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Strongly dense simultaneous similarity orbits of operators

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Introduction

Let X be a (real or complex) Banach space and let $B(X)$ denote the algebra of all bounded linear operators on X. Let $B^{(n)}(X)$ denote the product $B(X)\times\ldots\times B(X)$ of n copies of $B(X)$. The group of invertible operators in $B(X)$ acts on $B^{(n)}(X)$ by conjugation $A^{-1}(T_1, \ldots, T_n)A=(A^{-1}T_1A, \ldots, A^{-1}T_nA)$. For $(T_1, \ldots, T_n)$ in $B^{(n)}(X)$ denote by $S(T_1, \ldots, T_n)$ the orbit of $(T_1, \ldots, T_n)$ in $B^{(n)}(X)$,

$$S(T_1, \ldots, T_n) =$$
$$= \{A^{-1}(T_1, \ldots, T_n)A = (A^{-1}T_1A, \ldots, A^{-1}T_nA)\colon A \text{ is invertible in } B(X)\}.$$

The purpose of this paper is to describe those orbits $S(T_1, \ldots, T_n)$ which are strongly dense in $B^{(n)}(X)$. Recall that a net $\{S_\lambda\}$ in $B(X)$ converges strongly to an operator S in $B(X)$ if and only if $\lim_\lambda S_\lambda f=Sf$ for all f in X. If X is finite-dimensional then the strong topology coincides with the norm topology, and therefore $S(T_1, \ldots, T_n)$ is never dense in $B^{(n)}(X)$. If X is infinite-dimensional (and $n=1$), then $S(T)$ is strongly dense in $B(X)$ for a very large set of T's. More precisely, in [2] it was shown that $S(T)$ is strongly dense if and only if T is in the complement of the set $\{\lambda I+F\colon \lambda\in\mathbf{K},\ F \text{ has finite rank}\}$ ($\mathbf{K}$ is the field of scalars and I is the identity operator on X). Observe that an operator T is not a scalar plus a finite rank operator if and only if $\alpha_0I+\alpha_1T$ has infinite rank for all nonzero (α_0, α_1) in $\mathbf{K}^2$. This suggests to consider those n-tuples $(T_1, \ldots, T_n)$ such that $\alpha_0I+\alpha_1T_1+\ldots+\alpha_nT_n$ has infinite rank for all nonzero $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ in $\mathbf{K}^{n+1}$. In this paper we show that this condition on $(T_1, \ldots, T_n)$ characterizes the strong density of $S(T_1, \ldots, T_n)$ in $B^{(n)}(X)$. Another result from [2] states that $S(T)$ is strongly dense if and only if $S(T)$ is weakly dense. The corresponding generalization to n-tuples is also true. From [1] it follows that the strong density of $S(T)$ can be described in terms of compressions. If P is an idempotent in $B(X)$ with range X_0, then the

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compression of $S(T_1, \ldots, T_n)$ to X_0 is defined as the restriction of $PS(T_1, \ldots, P_n)P$ to X_0. Then for n-tuples the density of $S(T_1, \ldots, T_n)$ is characterized by the condition that the compression of $S(T_1, \ldots, T_n)$ to any finite-dimensional subspace $X_0(\subseteqq X)$ is equal to the full algebra $B^{(n)}(X_0)$.

Preliminaries

Lemma 1. *Let n be a fixed positive integer. For $1 \leqq i \leqq n$ and $m \geqq 1$, let $f_m^{(i)}$, $f^{(i)}$ be vectors in X such that $f_m^{(i)} \to f^{(i)}$ $(m \to \infty)$. Let $g_m = \alpha_m^{(1)} f_m^{(1)} + \ldots + \alpha_m^{(n)} f_m^{(n)}$, with $\alpha_m^{(i)} \in \mathbf{K}$. If $f^{(1)}, \ldots, f^{(n)}$ are linearly independent and if the sequence $\{g_m\}_{m=1}^{\infty}$ converges, then there are scalars $\alpha^{(1)}, \ldots, \alpha^{(n)}$ such that $\alpha_m^{(i)} \to \alpha^{(i)}$ $(m \to \infty)$ for $i = 1, \ldots, n$.*

