

## Linear operators with a normal factorization through Hilbert space

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**Introduction.** Let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space, and suppose that  $K(x, t)$  is a kernel on  $\Omega \times \Omega$  which is selfadjoint, that is,  $K(x, t) = \overline{K(t, x)}$  a.e. on  $\Omega \times \Omega$ . Let  $X$  be some Banach space of functions on  $\Omega$ , and assume that the integral operator

$$S(f)(x) = \int_{\Omega} K(x, t)f(t) d\mu(t) \quad (f \in X)$$

is a bounded linear operator on  $X$ . When  $X = L^2(\Omega)$  and the kernel  $|K|$  determines a bounded linear operator on  $X$ , then  $S$  is a selfadjoint operator on  $X$ . However, in general, the operator  $S$  may not have properties analogous to those of a selfadjoint operator. The purpose of this paper is to study a large class of operators which in many respects do behave like selfadjoint (or normal) operators. One motivation here is to find conditions under which selfadjoint kernels determine operators which have many of the properties of selfadjoint operators. This question is addressed implicitly in the context of the examples considered in Section 3.

There is a long history of interest in operators on a Banach space that have some properties in common with selfadjoint operators. Examples include symmetrizable operators [10], [11], the quasi-hermitian operators studied by J. DIENDONNÉ [7], and hermitian operators on Banach spaces [5], [6, Part 3]. The class of operators we study has some overlap with these classes. We consider linear operators that have a selfadjoint (or normal) factorization through a Hilbert space in the following sense.

**Definition 0.1.** An operator  $S \in \mathcal{B}(X)$  has a *selfadjoint (normal) factorization through a Hilbert space  $H$* , if there exist bounded linear maps  $A$  and  $T$ ,

$$T: X \rightarrow H, \quad A: H \rightarrow X,$$

with  $S = AT$  and  $TA$  selfadjoint (normal) on  $H$ .

When  $S=AT$  is a factorization of  $S$  with  $TA$  normal, then many properties of  $S$  and  $TA$  are closely linked. In particular, the spectral theory of the two operators is very much the same. For example, using the operational calculus of the normal operator  $TA$ , a rich operational calculus may be defined for  $S$ . This is done in Section 2. There is a large collection of examples in Section 3 which makes it clear that the theory has broad application.

Now we establish some notation. Throughout  $X$  is a Banach space and  $H$  is a Hilbert space. The algebra of all bounded linear operators on  $X$  is denoted  $\mathcal{B}(X)$ . For  $S \in \mathcal{B}(X)$ , let  $\sigma(S)$  be the spectrum of  $S$ . If  $T$  is a linear map, then let  $\mathfrak{N}(T)$  be the null space of  $T$ , and let  $\mathfrak{R}(T)$  be the range of  $T$ .

**1. Some preliminaries.** In this section we derive some preliminary results concerning factorizations. We assume throughout that  $S \in \mathcal{B}(X)$  has a factorization  $S=AT$  where  $T: X \rightarrow H$ ,  $A: H \rightarrow X$  and  $TA$  is normal on  $H$ .

**Definition 1.1.** Let  $E_0$  be the selfadjoint projection in  $\mathcal{B}(H)$  with range  $\mathfrak{R}(TA)$ . Set  $N=AE_0T$ . Then  $N$  is called the nilpotent part of  $S$ . Note that  $NA=AE_0TA=0$ , and  $SN=NS=0$ .

**Proposition 1.2.** Let  $E_0$  and  $N$  be as above and set  $\tilde{S}=S-N$ . Then  $\tilde{S}$  has a normal factorization  $\tilde{S}=\tilde{A}\tilde{T}$  through a Hilbert space  $\tilde{H}$  with the property that  $\mathfrak{N}(\tilde{T}\tilde{A})=\{0\}$ .

*Proof.* Set  $\tilde{H}=(I-E_0)H$ , and define  $\tilde{T}: X \rightarrow \tilde{H}$  by  $\tilde{T}(x)=(I-E_0)Tx$  and  $\tilde{A}: \tilde{H} \rightarrow X$  by  $\tilde{A}(y)=Ay$ . For  $x \in X$ ,  $\tilde{A}\tilde{T}(x)=A(I-E_0)Tx=ATx-AE_0Tx=Sx-Nx=\tilde{S}(x)$ . For  $y \in \tilde{H}$ ,  $\tilde{T}\tilde{A}y=(I-E_0)TAy=TAy$ . Since  $TA$  restricted to  $(I-E_0)H$  is normal, we have  $\tilde{T}\tilde{A}$  is normal on  $\tilde{H}$ .

Next we verify that  $\mathfrak{N}(\tilde{T}\tilde{A})=\{0\}$ . Assume  $y \in \tilde{H}$  and  $\tilde{T}\tilde{A}y=0$ . From the previous computation, we have  $TAy=\tilde{T}\tilde{A}y=0$ . Then by definition,  $E_0y=y$ , so  $y=(I-E_0)y=0$ .

Let  $\tilde{S}=S-N$  as in Proposition 1.2. Then the spectral theory of  $\tilde{S}$  is essentially the same as that of  $S$ . Now by Proposition 1.2  $\tilde{S}$  has a normal factorization with the property that  $\mathfrak{N}(\tilde{T}\tilde{A})=\{0\}$ . This means that from the point of view of spectral theory, we may make the following assumption without loss of generality.

$$(A1) \quad \mathfrak{N}(TA) = \{0\}.$$

An operator  $R \in \mathcal{B}(X)$  is similar to a normal operator  $W \in \mathcal{B}(H)$  if  $\exists U: X \rightarrow H$  such that  $U$  is a bicontinuous linear isomorphism of  $X$  onto  $H$  with  $R=U^{-1}WU$ . In this situation  $X$  is a Hilbert space in an equivalent renorming, and the spectral theory of  $R$  is completely determined by that of the normal operator  $W$ .

**Proposition 1.3.** *If  $TA$  is invertible and  $\mathfrak{N}(S)=\{0\}$  then  $S$  is invertible. When (A1) holds and  $S$  is invertible, then  $TA$  is invertible. Furthermore, in this case  $S$  is similar to the normal operator  $TA$ .*

*Proof.* Assume (A1) holds and  $S$  is invertible. The  $\mathfrak{R}(A)=X$  and  $\mathfrak{N}(T)=\{0\}$ . Also, since  $\mathfrak{N}(TA)=\{0\}$ ,  $\mathfrak{N}(A)=\{0\}$ . We verify that  $\mathfrak{R}(T)=H$ . For suppose  $y \in H$ . We have  $A(\mathfrak{R}(T))=X$ , so  $\exists z \in X$  with  $ATz=Ay$ . Then  $A(Tz-y)=0$ , so  $Tz=y$ . This proves that both  $A$  and  $T$  are one-to-one and onto maps. Thus,  $TA$  is invertible with  $(TA)^{-1}=A^{-1}T^{-1}$ . Also, in this case, setting  $U=T$ ,  $S=AT=U^{-1}(TA)U$ .

The proof that when  $TA$  is invertible and  $\mathfrak{N}(S)=\{0\}$ , then  $S$  is invertible, is similar to the proof above.

Suppose  $S=AT$  with  $TA$  invertible, but  $\mathfrak{N}(S) \neq \{0\}$ . We show that in this case  $S$  is the direct sum of the zero operator and an operator which is similar to a normal operator. Let  $R=(TA)^{-1}$ , and let  $P=ART$ . Elementary computations show that  $P^2=P$  and  $SP=PS=S$ . It follows that  $S(I-P)(X)=\{0\}$ . Also, if  $Sx=ATx=0$ , then since  $\mathfrak{N}(A)=\{0\}$ ,  $Tx=0$ , and thus  $Px=0$ . This implies that  $\mathfrak{N}(S)=(I-P)X$ . Therefore  $X=P(X) \oplus \mathfrak{N}(S)$ , and  $S=SP \oplus 0$ . Define  $U: P(X) \rightarrow H$  by  $UPx=TPx=Tx$ . Since  $T(X)=H$ ,  $U$  is onto, and when  $Px \in \mathfrak{N}(U)$ , then  $TPx=0$ , so  $SPx=0$ , and finally,  $Px=0$ . Therefore  $U$  has a bounded inverse. An easy computation shows  $SP=U^{-1}TAU$  on  $P(X)$ . Therefore  $S$  is the direct sum of an operator similar to a normal operator ( $SP$  on  $P(X)$  and  $0$  on  $(I-P)(X)$ ). In this case the spectral theory of  $S$  is easily derived from that of  $TA$ . Thus, in studying the spectral theory of  $S$ , we can make the following assumption without loss:

(A2)  $TA$  is not invertible.

