_\nu)-H(0)+U(n_\nu) = 0. \tag{3.6}$$

Since G is sequentially compact, therefore the sequence $\{f(n_\nu)\}_{\nu=1}^\infty$ contains at least one limit point. Let

$$f(n_{\nu_j}) \to g \quad (\in X).$$

Then, by (3.6) and using the fact H is continuous, we get

$$H(g)-g-H(0)+\delta = 0,$$

which with $H(g)-H(0)+\delta\in X_1$ implies that $g\in X_1$. So, by Lemma 5

$$\delta = g+H(0)-H(g) = 0.$$

Thus, we have proved that $X_1\cap\Gamma=\{0\}$. This completes the proof of (iv).

The proof of Lemma 8 is finished.

We now prove the theorem. We first show that

$$f(An+1)-H\big(f(n)\big) \to 0 \quad \text{as} \quad n\to\infty. \tag{3.7}$$

Assume the contrary. Let

$$f(An_\nu+1)-H\big(f(n_\nu)\big) \to \lambda \neq 0 \quad \text{as} \quad \nu\to\infty. \tag{3.8}$$

Since the sequence $\{f(n_\nu)\}_{\nu=1}^\infty$ contains at least one limit point, we can find a subsequence $\{n_{\nu j}\}_{j=1}^\infty$ of the sequence $\{n_\nu\}_{\nu=1}^\infty$ such that $f(n_{\nu_j})\to g$ $(\in X)$ as $j\to\infty$. Using the continuity of H, by (3.8) we have

$$H(g)-H(g) = \lambda,$$

which is contradiction. Thus, we have proved (3.7). From (3.2) and (3.7) we get immediately

$$f(An+1)-f(n)-H(0)+U(n) \to 0 \quad \text{as} \quad n\to\infty. \tag{3.9}$$

Let

$$F(n):=f(n)-U(n) \quad (n\in\mathbf{N}).$$

It is obvious by Lemma 8 that $F\in\mathcal{A}_G^*$ and

$$F(An+1) = f(An+1)-U(An+1) = f(An+1)$$

for all $n\in\mathbf{N}$. This with (3.9) implies

$$F(An+1)-F(n)-H(0) \to 0 \quad \text{as} \quad n\to\infty,$$

consequently $F\in\mathcal{A}_G^*(\Delta[A, 1])$. It was proved in [5] that if $F\in\mathcal{A}_G^*(\Delta[A, 1])$, then there is a continuous homomorphism $\Phi\colon \mathbf{R}_x\to G$ such that $F(n)=\Phi(n)$ for all $n\in\mathbf{N}$, where $\mathbf{R}_x$ denotes the multiplicative group of positive reals. Thus, we have proved that

$$f(n) = \Phi(n)+U(n), \tag{3.10}$$

where U satisfies the conditions (i)—(iv) of Lemma 8. By (3.2) and (3.10) we also have

$$\Phi(n) = H\big(f(n)\big)-H(0) \quad \text{for all} \quad n\in\mathbf{N},$$

therefore it follows from (3.5) that the set of all limit points of $\{\Phi(n)|n\in\mathbf{N}\}$ is X_1.

So we have proved the first part of our theorem.

Finally, let $\Phi\colon \mathbf{R}_x\to G$ be a continuous homomorphism and let $U\in\mathcal{A}_G^*$ be so chosen that

$$U(n+A) = U(n) \quad \text{for all} \quad n\in\mathbf{N}, \quad (n, A) = 1 \tag{3.11}$$

and

$$X_1\cap\Gamma = \{0\}$$

where X_1, Γ denote the smallest closed groups in G which are generated by $\Phi(\mathbf{N})$ and $U(\mathbf{N})$, respectively.

Let

$$f(n):= \Phi(n)+U(n)\in\mathcal{A}_G^*.$$

Assume that for some subsequence $\{n_\nu\}_{\nu=1}^\infty$ of positive integers the sequence $\{f(n_\nu)\}_{\nu=1}^\infty$ converges. Then, by using $\Phi(n_\nu)\in X_1$, $U(n_\nu)\in\Gamma$ and $X_1\cap\Gamma=\{0\}$, we deduce that the sequences $\{\Phi(n_\nu)\}_{\nu=1}^\infty$ and $\{U(n_\nu)\}_{\nu=1}^\infty$ are convergent, therefore by (3.11) and $(A, B)=1$ we see that

$$\lim_{\nu\to\infty} f(An_\nu+B) = \lim_{\nu\to\infty} \{\Phi(An_\nu+B)+U(An_\nu+B)\} =$$

$$= \lim_{\nu\to\infty} \Phi(An_\nu+B)+U(B) = \Phi(A)+U(B)+\lim_{\nu\to\infty} \Phi(n_\nu)$$

exists as well. So we have proved that $f\in\mathcal{A}_G^*(D[A, B])$.

The proof of our theorem is finished.

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