Proof. If $n=1$ we choose a bounded linear functional Φ on X such that $\Phi(f_1)=1$, then $g_m = \alpha_m^{(1)} f_m^{(1)}$ implies that $\lim_{m\to\infty} \alpha_m^{(1)} = \lim_{m\to\infty} \Phi(g_m)$. Now we assume that $n \geqq 2$. The next step is to show that $\{|\alpha_m^{(1)}|\}_{m=1}^{\infty}$ cannot converge to infinity. Indeed, if $|\alpha_m^{(1)}| \to \infty$ $(m \to \infty)$, then the left hand side of

$$\frac{g_m}{\alpha_m^{(1)}} - f_m^{(1)} = \frac{\alpha_m^{(2)}}{\alpha_m^{(1)}} f_m^{(2)} + \ldots + \frac{\alpha_m^{(n)}}{\alpha_m^{(1)}} f_m^{(n)}$$

converges to $-f^{(1)}$ and then the induction hypothesis can be applied to $f_m^{(2)}, \ldots, f_m^{(n)}$ to conclude that there are scalars $\beta^{(2)}, \ldots, \beta^{(n)}$ such that $-f^{(1)} = \beta^{(2)} f^{(2)} + \ldots + \beta^{(n)} f^{(n)}$. This contradicts the fact that $f^{(1)}, \ldots, f^{(n)}$ are linearly independent. The same reasoning applies to any subsequence of $\{|\alpha_m^{(1)}|\}_{m=1}^{\infty}$, therefore $\{\alpha_m^{(1)}\}_{m=1}^{\infty}$ is bounded. Next, let $\{m_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers such that $\alpha_{m_k}^{(1)} \to \alpha^{(1)}$ $(k \to \infty)$ for some scalar $\alpha^{(1)}$. Then from the induction hypothesis it follows that there are scalars $\alpha^{(2)}, \ldots, \alpha^{(n)}$ such that $\alpha_{m_k}^{(i)} \to \alpha^{(i)}$ $(k \to \infty)$ for $i = 1, \ldots, n$. Since $f^{(1)}, \ldots, f^{(n)}$ are linearly independent, the scalars $\alpha^{(1)}, \ldots, \alpha^{(n)}$ are independent of the sequence $\{m_k\}_{k=1}^{\infty}$. Then it follows that $\alpha_m^{(i)} \to \alpha^{(i)}$ $(m \to \infty)$ for $i = 1, \ldots, n$.

Lemma 2. *Let $T_1, T_2, \ldots, T_n \in B(X)$. Assume that for every vector f in X the set $\{T_1 f, T_2 f, \ldots, T_n f\}$ is linearly dependent. Then there is a nonzero n-tuple $(\alpha_1, \ldots, \alpha_n)$ in $\mathbf{K}^n$ such that $\alpha_1 T_1 + \ldots + \alpha_n T_n$ has rank less than or equal to $n-1$.*

Proof. If $n=1$ then the hypothesis reduces to $T_1 f = 0$ for all f in X, and the conclusion holds. Assume that $n \geqq 2$. Let D be the set of all vectors f in X such that $\{T_1 f, \ldots, T_{n-1} f\}$ is linearly dependent. If $D = X$ then the conclusion follows by induction. Assume that $D \neq X$. An easy compactness argument in $\mathbf{K}^n$ implies that D is a closed set. For every vector h in $X \setminus D$ (the complement of D) the set $\{T_1 h, \ldots, T_{n-1} h\}$ is linearly independent; then from the linear dependence of $\{T_1 h, \ldots, T_{n-1} h, T_n h\}$ it follows that there are functions $\alpha_1, \ldots, \alpha_{n-1}$ from $X \setminus D$

to **K** such that

(1) $$\alpha_1(h)T_1h+\dots+\alpha_{n-1}(h)T_{n-1}h+T_nh=0 \quad \text{for all } h \text{ in} X\setminus D.$$

Let f be a fixed vector in $X\setminus D$, and let M be the subspace spanned by $\{T_1f, \dots, T_{n-1}f\}$. The proof will be completed by showing that the range of $\alpha_1(f)T_1+\dots+\alpha_{n-1}(f)T_{n-1}+T_n$ is contained in M. Let g be an arbitrary vector in X. Since $X\setminus D$ is open, there is a positive δ such that $f+\lambda g\in X\setminus D$ for $|\lambda|<\delta$. If $|\lambda|<\delta$, from (1) we obtain

(2) $$\alpha_1(f+\lambda g)T_1(f+\lambda g)+\dots+\alpha_{n-1}(f+\lambda g)T_{n-1}(f+\lambda g)+T_n(f+\lambda g)=0,$$

and (with $\lambda=0$)