Note that when (A1) and (A2) hold then Proposition 1.3 implies that  $S$  is not invertible.

**2. Spectral theory.** *Throughout this section it is assumed that  $S$  has a normal factorization,  $S=AT$  with  $TA$  normal. Most of the properties of normal operators used in this paper can be found in M. SCHECHTER's book [13].*

**Theorem 2.1.**

- (1)  $\sigma(S) \cup \{0\} = \sigma(TA) \cup \{0\}$ . When (A1) and (A2) hold, then  $0 \in \sigma(S) = \sigma(TA)$ .
- (2) If  $\lambda \neq 0$  with  $\lambda \notin \sigma(TA)$ , then

$$(\lambda - S)^{-1} = \lambda^{-1} + \lambda^{-1}A(\lambda - TA)^{-1}T.$$

*Proof.* Assume  $\lambda \notin \sigma(TA)$ ,  $\lambda \neq 0$ . The formula in (2) is verified by direct computation:

$$\begin{aligned} & (\lambda - AT)\{\lambda^{-1} + \lambda^{-1}A(\lambda - TA)^{-1}T\} = \\ & = I + A(\lambda - TA)^{-1}T - \lambda^{-1}AT - \lambda^{-1}ATA(\lambda - TA)^{-1}T. \end{aligned}$$

Now the last term

$$-\lambda^{-1}ATA(\lambda-TA)^{-1}T = \lambda^{-1}A(\lambda-TA)(\lambda-TA)^{-1}T - A(\lambda-TA)^{-1}T,$$

and substituting the expression on the right for this term yields the result. This proves (2).

To prove (1) note that the same computation that establishes (2) shows that when  $\lambda \neq 0$  is in the resolvent of  $S$ , then

$$(\lambda-TA)^{-1} = \lambda^{-1} + \lambda^{-1}T(\lambda-S)^{-1}A.$$

Now (1) follows from (2) and the remark following the statement of (A2).

**Corollary 2.2.** *Assume  $S \in \mathcal{B}(X)$  has a normal factorization through Hilbert space. Then  $\exists M > 0$  such that when  $\lambda \notin \sigma(S)$ ,  $\lambda \neq 0$ ,*

$$\|(\lambda-S)^{-1}\| \leq M|\lambda|^{-1}(1+d(\lambda)^{-1})$$

where  $d(\lambda) = \inf \{|\lambda-\mu| : \mu \in \sigma(S)\}$ .

Assume  $\Delta$  is a compact subset of  $\mathbb{C}$ . Let  $\text{BM}(\Delta)$  be the algebra of all bounded Borel measurable functions on  $\Delta$ . Define  $\mathfrak{M}(\Delta)$  to be the set of all  $f \in \text{BM}(\Delta)$  such that  $\exists g \in \text{BM}(\Delta)$  with  $f(\lambda) = \lambda g(\lambda)$  for all  $\lambda \in \Delta$ . Now assume that (A1) and (A2) hold. Set  $\Delta = \sigma(S) = \sigma(TA)$ . Using the fact that the normal operator  $TA$  has an operational calculus  $g \rightarrow g(TA)$  for all  $g \in \text{BM}(\Delta)$ , we construct an operational calculus  $f \rightarrow f(S)$  for functions  $f \in \mathfrak{M}(\Delta)$ .

**Definition 2.3.** For  $f \in \mathfrak{M}(\Delta)$  with  $f(\lambda) = \lambda g(\lambda)$  for all  $\lambda \in \Delta$ , and where  $g \in \text{BM}(\Delta)$ , define

$$f(S) = Ag(TA)T.$$

By assumption (A2),  $0 \in \Delta$ . This means that  $g(0)$  is not uniquely determined by the requirement  $f(\lambda) = \lambda g(\lambda)$  on  $\Delta$ . Nevertheless,  $f(S)$  is well-defined. To check this it suffices to show that when  $e(\lambda) = 0$ ,  $\lambda \in \Delta \setminus \{0\}$ , and  $e(0) = 1$ , then  $e(TA) = 0$ . Since  $\lambda e(\lambda) = 0$  for all  $\lambda \in \Delta$ , we have  $TAe(TA) = 0$ . Then by (A1),  $e(TA) = 0$ .

**Theorem 2.4.** *Assume (A1) and (A2) hold. Let  $\Delta = \sigma(S) = \sigma(TA)$ .*

(1) *The operational calculus  $f \rightarrow f(S)$  is an algebra homomorphism of  $\mathfrak{M}(\Delta)$  into  $\mathcal{B}(X)$ .*

(2) [The Spectral Mapping Theorem.] *For  $f \in \mathfrak{M}(\Delta)$*

$$\sigma(f(S)) = \sigma(f(TA)).$$

*In particular, if  $f \in \mathfrak{M}(\Delta)$  and  $f$  is continuous on  $\Delta$ , then*

$$\sigma(f(S)) = \{f(\lambda) : \lambda \in \Delta\}.$$

(3) Assume  $\{f_n\}$  is a sequence in  $\mathfrak{M}(\Delta)$  with  $f_n(\lambda) = \lambda g_n(\lambda)$ ,  $\{g_n\} \subseteq \text{BM}(\Delta)$  and  $g_n \rightarrow g$  uniformly on  $\Delta$ . Then  $f_n(S) \rightarrow f(S)$  in  $\mathcal{B}(X)$ .

(4) Assume either  $\mathfrak{N}(S) = \{0\}$  or  $\mathcal{R}(S)$  is dense in  $X$ . If  $P \in \mathcal{B}(X)$  with  $PS = SP$ , then for every  $f \in \mathfrak{M}(\Delta)$ ,  $Pf(S) = f(S)P$ .

Proof. Part (1) follows from the fact that  $g \rightarrow g(TA)$  is an algebra homomorphism of  $\text{BM}(\Delta)$  into  $\mathcal{B}(H)$ . We check the property that when  $f_1$  and  $f_2$  are in  $\mathfrak{M}(\Delta)$ , then  $f_1 f_2(S) = f_1(S) f_2(S)$ . Write  $f_k(\lambda) = \lambda g_k(\lambda)$  on  $\Delta$  for  $k=1, 2$ . Then  $f_1 f_2(\lambda) = \lambda g(\lambda)$  on  $\Delta$  where  $g(\lambda) = g_1(\lambda) \lambda g_2(\lambda)$ . Therefore  $f_1 f_2(S) = Ag(TA)T = Ag_1(TA)TAg_2(TA)T = f_1(S)f_2(S)$ .

To prove (2), note that  $f(S)$  factors through  $H$  where the factors are  $T: X \rightarrow H$  and  $Ag(TA): H \rightarrow X$ . We have  $f(S) = (Ag(TA))T$  and  $T(Ag(TA)) = f(TA)$ . Therefore Theorem 2.1 implies that the nonzero spectrum of  $f(S)$  and  $f(TA)$  is the same. But also, by (A2)  $TA$  is not invertible, so  $f(TA) = TAg(TA)$  is not invertible. By Proposition 1.3 it follows that  $f(S)$  is not invertible. This proves  $0 \in \sigma(f(TA))$  and  $0 \in \sigma(f(S))$ .

The proof of (3) is elementary. Assuming the hypothesis in (3), it follows  $g_n(TA) \rightarrow g(TA)$  in  $\mathcal{B}(H)$ . Therefore  $f_n(S) = Ag_n(TA)T \rightarrow Ag(TA)T = f(S)$  in  $\mathcal{B}(X)$ .