(3) $$\alpha_1(f)T_1f+\dots+\alpha_{n-1}(f)T_{n-1}f+T_nf=0.$$

Subtracting (3) from (2) we get

$$\lambda[\alpha_1(f+\lambda g)T_1g+\dots+\alpha_{n-1}(f+\lambda g)T_{n-1}g+T_ng]=$$
$$=[\alpha_1(f)-\alpha_1(f+\lambda g)]T_1f+\dots+[\alpha_{n-1}(f)-\alpha_{n-1}(f+\lambda g)]T_{n-1}f$$

which implies that

(4) $$\alpha_1(f+\lambda g)T_1g+\dots+\alpha_{n-1}(f+\lambda g)T_{n-1}g+T_ng\in M \quad \text{for} \quad 0<|\lambda|<\delta.$$

Let $\{\lambda_m\}_{m=1}^{\infty}$ be a sequence of scalars such that $\lambda_m\to 0$ $(m\to\infty)$. If we define $f_m^{(i)}=$ $=T_i(f+\lambda_m g)$ $(1\leqq i\leqq n-1)$, then $f_m^{(i)}\to T_if$ $(m\to\infty)$, and $T_1f, \dots, T_{n-1}f$ are linearly independent. Then, using (2), we can apply Lemma 1, with $g_m=-T_n(f+\lambda_m g)$, to conclude that $\alpha_i(f+\lambda_m g)\to\alpha^{(i)}$ $(m\to\infty)$ for $i=1, \dots, n-1$. Then, from (2) again, $\alpha^{(1)}T_1f+\dots+\alpha^{(n-1)}T_{n-1}f+T_nf=0$, and comparing with (3) it follows that $\alpha^{(i)}=\alpha_i(f)$ for $i=1, \dots, \dots, n-1$. This shows that the functions $\lambda\to\alpha_i(f+\lambda g)(|\lambda|<\delta)$ are continuous at $\lambda=0$ in every direction. Since M is a closed subspace, from (4) we conclude that $\alpha_1(f)T_1g+\dots+\alpha_{n-1}(f)T_{n-1}g+T_ng\in M$. Since g is an arbitrary vector, then the range of $\alpha_1(f)T_1+\dots+\alpha_{n-1}(f)T_{n-1}+T_n$ is contained in M.

Lemma 3. *Let $T_1, T_2, \dots, T_n\in B(X)$. Assume that every nontrivial linear combination of $T_1, \dots, T_n$ has infinite rank. Then given a positive integer m there are vectors $f_1, \dots, f_m$ in X such that $\{T_if_j\colon 1\leqq i\leqq n, 1\leqq j\leqq m\}$ is a linearly independent set.*

Proof. If $f_1, \dots, f_m$ are vectors in X then we denote by $L(f_1, \dots, f_m)$ the set $\{T_if_j\colon 1\leqq i\leqq n, 1\leqq j\leqq m\}$. If $m=1$, then what is wanted is a vector f in X such that $T_1f, \dots, T_nf$ are linearly independent. If this is not true then Lemma 2 implies that some nontrivial linear combination of $T_1, \dots, T_n$ has finite rank. Since this contradicts the hypothesis, the lemma holds for $m=1$. Now we assume that $L(f_1, \dots, f_m)$ is a linearly independent set for some vectors $f_1, \dots, f_m$. Let M be the subspace spanned by $L(f_1, \dots, f_m)$ and let N be a closed subspace which is a complement of

M (i.e., $X=M+N$ and $M\cap N=(0)$). Let P be the idempotent in $B(X)$ with range N and null space M. Since $T_i=(I-P)T_i+PT_i(I-P)+PT_iP$, and since $I-P$ has finite rank, then every nontrivial linear combination of $PT_1P, \dots, PT_nP$ has infinite rank. Now from the first part of the proof it follows that there is a vector g in N such that $PT_1g, \dots, PT_ng$ are linearly independent. If we define $f_{m+1}=g$, then $L(f_1, \dots, f_m, f_{m+1})$ is linearly independent. Indeed, if $\sum_{i=1}^{n}\sum_{j=1}^{m+1} \alpha_{ij}T_if_j=0$, and since P annihilates $L(f_1, \dots, f_m)$, it follows that $\sum_{i=1}^{n} \alpha_{i,m+1}PT_ig=0$, and therefore $\alpha_{i,m+1}=0$ for $i=1, \dots, n$; finally, since $L(f_1, \dots, f_m)$ is linearly independent we conclude that $\alpha_{ij}=0$ for all i and j.