Now assume  $P \in \mathcal{B}(X)$  and  $P(AT) = (AT)P$ . Then  $(TPA)(TA) = (TA)(TPA)$ . Assume  $f \in \mathfrak{M}(\Delta)$  with  $f(\lambda) = \lambda g(\lambda)$  on  $\Delta$ . Then

$$(1) \quad (TPA)g(TA) = g(TA)(TPA).$$

Applying the operator  $T$  on the right to the equality in (1), we have

$$TPAg(TA)T = g(TA)TPAT = g(TA)TATP = TAg(TA)TP.$$

When  $\mathfrak{N}(S) = \{0\}$ , then  $\mathfrak{N}(T) = \{0\}$ . Thus,

$$P(Ag(TA)T) = (Ag(TA)T)P$$

which proves (4) in this case. When  $\mathcal{R}(S)$  is dense, apply  $A$  on the left in equality (1), make a computation analogous to the one above, and use the fact that  $\mathcal{R}(A)$  must be dense to arrive at the conclusion.

Corollary 2.5 Assume that  $S \in \mathcal{B}(X)$  has a selfadjoint factorization through Hilbert space and (A1) and (A2) hold. Then  $\exists M > 0$  such that for all  $t \in \mathbf{R}$

$$\|e^{itS}\| \leq M|t|.$$

Therefore, if  $f(t)$  and  $tf(t)$  are in  $L^1(\mathbf{R})$ , then  $\int_{-\infty}^{+\infty} f(t)e^{itS} dt$  converges in  $\mathcal{B}(X)$ .

Proof. Assume  $S = AT$  with  $TA$  selfadjoint.  $\exists J > 0$  such that  $|w^{-1}(e^{iw} - 1)| \leq J$  for all  $w \in \mathbf{R}$ ,  $w \neq 0$ . For  $\lambda \in \mathbf{R}$ ,  $\lambda \neq 0$ , let  $g(\lambda) = \lambda^{-1}(e^{i\lambda t} - 1)$ . Then  $|g(\lambda)| \leq J|t|$

on  $\mathbb{R}$ . Thus

$$\|e^{itS}\| = \|Ag(TA)T\| \cong \|A\| \|T\| |J| |t|.$$

Corollary 2.5 shows that when  $S$  has a selfadjoint factorization through Hilbert space, then  $S$  is in the class of operators studied in [2].

Corollary 2.6. *Assume (A1) and (A2) hold. Assume  $f \in \mathfrak{M}(\Delta)$  with  $f(\lambda) = \lambda g(\lambda)$   $\lambda \in \Delta$ ,  $g \in \text{BM}(\Delta)$ , where in addition  $\lim_{\lambda \rightarrow 0} g(\lambda) = g(0) = 0$ . Then  $\exists \{f_n\}$  a sequence of simple functions in  $\mathfrak{M}(\Delta)$  with  $f_n(S) \rightarrow f(S)$  in  $\mathcal{B}(X)$ . In particular such a sequence exists for  $f(S) = S^2$ .*

Proof. Let

$$\varepsilon_n = \sup \{|g(\lambda)| : \lambda \in \Delta, |\lambda| < n^{-1}\}.$$

Then by hypothesis,  $\varepsilon_n \rightarrow 0$ . Choose  $\{t_n\}$  a sequence of simple functions such that for each  $n \geq 1$ ,

$$|f(\omega) - t_n(\omega)| \leq n^{-2} \quad (\omega \in \Delta).$$

Thus,

$$|(f(\lambda)/\lambda) - (t_n(\lambda)/\lambda)| \leq n^{-1}$$

whenever  $\lambda \in \Delta$  and  $|\lambda| \geq n^{-1}$ . Let  $\chi_n$  be the characteristic function of the  $\{\lambda \in \Delta : |\lambda| \geq n^{-1}\}$ . Define  $f_n$  to be the simple function  $f_n = \chi_n t_n$ ,  $n \geq 1$ . Then

$$|g(\lambda) - (f_n(\lambda)/\lambda)| \leq n^{-1} + \varepsilon_n$$

for all  $\lambda \in \Delta$ . Therefore  $(f_n(\lambda)/\lambda) \rightarrow g(\lambda)$  uniformly on  $\Delta$ , so  $f_n(S) \rightarrow f(S)$  in  $\mathcal{B}(X)$  by Theorem 2.4 (3).

Assume  $S = AT$  with  $TA$  normal, and assume  $0 \in \Delta = \sigma(TA)$ . Let  $U$  be an open set with  $\Delta \subseteq U$  and suppose  $f$  is holomorphic on  $U$  with  $f(0) = 0$ . Then  $g(\lambda) = f(\lambda)/\lambda$  is holomorphic on  $U$  ( $g(0) = f'(0)$ ), thus  $f \in \mathfrak{M}(\Delta)$ . Let  $f(S)$  be the operator in  $\mathcal{B}(X)$  defined by the operational calculus constructed above. Now  $f(S)$  has another meaning defined in terms of the usual holomorphic operational calculus. In fact, in this case the two possible meanings of  $f(S)$  are the same. For let  $\gamma$  be an appropriate curve in  $U$  surrounding  $\Delta$ . Then using Theorem 2.1 we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} f(\lambda)(\lambda - S)^{-1} d\lambda &= \frac{1}{2\pi i} \int_{\gamma} f(\lambda)[\lambda^{-1} + \lambda^{-1} A(\lambda - TA)^{-1} T] d\lambda = \\ &= A \left\{ \frac{1}{2\pi i} \int_{\gamma} g(\lambda)(\lambda - TA)^{-1} d\lambda \right\} T = Ag(TA)T = f(AT). \end{aligned}$$

Here we have used the fact that the operational calculus determined by functions in  $\text{BM}(\Delta)$  and the holomorphic operational calculus coincide for normal operators.

As a consequence of the coincidence of the two operational calculi, it follows that when  $\Gamma$  is an open and closed subset of  $\Delta$  with  $0 \notin \Gamma$ , then the spectral idempotent  $P_\Gamma$  determined by the usual holomorphic operational calculus satisfies  $P_\Gamma = \chi_\Gamma(S)$  where  $\chi_\Gamma$  denotes the characteristic function of  $\Gamma$ .

Next we turn to some results concerning eigenvalues and eigenspaces.

**Proposition 2.7.** *If  $\lambda_0 \in \mathbb{C}$ ,  $\lambda_0 \neq 0$ , then*

$$\mathfrak{N}((\lambda_0 - S)^n) = \mathfrak{N}(\lambda_0 - S) = A\{\mathfrak{N}(\lambda_0 - TA)\}.$$

**Proof.** If  $TAx = \lambda_0 x$ , then  $ATAx = \lambda_0 Ax$ . Thus  $A\{\mathfrak{N}(\lambda_0 - TA)\} \subseteq \mathfrak{N}(\lambda_0 - S)$ . Conversely, if  $ATy = \lambda_0 y$ , then  $TATy = \lambda_0 Ty$ , so  $Ty \in \mathfrak{N}(\lambda_0 - TA)$ . Also  $y = A(\lambda_0^{-1}Ty) \in A\{\mathfrak{N}(\lambda_0 - TA)\}$ . This proves

$$(1) \quad \mathfrak{N}(\lambda_0 - S) = A\{\mathfrak{N}(\lambda_0 - TA)\}.$$

To show  $\mathfrak{N}((\lambda_0 - S)^n) = \mathfrak{N}(\lambda_0 - S)$ , it suffices to prove this for  $n=2$ . Suppose  $x \in \mathfrak{N}((\lambda_0 - S)^2)$ , so  $(\lambda_0 - S)x \in \mathfrak{N}(\lambda_0 - S)$ . By (1),  $\exists y \in \mathfrak{N}(\lambda_0 - TA)$  with  $(\lambda_0 - S)x = Ay$ . Then  $(\lambda_0 - TA)Tx = T(\lambda_0 - AT)x = T Ay = \lambda_0 y$ . Therefore  $(\lambda_0 - TA)^2 Tx = \lambda_0(\lambda_0 - TA)y = 0$ . Since  $TA$  is normal, this implies  $0 = (\lambda_0 - TA)Tx = T(\lambda_0 - AT)x$ . Then as  $(\lambda_0 - S)x \in \mathfrak{N}(\lambda_0 - S)$ , we have  $0 = AT(\lambda_0 - AT)x = \lambda_0(\lambda_0 - AT)x$ . Thus,  $(\lambda_0 - AT)x = 0$ .

**Proposition 2.8.** *Assume (A1) and (A2) hold.*

(1) *If  $\lambda_0 \neq 0$  is an isolated point of  $\sigma(S)$ , then  $\lambda_0$  is an eigenvalue of  $S$ .*

(2) *Assume  $\lambda_0 \neq 0$  is an eigenvalue of  $S$ . Let  $X_0$  be the corresponding eigenspace.*

*Let  $\chi_0$  be the characteristic function of  $\{\lambda_0\}$ , so  $\chi_0 \in \mathfrak{M}(\Delta)$ . Then  $P_0 = \chi_0(S)$  is a projection with  $\mathfrak{R}(P_0) = X_0$  and  $\mathfrak{R}(\lambda_0 - S) \subseteq \mathfrak{N}(P_0)$ .*

**Proof.** Assume  $\lambda_0 \neq 0$  is an isolated point of  $\sigma(S)$ . Then  $\lambda_0$  is an isolated point of  $\sigma(TA)$ , and since  $TA$  is normal, it follows that  $\lambda_0$  is an eigenvalue of  $TA$ . By Proposition 2.7  $\lambda_0$  is an eigenvalue of  $S$ .

Now assume  $\lambda_0 \neq 0$  is an eigenvalue of  $S$ . Let  $X_0, \chi_0$ , and  $P_0$  be as in (2). By Proposition 2.7  $\lambda_0$  is an eigenvalue of  $TA$ . Let  $H_0$  be the corresponding eigenspace. Since  $TA$  is normal,  $Q_0 = \chi_0(TA)$  is the orthogonal projection with  $\mathfrak{R}(Q_0) = H_0$ . Now  $\lambda \chi_0(\lambda) = \lambda_0 \chi_0(\lambda)$  on  $\Delta$ , so  $P_0(\lambda_0 - S) = (\lambda_0 - S)P_0 = 0$ . This proves  $\mathfrak{R}(P_0) \subseteq X_0$  and  $\mathfrak{R}(\lambda_0 - S) \subseteq \mathfrak{N}(P_0)$ . By Proposition 2.7  $AH_0 = X_0$ . We prove that  $AH_0 \subseteq \mathfrak{R}(P_0)$  to complete the proof of (2). Set  $g(\lambda) = \lambda^{-1}\chi_0(\lambda)$ ,  $\lambda \in \Delta$ . Then  $P_0 = Ag(TA)T$ , and for  $x \in H_0$ ,  $P_0 Ax = Ag(TA)TAx = A Q_0 x = Ax$ .

A number  $\lambda \in \mathbb{C}$  is a Fredholm point of  $T \in B(X)$  if  $\lambda - T$  is a Fredholm operator. Let  $\pi_{00}(T)$  denote the set of eigenvalues of  $T$  of finite multiplicity. When  $T$  is a normal operator on Hilbert space, then a Fredholm point of  $T$  with  $\lambda \in \sigma(T)$  is an isolated point of  $\sigma(T)$  and  $\lambda \in \pi_{00}(T)$ .

In the next theorem we prove some results concerning eigenvectors and Fredholm points of  $S$ .

**Theorem 2.9.**

(1) Assume that  $\mathcal{R}(S)$  is dense in  $X$ . For  $\lambda_0 \in \sigma(S)$ ,  $\lambda_0 \neq 0$ , either  $\lambda_0$  is an eigenvalue of  $S$ , or  $\mathcal{R}(\lambda_0 - S)$  is dense in  $X$ . When (A1) and (A2) hold,  $\sigma(S)$  is the union of the point spectrum and continuous spectrum of  $S$ .

(2) Assume  $\lambda \in \mathbb{C}$ ,  $\lambda \neq 0$ . Then  $\lambda$  is a Fredholm point of  $S$  if and only if  $\lambda$  is a Fredholm point of  $TA$ .

(3) When  $\lambda \in \sigma(S)$ ,  $\lambda \neq 0$ , and  $\lambda$  is a Fredholm point of  $S$ , then  $\lambda$  is an isolated point of  $\sigma(S)$  and  $\lambda \in \pi_{00}(S)$ .

(4) If  $\lambda \in \pi_{00}(S)$ ,  $\lambda \neq 0$ , then  $\lambda$  is a Fredholm point of  $S$  and  $(\lambda - S)$  has index zero.

**Proof.** First we prove (1). Assume  $\lambda_0 \neq 0$  and  $\mathfrak{N}(\lambda_0 - S) = \{0\}$ . Then by Proposition 2.7  $\mathfrak{N}(\lambda_0 - TA) = \{0\}$ , and since  $TA$  is normal, we have  $(\lambda_0 - TA)H$  is dense in  $H$ . Since  $\mathcal{R}(S)$  is dense in  $X$ , it follows that  $\mathcal{R}(A)$  is dense in  $X$ . Therefore  $A(\lambda_0 - TA)H = (\lambda_0 - AT)AH$  is dense in  $X$ . Thus,  $\mathcal{R}(\lambda_0 - S)$  is dense in  $X$ .

Now we prove (2). Assume  $\lambda \neq 0$  is a Fredholm point of  $TA$ . Then  $\exists R \in \mathcal{B}(H)$  and  $\exists F, G \in \mathcal{B}(H)$  with  $\mathcal{R}(F)$  and  $\mathcal{R}(G)$  finite dimensional so that

$$(\lambda - TA)R = I - F \quad \text{and} \quad R(\lambda - TA) = I - G.$$

Then

$$\begin{aligned} (\lambda - AT)(\lambda^{-1} + \lambda^{-1}ART) &= I + ART - \lambda^{-1}AT - \lambda^{-1}ATART = \\ &= I - \lambda^{-1}AT + \lambda^{-1}A(\lambda - TA)RT = I - \lambda^{-1}AT + \lambda^{-1}A(I - F)T = I - \lambda^{-1}AFT. \end{aligned}$$

Similarly,

$$(\lambda^{-1} + \lambda^{-1}ART)(\lambda - AT) = I - \lambda^{-1}AGT.$$

Therefore  $\lambda$  is a Fredholm point of  $S = AT$ . The converse is proved in exactly the same way.

Now assume as in (3) that  $\lambda \in \sigma(S)$ ,  $\lambda \neq 0$ , and  $\lambda$  is a Fredholm point of  $S$ . By (2),  $\lambda$  is a Fredholm point of  $TA$ . Since  $TA$  is normal, this implies that  $\lambda$  is an isolated point of  $\sigma(TA)$  and  $\lambda \in \pi_{00}(TA)$ . Then by Theorem 2.1  $\lambda$  is an isolated point of  $\sigma(S)$ . Also, by Proposition 2.7  $\mathfrak{N}(\lambda - S) = A\mathfrak{N}(\lambda - TA)$ . Since  $A$  is one-to-one on  $\mathfrak{N}(\lambda - TA)$ ,  $\mathfrak{N}(\lambda - S)$  has finite dimension. Therefore  $\lambda \in \pi_{00}(S)$ .

Assume  $\lambda \in \pi_{00}(S)$ ,  $\lambda \neq 0$ . Then just as above,  $\lambda \in \pi_{00}(TA)$ . Since  $TA$  is normal  $\lambda$  is a Fredholm point of  $TA$  and an isolated point of  $\sigma(TA)$ . Thus, by part (2),  $\lambda$  is a Fredholm point of  $S$  and an isolated point of  $\sigma(S)$ . It follows that  $\lambda - S$  has index zero [13, VI, Theorem 4.5]. This proves (4).

For an operator  $T \in B(X)$ , let

$$W(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not a Fredholm operator with index zero}\}.$$



The set  $W(T)$  is called the Weyl spectrum of  $T$ . When  $T$  is a normal operator on Hilbert space,

$$W(T) = \sigma(T) \setminus \pi_{00}(T).$$

When this equality holds for some  $T \in \mathcal{B}(X)$ , then one says that Weyl's Theorem holds for  $T$ ; see [3].