Density

Theorem 4. *Let $T_1, T_2, \dots, T_n \in B(X)$. Assume that every nontrivial linear combination of $I, T_1, \dots, T_n$ has infinite rank. Then the similarity orbit $S(T_1, \dots, T_n)$ is strongly dense in $B^{(n)}(X)$.*

Proof. Let $\tilde{S}=(S_1, \dots, S_n)\in B^{(n)}(X)$ and let U be a strong neighborhood of $\tilde{S}$. Then there are linearly independent vectors $e_1, \dots, e_m$ in X and a positive number ε such that U contains

$$\{(A_1, \dots, A_n)\in B^{(n)}(X)\colon \|(A_i-S_i)e_j\| < \varepsilon,\ 1 \leqq i \leqq n,\ 1 \leqq j \leqq m\}.$$

Let M be the span of $\{e_1, \dots, e_m\}$. Let N be a complement of the subspace $M+S_1M+\dots+S_nM$. Since N is infinite-dimensional, we can choose in N a set $\{h_{ij}\colon 1\leqq i\leqq n, 1\leqq j\leqq m\}$ of linearly independent vectors such that $\|h_{ij}\|<\varepsilon$ for all i, j. Let $f_{ij}=S_ie_j+h_{ij}$. Then the set $\{e_i, f_{ij}\colon 1\leqq i\leqq n, 1\leqq j\leqq m\}$ is linearly independent and $\|S_ie_j-f_{ij}\|<\varepsilon$ for all i and j. We apply Lemma 3 to $I, T_1, \dots, T_n$ to find vectors $f_1, \dots, f_m$ in X such that $\{f_j, T_if_j\colon 1\leqq i\leqq n, 1\leqq j\leqq m\}$ is a linearly independent set. If A is an invertible operator on X such that $Ae_j=f_j$ and $Af_{ij}=T_if_j$ for $1\leqq i\leqq n$ and $1\leqq j\leqq m$, then

$$\|(A^{-1}T_iA-S_i)e_j\| = \|A^{-1}T_if_j-S_ie_j\| = \|A^{-1}Af_{ij}-S_ie_j\| = \|f_{ij}-S_ie_j\| < \varepsilon$$

for all i and j. Therefore $(A^{-1}T_1A, \dots, A^{-1}T_nA)\in U$, and $S(T_1, \dots, T_n)$ is strongly dense in $B^{(n)}(X)$.

Theorem 5. *Let $T_1, T_2, \dots, T_n\in B(X)$. Assume that every nontrivial linear combination of $I, T_1, \dots, T_n$ has infinite rank. Then the compression of $S(T_1, \dots, T_n)$ to a given finite-dimensional subspace M is equal to $B^{(n)}(M)$. More precisely, if P is an idempotent in $B(X)$ with range M, then the restriction of $PS(T_1, \dots, T_n)P$ to M is $B^{(n)}(M)$.*

Proof. Let P be a fixed idempotent in $B(X)$ with range M. Let $(F_1, \ldots, F_n)$ be arbitrary in $B^{(n)}(M)$. Let $T_0=I$ and $m=\dim M$. By Lemma 3 there are vectors $f_1, \ldots, f_m$ such that $\{T_i f_j: 0 \leqq i \leqq n, 1 \leqq j \leqq m\}$ is a linearly independent set. For $0 \leqq i \leqq n$ let N_i be the subspace spanned by $\{T_i f_1, \ldots, T_i f_m\}$. We choose linearly independent subspaces $M_0, M_1, \ldots, M_n$ (i.e., $g_i \in M_i$ and $g_0+g_1+\ldots+g_n=0$ imply that $g_i=0$ for all i) satisfying the following conditions: $M_0=M$, $M_i \subset \ker P$ for $1 \leqq i \leqq n$, and $\dim M_i=m$ for all i. Let $B \in B(X)$ be an invertible operator such that $BM_i=N_i$ for $0 \leqq i \leqq n$. Let $S_i=B^{-1}T_iB$ $(1 \leqq i \leqq n)$. Then

$$BS_i(M) = T_iBM_0 = T_iN_0 = N_i = BM_i$$

and therefore $S_iM=M_i$. In particular, S_i is injective on M, and we can find $C_i \in B(M_i, M)$ such that $C_iS_if=-F_if$ for all f in M. Let M_{n+1} be a subspace of $\ker P$ which is a complement (in $\ker P$) of the subspace $M_1+M_2+\ldots+M_n$. Then $X=M_0+M_1+\ldots+M_{n+1}$, and we use this decomposition of X to define the operator C on X given by the $(n+2)\times(n+2)$ operator matrix,

$$C = \begin{bmatrix} I & C_1 & C_2 \ldots C_n & 0 \\ 0 & I & 0 \ldots 0 & 0 \\ & & \ddots \quad \vdots & \vdots \\ & 0 & I & 0 \\ & & 0 & I \end{bmatrix}.$$