Parts (3) and (4) of Theorem 2.9 imply the following corollary.

**Corollary 2.10.** *When  $0 \notin \pi_{00}(S)$ , then  $W(S) = \sigma(S) \setminus \pi_{00}(S)$ . Therefore in this case Weyl's Theorem holds for  $S$ .*

An operator  $T \in \mathcal{B}(X)$  is a Riesz operator if the nonzero spectrum of  $T$  consists of poles of finite rank of the resolvent of  $T$ . This implies that  $\sigma(T)$  is either finite or a sequence converging to zero, and  $\sigma(T) \setminus \{0\} \subseteq \pi_{00}(T)$ . Every compact operator is a Riesz operator.

**Proposition 2.11.** *If  $S$  is a Riesz operator, then  $TA$  is compact and  $S^2$  is compact.*

*Proof.* Assume  $S$  is a Riesz operator. If  $S$  has no nonzero eigenvalue, then  $\sigma(S) = \{0\}$ , which implies  $\sigma(TA) = 0$ . In this case  $TA = 0$  and  $S^2 = A(TA)T = 0$ .

Now assume  $S$  has a nonzero eigenvalue, and let  $\{\lambda_k\}_{k \geq 1}$  be the sequence of distinct nonzero eigenvalues of  $S$  (of course, this set may be finite). For each  $k$  let  $X_k$  be the eigenspace of  $S$  corresponding to the eigenvalue  $\lambda_k$ . Since  $S$  is a Riesz operator,  $\lambda_k \rightarrow 0$  and each  $X_k$  is finite dimensional. By Proposition 2.7  $\lambda_k$  is an eigenvalue of  $TA$  and  $X_k = A\mathfrak{R}(\lambda_k - TA)$ ,  $k \geq 1$ . Clearly  $A$  is one-to-one on  $\mathfrak{R}(\lambda_k - TA)$ , so  $\mathfrak{R}(\lambda_k - TA)$  is finite dimensional. Then as  $TA$  is normal,  $TA$  must be compact. It follows that  $S^2 = A(TA)T$  is compact.

**Theorem 2.12.** *Assume  $S$  is a Riesz operator. Let  $\{\lambda_k\}_{k \geq 1}$  be the sequence of distinct nonzero eigenvalues of  $S$ , and let  $X_k$  be the eigenspace of  $S$  corresponding to the eigenvalue  $\lambda_k$ ,  $k \geq 1$ . Then there exists a sequence of projection operators,  $\{P_k\} \subseteq \mathcal{B}(X)$  with  $P_k P_j = 0$  if  $k \neq j$ ,  $S P_k = P_k S = \lambda_k P_k$ , and  $\mathcal{R}(P_k) = X_k$ ,  $k \geq 1$ , such that for all  $x \in X$ ,*

$$Sx = \sum_{k \geq 1} \lambda_k P_k x + Nx.$$

Here  $N$  is the nilpotent part of  $S$ . Furthermore, for  $n \geq 2$ ,

$$S^n = \sum_{k \geq 1} \lambda_k^n P_k$$

where convergence is in the operator norm.

*Proof.* By Proposition 2.11  $TA$  is compact. Let  $E_k$  be the orthogonal projection with range the eigenspace of  $TA$  corresponding to  $\lambda_k$ . Define  $P_k = \lambda_k^{-1} A E_k T$ ,

$k \geq 1$ . Then

$$P_k P_j = \lambda_k^{-1} \lambda_j^{-1} A E_k T A E_j T = \lambda_k^{-1} A E_k E_j T = \begin{cases} 0 & \text{if } k \neq j; \\ P_k & \text{if } k = j. \end{cases}$$

Also,  $SP_k = \lambda_k^{-1} A T A E_k T = A E_k T = \lambda_k P$ , and similarly,  $P_k S = \lambda_k P$ .

Since  $SP_k = \lambda_k P_k$ , it follows that  $\mathcal{R}(P_k) \subseteq X_k$ . Now by Proposition 2.7  $X_k = A\mathfrak{N}(\lambda_k - TA)$ . If  $x \in X_k$ , then choose  $y$  with  $TAy = \lambda_k y$  and  $x = Ay$ . Then  $P_k x = \lambda_k^{-1} A E_k T x = \lambda_k^{-1} A E_k T A y = A E_k y = Ay = x$ . This proves  $\mathcal{R}(P_k) = X_k$ .

Let  $E_0$  be the orthogonal projection with range  $\mathfrak{N}(TA)$ . Since  $TA$  is normal and compact, for every  $y \in H$  we have

$$y = \sum_{k \geq 1} E_k y + E_0 y.$$

Thus, for  $x \in X$ ,

$$Tx = \sum_{k \geq 1} E_k T x + E_0 T x,$$

and applying  $A$ ,

$$Sx = ATx = \sum_{k \geq 1} A E_k T x + A E_0 T x = \sum_{k \geq 1} \lambda_k P_k x + N x.$$

Finally for  $n \geq 2$ ,  $(TA)^{n-1} = \sum_{k \geq 1} \lambda_k^{n-1} E_k$ , so

$$S^n = A(TA)^{n-1}T = \sum_{k \geq 1} \lambda_k^n P_k.$$

The next result concerns the restriction of  $S$  to a closed  $S$ -invariant subspace of  $X$ . It has application to the situation when  $X = L^\infty(\Omega, \mu)$ , where  $\Omega$  is a locally compact Hausdorff space and  $\mu$  is a regular Borel measure, and  $S \in \mathcal{B}(X)$  leaves invariant the subspace of bounded continuous functions on  $\Omega$ .

**Proposition 2.13.** *Assume  $S = AT$  where  $TA$  is selfadjoint. Assume  $Y$  is a closed  $S$ -invariant subspace of  $X$ . Let  $\tilde{S}$  be the restriction of  $S$  to  $Y$ , so  $\tilde{S} \in \mathcal{B}(Y)$ . Then  $\tilde{S}$  has a selfadjoint factorization through Hilbert space. Furthermore,  $\sigma(\tilde{S}) \subseteq \subseteq \sigma(S) \cup \{0\}$ .*

*Proof.* Let  $\tilde{H}$  be the closure of  $T(Y)$  in  $H$ . Define  $\tilde{T}: Y \rightarrow \tilde{H}$  by  $\tilde{T}(y) = Ty$  for  $y \in Y$ . Define  $\tilde{A}: \tilde{H} \rightarrow Y$ ,  $\tilde{A}(z) = Az$  for  $z \in \tilde{H}$ . Here one notes that  $A(T(Y)) \subseteq Y$ , so  $A(\tilde{H}) \subseteq Y$ . Then  $\tilde{S} = \tilde{A}\tilde{T}$  and  $\tilde{T}\tilde{A}$  is selfadjoint in  $\tilde{H}$ . In fact, since  $A(\tilde{H}) \subseteq Y$ , we have  $TA(\tilde{H}) \subseteq T(Y) \subseteq \tilde{H}$ . This last inclusion shows that  $\tilde{H}$  is  $TA$ -invariant. It follows that  $\sigma(\tilde{T}\tilde{A}) \subseteq \sigma(TA) \cup \{0\}$ . Therefore  $\sigma(\tilde{S}) \subseteq \sigma(S) \cup \{0\}$ .

**3. Examples.** This section is devoted to examples of classes of operators on Banach spaces which have selfadjoint or normal factorizations through a Hilbert space. The specific operators involved are of the type that occur commonly in oper-

ator theory and the applications of operator theory. There are also a few examples of operators which are closely related to selfadjoint operators, but which do not have a selfadjoint factorization on Hilbert space.

Example I. Let  $H$  be a Hilbert space. Assume  $V, W, R \in \mathcal{B}(H)$  with  $R \geq 0$  and  $WV$  selfadjoint. Then  $S = VRW$  has a selfadjoint factorization through  $H$ . For set  $T = R^{1/2}W$  and  $A = VR^{1/2}$ . Then  $A, T \in \mathcal{B}(H)$ ,  $S = AT$  and  $TA = R^{1/2}WVR^{1/2}$  is a selfadjoint operator.

Specific examples of operators  $S$  of the type considered above are well known in operator theory; see [10, p. 345].

Example II. Let  $X$  be a Banach space with a bounded pre-innerproduct  $(x, y)$ ,  $x, y \in X$ . This means that the form  $(x, y)$  has all the properties of an innerproduct except that

$$K = \{x \in X : (x, y) = 0 \text{ for all } y \in X\}$$

may be nonzero. Also, that the form is bounded means  $\exists C > 0$  such that

$$|(x, y)| \leq C \|x\| \|y\| \quad (x, y \in X).$$

The quotient space  $X/K$  has an innerproduct determined in the natural way

$$(x + K, y + K) = (x, y) \quad (x, y \in X).$$

Let  $H$  be the completion of  $X/K$  in the norm determined by the innerproduct. Many authors study operators in  $\mathcal{B}(X)$  which are selfadjoint with respect to a given bounded innerproduct on  $X$ ; see for example [10, Chapter 9]. We consider the case where  $S \in \mathcal{B}(X)$ ,  $S$  has an adjoint  $S^* \in \mathcal{B}(X)$  where  $(Sx, y) = (x, S^*y)$  for all  $x, y \in X$ , and  $\exists J > 0$  with

$$(\#) \quad \|Sx\|_X \leq J(x, x)^{1/2} \quad (x \in X).$$

Using the special assumption  $(\#)$ , we will show that  $S$  has a factorization through  $H$ . Note that  $(\#)$  implies that  $S(K) = \{0\}$ . Then  $S$  determines an operator  $\tilde{S}: X/K \rightarrow X$  in the natural way

$$\tilde{S}(x + K) = Sx \quad (x \in X).$$

$(\#)$  implies that

$$\|\tilde{S}(x + K)\|_X \leq J(x + K, x + K)^{1/2} \quad (x \in X),$$

and it follows that  $\tilde{S}$  has an extension to a bounded linear operator  $A: H \rightarrow X$  with  $A(x + K) = Sx$  for all  $x \in X$ . Let  $T: X \rightarrow H$  be given by  $Tx = x + K$ . The fact that the pre-innerproduct is bounded implies the continuity of  $T$ . Then  $Sx = ATx$  for all  $x \in X$ , and

$$TA(x + K) = Sx + K \quad (x \in X).$$

It follows immediately that when  $S = S^*$ , then  $TA$  is selfadjoint. When  $S$  is normal further argument is necessary. First, let  $W$  be the adjoint of  $TA$  on  $H$ . Note that

$S^*(K) = \{0\}$  since  $(Sx, Sx) = (S^*Sx, x) = (SS^*x, x) = (S^*x, S^*x)$  for all  $x \in X$ . Let  $S^*$  be defined on  $X/K$  in the usual way,  $S^*(x+K) = S^*x + K$ . For  $x, y \in X$ ,  $(x+K, W(y+K)) = (Sx+K, y+K) = (x+K, S^*y+K)$ . Therefore  $W(y+K) = S^*y+K$  for all  $y \in X$ . Thus, for  $y \in X$ ,

$$(TA)W(y+K) = SS^*y+K = S^*Sy+K = W(TA)(y+K).$$

This proves  $TA$  is normal on  $H$ .

Note that in the situation describe above,  $TA$  is the unique extension of  $S$  to an operator on  $H$ . By the theory in Section 2,  $S$  and this extension have essentially the same spectral theory.

Now we consider a specific class of examples where this discussion applies. Let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space, and let  $X = L^2 = L^2(\Omega, \mu)$ . Assume  $\varrho \in L^\infty(\Omega)$  with  $\varrho(t) \geq 0$  a.e. on  $\Omega$ . Define a pre-innerproduct on  $X$  by

$$(f, g)_\varrho = \int_\Omega f(x)\overline{g(x)}\varrho(x) d\mu(x) \quad (f, g \in X).$$

Then

$$|(f, g)_\varrho| \leq \|\varrho\|_\infty \|f\| \|g\| \quad (f, g \in X),$$

so this pre-innerproduct is bounded. Let  $V \in \mathcal{B}(X)$  be selfadjoint, and define  $S \in \mathcal{B}(X)$  by

$$S(f) = V(\varrho f) \quad (f \in X).$$

It is easy to verify that  $S$  satisfies (#): For  $f \in X$ ,  $\|Sf\| = \|V(\varrho f)\| \leq \|V\| \|\varrho f\| \leq \|V\| \|\varrho\|_\infty^{1/2} \left( \int_\Omega |f(x)|^2 \varrho(x) d\mu(x) \right)^{1/2}$ . Also,  $S$  is symmetric with respect to the preinnerproduct:

$$(S(f), g)_\varrho = \int_\Omega V(\varrho f)\overline{g}\varrho d\mu = (V(\varrho f), \varrho g) = (\varrho f, V(\varrho g)) = (f, S(g))_\varrho.$$

In this case  $H = L^2(\varrho)$ , the  $L^2$ -space corresponding to the measure  $\varrho d\mu$ . Then  $S$  has a selfadjoint factorization  $S = AT$  with  $TA$  the unique extension of  $S$  to a bounded selfadjoint operator on  $L^2(\varrho)$ . As noted before, the spectral theory of  $S$  on  $L^2$  is essentially the same as that of the selfadjoint operator  $TA$  on  $L^2(\varrho)$ .

Now we give an example of an operator selfadjoint with respect to an innerproduct which does not have a selfadjoint factorization through Hilbert space. Let  $X$  be the disk algebra; the algebra of all continuous complex-valued functions defined on the closed unit disk  $D$ , and holomorphic on the interior of  $D$ . Define a bounded innerproduct on  $X$  by

$$(f, g) = \sum_{n=1}^{\infty} f(n^{-1})\overline{g(n^{-1})}n^{-2} \quad (f, g \in X).$$

Let  $S \in \mathcal{B}(X)$  be given by

$$S(f)(z) = zf(z) \quad (z \in D, f \in X).$$

Then  $S$  is selfadjoint with respect to the given innerproduct. But  $\sigma(S) = D$ , so  $S$  has no selfadjoint factorization on a Hilbert space.

There is an extension  $\bar{S}$  of  $S$  to a selfadjoint operator on  $H$ , the completion of  $X$  with respect to the innerproduct. The vectors in  $H$  are sequences in  $l^2(n^{-2})$ , and

$$\bar{S}(\{a_n\}_{n \geq 1}) = \{n^{-1}a_n\}_{n \geq 1}.$$

It is easy to see that  $\bar{S}$  is a Hilbert—Schmidt operator on  $H$ . Thus, there exists a Hilbert—Schmidt operator on  $H$  which when restricted to an invariant Banach subspace of  $H$  no longer has the properties of a selfadjoint operator.

Example III. Let  $(\Omega, \mu)$  be a measure space with  $\mu$  a finite measure. We set  $L^p = L^p(\Omega, \mu)$  for  $1 \leq p \leq \infty$ . Assume  $S: L^1 \rightarrow L^\infty$ . Then for  $1 \leq p \leq \infty$ ,

$$S(L^p) \subseteq S(L^1) \subseteq L^\infty \subseteq L^p.$$

It is straightforward to check that  $S$  is closed as an operator from  $L^p$  to  $L^p$ . Thus for each  $p$ ,  $S$  determines a bounded linear operator  $S_p: L^p \rightarrow L^p$ . We prove that for each  $p$ ,  $S_p$  has a factorization through Hilbert space. First consider the case where  $1 \leq p < 2$ . Then

$$S(L^p) \subseteq L^\infty \subseteq L^2, \quad \text{and} \quad L^2 \subseteq L^p.$$

Let  $T: L^p \rightarrow L^2$  be determined by  $S$  (again,  $T$  is closed, hence continuous). Let  $A$  be the continuous embedding of  $L^2$  into  $L^p$ . Then  $S_p = AT$  is a factorization of  $S_p$  through  $L^2$ . Note that  $TA(f) = S(f)$  for all  $f \in L^2$ , so  $TA = S_2$ .