Then C is invertible, and C^{-1} is the operator matrix whose first row is $[I, -C_1, -C_2, \ldots, -C_n, 0]$, and the other rows are identical to the corresponding rows of C. Now for $f \in M$ and $1 \leqq i \leqq n$ we have (denoting the $(i+1)$-th component of the vector f by S_if)

$$C^{-1}S_iCf = C^{-1}S_iC\langle f, 0, \ldots, 0\rangle = C^{-1}S_i\langle f, 0, \ldots, 0\rangle =$$
$$= C^{-1}\langle 0, \ldots, 0, S_if, 0, \ldots, 0\rangle = \langle -C_iS_if, *, \ldots, *\rangle$$

(the third equality follows from $S_iM=M_i$), and therefore $PC^{-1}S_iCf=-C_iS_if=$ $=F_if$. Finally, with $A=BC$, the restriction of $PA^{-1}T_iA$ to M is F_i for $i=1, \ldots, n$.

Corollary 6. *Let $T_1, T_2, \ldots, T_n \in B(X)$. The following statements are equivalent:*

(1) *$S(T_1, \ldots, T_n)$ is strongly dense in $B^{(n)}(X)$.*

(2) *$S(T_1, \ldots, T_n)$ is weakly dense in $B^{(n)}(X)$.*

(3) *Every nontrivial linear combination of $I, T_1, \ldots, T_n$ has infinite rank.*

(4) *For every finite-dimensional subspace M of X the compression of $S(T_1, \ldots, T_n)$ to M is equal to $B^{(n)}(M)$.*

Proof. Since the strong topology is finer than the weak topology, then (1) implies (2). Next we assume that some linear combination $\alpha_0I+\alpha_1T_1+\ldots+\alpha_nT_n=F$

has finite rank and $(\alpha_0, \alpha_1, \ldots, \alpha_n) \neq 0$. Let $(S_1, \ldots, S_n) \in S(T_1, \ldots, T_n)$. Then there is an invertible operator A on X such that $S_i = A^{-1}T_iA$ for $1 \leqq i \leqq n$. Therefore $\alpha_0 I + \alpha_1 S_1 + \ldots + \alpha_n S_n = A^{-1}FA$ and $\operatorname{rank}(\alpha_0 I + \alpha_1 S_1 + \ldots + \alpha_n S_n) = \operatorname{rank} F < \infty$. Since the set $\{S \in B(X) \colon \operatorname{rank} S \leqq \operatorname{rank} F\}$ is weakly closed, it follows that the weak closure of $S(T_1, \ldots, T_n)$ is contained in the set

$$\{(S_1, \ldots, S_n) \in B^{(n)}(X) \colon \operatorname{rank}(\alpha_0 I + \alpha_1 S_1 + \ldots + \alpha_n S_n) \leqq \operatorname{rank} F\},$$

and this set is smaller than $B^{(n)}(X)$. Hence (2) implies (3). Now by Theorem 4 we conclude that (1), (2), and (3) are equivalent. By Theorem 5, (3) implies (4). Now we assume that (4) holds. Let $(\alpha_0, \alpha_1, \ldots, \alpha_n) \neq 0$. Let M be an arbitrary finite-dimensional subspace of X. Choose $(F_1, \ldots, F_n)$ in $B^{(n)}(M)$ such that $\alpha_0 I + \alpha_1 F_1 + \ldots + \alpha_n F_n = I$ (the identity on M). By (4), there is an invertible operator A on X such that the compression of $A^{-1}T_iA$ to M is F_i $(1 \leqq i \leqq n)$. Then

$$\operatorname{rank}(\alpha_0 I + \alpha_1 T_1 + \ldots + \alpha_n T_n) = \operatorname{rank} A^{-1}(\alpha_0 I + \alpha_1 T_1 + \ldots + \alpha_n T_n)A \geqq$$
$$\geqq \operatorname{rank}(\alpha_0 I + \alpha_1 F_1 + \ldots + \alpha_n F_n) = \dim M.$$

Since M is arbitrary, we conclude that $\alpha_0 I + \alpha_1 T_1 + \ldots + \alpha_n T_n$ has infinite rank. This shows that (4) implies (3).

References

[1] J. Barría and P. R. Halmos, Weakly transitive matrices, *Illinois J. Math.*, **28** (1984), 370—378.

[2] D. W. Hadwin, E. A. Nordgren, H. Radjavi and P. Rosenthal, Most similarity orbits are strongly dense, *Proc. Amer. Math. Soc.*, 76 (1979), 250—252.

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