Now suppose  $2 < p \leq \infty$ , in which case

$$S(L^2) \subseteq L^\infty \subseteq L^p, \quad \text{and} \quad L^p \subseteq L^2.$$

Let  $T$  be the continuous embedding of  $L^p$  into  $L^2$ , and let  $A$  be the bounded linear operator from  $L^2$  into  $L^p$  determined by  $S$ . Then  $S_p = AT$  is a factorization of  $S_p$  through  $L^2$ , and  $TA = S_2$  on  $L^2$ . We summarize these results in a theorem.

**Theorem 3.1.** *Let  $(\Omega, \mu)$  be a finite measure space. Assume  $S: L^1 \rightarrow L^\infty$ . Then for each  $p$ ,  $1 \leq p \leq \infty$ ,  $S$  determines an operator  $S_p \in \mathcal{B}(L^p)$ ,  $S_p$  has a factorization  $S_p = AT$  through  $L^2$ , and  $TA = S_2$ . Therefore if  $S_2$  is normal then the factorization is normal.*

Now we look at two specific classes of examples where Theorem 3.1 applies.

**Corollary 3.2.** *Assume  $(\Omega, \mu)$  is a finite measure space and that  $K \in L^\infty(\Omega \times \Omega)$ . Let  $S$  be defined by*

$$S(f)(x) = \int_{\Omega} K(x, t) f(t) d\mu(t) \quad (f \in L^1).$$

*Then  $S(L^1) \subseteq L^\infty$ . If  $S_2$  is a normal operator, then  $S_p$  has a normal factorization through  $L^2$  for  $1 \leq p \leq \infty$ . In particular, if  $K(x, t) = \overline{K(t, x)}$  a.e. on  $\Omega \times \Omega$ , then  $S_p$  has a selfadjoint factorization through  $L^2$  for all  $p$ .*

**Corollary 3.3.** *Let  $\psi(t)$  and  $\varphi(t)$  be complex-valued measurable functions on  $(a, b)$  with*

- (i)  $|\psi(t)|$  increasing on  $(a, b)$ ;
- (ii)  $|\varphi(t)|$  decreasing on  $(a, b)$ ;
- (iii)  $\varphi\psi \in L^\infty[a, b]$ .

*Define*

$$K(x, t) = \begin{cases} \overline{\varphi(x)}\psi(t) & a \leq t \leq x \leq b, \\ \varphi(t)\overline{\psi(x)} & a \leq x \leq t \leq b. \end{cases}$$

*Let  $S$  be the integral operator determined by the kernel  $K$ . Then  $S: L^1[a, b] \rightarrow L^\infty[a, b]$  and  $S_2$  is selfadjoint.*

**Proof.** It is straightforward to check that  $K(x, t)$  is bounded.

**Example IV.** Let  $X$  be a Banach space which is a subspace of a Hilbert space  $H$  with  $X$  continuously embedded in  $H$ . Assume  $R \in \mathcal{B}(H)$  with  $R$  selfadjoint (or normal) and suppose  $R(H) \subseteq X$ . Let  $S$  be the restriction of  $R$  to  $X$ . It is easy to check that  $S$  is closed on  $X$  so  $S \in \mathcal{B}(X)$ . Let  $T: X \rightarrow H$  be the continuous embedding. Define  $A: H \rightarrow X$  by  $Ay = Ry, y \in H$ . Again,  $A$  is closed, hence continuous. Then  $S = AT$  and  $TA = R$ . Examples of this type are quite common.

Here is a specific example. Let  $G$  be a locally compact unimodular group with a fixed left Haar measure. Fix  $k \in L^1(G) \cap L^2(G)$  such that  $k(x^{-1}) = \overline{k(x)}, x \in G$ . Then  $k(xt^{-1})$  is a selfadjoint kernel, and the corresponding convolution operator

$$R(f)(x) = \int_G k(xt^{-1}) f(t) dt \quad (f \in L^2(G))$$

is selfadjoint on  $L^2(G)$ . Let  $X$  be the Banach subspace of  $L^2(G)$  consisting of all those  $f \in L^2(G)$  which are continuous and bounded on  $G$ . By [9, (20.19)(iii)]  $R(L^2(G)) \subseteq X$ . Thus, as indicated above the operator  $S \in \mathcal{B}(X)$  defined by

$$S(f)(x) = \int_G k(xt^{-1}) f(t) dt \quad (f \in X)$$

has a selfadjoint factorization through  $L^2(G)$ .

We give one more specific class of examples. Let  $X=L^2[0, \infty)\cap L^p[0, \infty)$  for some  $p, 1\leqq p\leqq \infty$ ; or let  $X$  be the set of  $f\in L^2[0, \infty)$  which are bounded and continuous on  $[0, \infty)$ . For  $\varphi, \psi\in X$ , let  $K(x, t)$  be the kernel

$$K(x, t) = \begin{cases} \overline{\varphi(x)}\psi(t) & 0\leqq t\leqq x, \\ \varphi(t)\overline{\psi(x)} & 0\leqq x\leqq t. \end{cases}$$

Let  $R$  be the selfadjoint operator on  $L^2[0, \infty)$  determined by this kernel. For  $f\in L^2[0, \infty)$ ,

$$R(f)(x) = \overline{\varphi(x)}\int_0^x \psi(t)f(t)dt + \overline{\psi(x)}\int_x^\infty \varphi(t)f(t)dt.$$

For  $x\geqq 0, f\in L^2[0, \infty)$ ,

$$|R(f)(x)| \leqq |\varphi(x)|\|\psi\|_2\|f\|_2 + |\psi(x)|\|\varphi\|_2\|f\|_2.$$

This inequality proves that  $R(f)\in X$ . Thus, as before, the integral operator  $S$  on  $X$  determined by the kernel  $K$  has a selfadjoint factorization through  $L^2[0, \infty)$ .

Example V. Let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space. We construct a class of operators on  $L^\infty$  (and later on  $L^1$ ) which have a selfadjoint factorization through  $L^2$ . If  $f$  and  $g$  are measurable functions on  $\Omega$  with  $\int fg\in L^1$ , then let  $(f, g) = \int_\Omega fg\,d\mu$ . Assume

$$(*) \quad V: L^1 \rightarrow L^\infty \quad \text{with} \quad (V(f), g) = (f, V(g)) \quad \text{for all} \quad f, g\in L^1.$$

Assume  $k\in L^1, k\geqq 0$  a.e. on  $\Omega$ . Define  $T: L^\infty \rightarrow L^2$  by

$$T(f) = k^{1/2}f \quad (f\in L^\infty).$$

Define  $A: L^2 \rightarrow L^\infty$  by

$$A(f) = V(k^{1/2}f) \quad (f\in L^2).$$

For  $f\in L^2$ ,

$$\|Af\|_\infty = \|V(k^{1/2}f)\|_\infty \leqq \|V\| \|k^{1/2}f\|_1 \leqq \|V\| \|k^{1/2}\|_1 \|f\|_2$$

where the last inequality follows by applying the Cauchy—Schwarz Inequality. Therefore  $A$  is bounded. Thus,  $S=AT: L^\infty \rightarrow L^\infty$ ,

$$S(f) = V(kf) \quad (f\in L^\infty),$$

has a factorization through  $L^2$ . We check that  $TA: L^2 \rightarrow L^2$  is selfadjoint. For  $f, g\in L^2, k^{1/2}f$  and  $k^{1/2}g$  are in  $L^1$ , so using  $(*)$  we have

$$(TA(f), g) = (k^{1/2}V(k^{1/2}f), g) = (V(k^{1/2}f), k^{1/2}g) = (k^{1/2}f, V(k^{1/2}g)) = (f, TA(g)).$$

Now we consider a related operator on  $L^1$  that factors. Again, assume  $V$  is as in  $(*)$ , and  $k\in L^1, k\geqq 0$  a.e. on  $\Omega$ . Define  $T: L^1 \rightarrow L^2$  by

$$T(f) = k^{1/2}V(f) \quad (f\in L^1),$$

and  $A: L^2 \rightarrow L^1$  by

$$A(f) = k^{1/2} f \quad (f \in L^2).$$

Then  $R = AT: L^1 \rightarrow L^1$ ,

$$R(f) = kV(f) \quad (f \in L^1),$$

and a computation similar to that above proves that  $TA$  is selfadjoint on  $L^2$ .

We summarize this discussion in a theorem.

**Theorem 3.4.** *Assume  $V: L^1 \rightarrow L^\infty$  with  $(V(f), g) = (f, V(g))$  for all  $f, g \in L^1$ . Assume  $k \in L^1$ ,  $k \geq 0$  a.e. on  $\Omega$ . Then  $S: L^\infty \rightarrow L^\infty$  and  $R: L^1 \rightarrow L^1$  defined by*

$$S(f) = V(kf) \quad (f \in L^\infty),$$

$$R(f) = kV(f) \quad (f \in L^1),$$

*have selfadjoint factorizations through  $L^2$ .*

Next we give some examples of operators  $V$  which satisfy (\*).

**Proposition 3.5.** *Let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space and assume  $K \in L^\infty(\Omega \times \Omega)$  with  $K(x, t) = \overline{K(t, x)}$  a.e. Let  $V$  be the corresponding integral operator*

$$V(f)(x) = \int_{\Omega} K(x, t) f(t) dt \quad (f \in L^1(\Omega)).$$

*Then  $V$  satisfies (\*).*

The proof is elementary, so it will not be given.

**Proposition 3.6.** *Assume  $\varphi$  and  $\psi$  are  $C$ -valued functions on  $(0, \infty)$  with*

- (i)  $|\psi(t)|$  is increasing on  $(0, +\infty)$ ;
- (ii)  $|\varphi(t)|$  is decreasing on  $(0, +\infty)$ ;
- (iii)  $\varphi\psi \in L^\infty(0, +\infty)$ .

*Let*

$$K(x, t) = \begin{cases} \overline{\varphi(x)}\psi(t) & 0 \leq t \leq x, \\ \varphi(t)\overline{\psi(x)} & 0 \leq x \leq t. \end{cases}$$

*Let  $V$  be the integral operator determined by the kernel  $K$ . Then  $V$  satisfies (\*).*

The proof of this proposition is straightforward.

Now, without providing the details, we discuss two concrete situations involving kernels of the types in Propositions 3.5 and 3.6. Let  $W$  be the space of all bounded  $C$ -valued continuous functions  $f$  on  $(0, \infty)$  such that  $f'$  exists and is continuous on  $(0, \infty)$ , and  $f''(x)$  exists for a.e.  $x$  in  $(0, \infty)$ . Assume  $\varrho(t) > 0$  on  $(0, \infty)$  and  $\varrho \in L^1(0, \infty)$ . Fix  $a > 0$ .

First consider the differential operator

$$L = \varrho(t)^{-1} \left( \frac{d^2}{dt^2} + a^2 \right)$$



with domain  $\mathcal{D}(L) \subseteq L^\infty(0, \infty)$  given by

$$\mathcal{D}(L) = \{f \in W: \varrho(t)^{-1}(f''(t) + a^2 f(t)) \in L^\infty(0, \infty)\},$$

and  $L(f) = \varrho(t)^{-1}(f''(t) + a^2 f(t))$  for  $f \in \mathcal{D}(L)$ .

Let

$$K(x, t) = -a^{-1} \begin{cases} \cos(ax) \sin(at) & 0 < t < x, \\ \sin(ax) \cos(at) & 0 < x < t. \end{cases}$$

$K(x, t)$  is a bounded kernel. Set  $J(x, t) = K(x, t)\varrho(t)$ ,  $x, t > 0$ , and let  $S$  be the integral operator on  $L^\infty(0, \infty)$  determined by  $J$ . By Theorem 3.4  $S$  has a self-adjoint factorization through  $L^2(0, \infty)$ . Also,  $S$  is a right inverse for  $L$ , meaning  $S(L^\infty) \subseteq \mathcal{D}(L)$  and  $L(S(f)) = f$  for  $f \in L^\infty$ . In addition,  $S$  is a Fredholm inverse for  $L$ .

Let  $W$ ,  $\varrho$ , and  $a$  be as above. We consider a second differential operator

$$L = \varrho(t)^{-1} \left( \frac{d^2}{dt^2} - a^2 \right)$$

with

$$\mathcal{D}(L) = \{f \in W: \varrho(t)^{-1}(f''(t) - a^2 f(t)) \in L^\infty(0, \infty)\}.$$

Let

$$K(x, t) = -(2a)^{-1} \begin{cases} e^{at} e^{-ax} & 0 < t < x, \\ e^{ax} e^{-at} & 0 < x < t. \end{cases}$$

$K$  is a kernel of the type considered in Proposition 3.6. Set  $J(x, t) = K(x, t)\varrho(t)$ ,  $x, t > 0$ , and let  $S$  be the integral operator on  $L^\infty(0, \infty)$  with kernel  $J$ . By Theorem 3.4  $S$  has a selfadjoint factorization on  $L^2(0, \infty)$ . Again in this case  $S$  is a right inverse of  $L$  and a Fredholm inverse for  $L$ .

Example VI. Let  $\varrho(t)$  be the weight function on  $[0, 1]$  defined by  $\varrho(t) = e^{1/t}$ ,  $0 < t \leq 1$ . Let  $L^2(\varrho)$  be the Hilbert space of  $L^2$ -functions on  $[0, 1]$  relative to the measure  $\varrho(t) dt$ . Let  $L^2 = L^2[0, 1]$ , and note  $L^2(\varrho) \subseteq L^2$ . We construct a selfadjoint Hilbert—Schmidt operator  $S$  on  $L^2(\varrho)$  such that  $S$  has an extension  $\bar{S} \in \mathcal{B}(L^2)$  such that  $\bar{S}$  is not compact and  $\sigma(\bar{S})$  is not a subset of  $\mathbf{R}$ .

Let  $K(x, t)$  be the kernel

$$K(x, t) = \begin{cases} x^{-1} \varrho(t)^{-1} & 0 \leq t \leq x \leq 1, \\ 0 & 0 \leq x < t \leq 1. \end{cases}$$

(1)  $K$  is a Hilbert—Schmidt kernel on  $L^2(\varrho)$ .

Proof. First note that

$$\int_0^x e^{-(1/t)} dt = \int_0^x t^2 (t^{-2} e^{-(1/t)}) dt \leq x^2 \int_0^x t^{-2} e^{-(1/t)} dt = x^2 e^{-(1/x)}.$$

Then

$$\begin{aligned} \int_0^1 \left( \int_0^1 K(x, t)^2 \varrho(t) dt \right) \varrho(x) dx &= \int_0^1 x^{-2} \varrho(x) \left[ \int_0^x e^{-(1/t)} dt \right] dx \cong \\ &\cong \int_0^1 x^{-2} \varrho(x) [x^2 e^{-(1/x)}] dx = 1. \end{aligned}$$

Let  $T$  be the Hilbert—Schmidt operator determined by the kernel  $K$ . The adjoint kernel of  $K$ ,  $K^*$ , is given by

$$K^*(x, t) = \begin{cases} 0 & 0 \leq t < x \leq 1, \\ t^{-1} \varrho(x)^{-1} & 0 \leq x \leq t \leq 1. \end{cases}$$

The corresponding operator is the adjoint of  $T$ . Let  $S = T + T^*$  on  $L^2(\varrho)$ .  $S$  is determined by the kernel  $K + K^*$ , so for  $f \in L^2(\varrho)$ ,

$$\begin{aligned} S(f)(x) &= \int_0^1 K(x, t) f(t) \varrho(t) dt + \int_0^1 K^*(x, t) f(t) \varrho(t) dt = \\ &= x^{-1} \int_0^x f(t) dt + \varrho(x)^{-1} \int_x^1 t^{-1} \varrho(t) f(t) dt. \end{aligned}$$

Let

$$J(x, t) = \begin{cases} 0 & 0 \leq t < x \leq 1, \\ \varrho(x)^{-1} t^{-1} \varrho(t) & 0 \leq x \leq t \leq 1. \end{cases}$$

(2)  $J$  is a Hilbert—Schmidt kernel on  $L^2$ .

Proof. For  $x > 0$

$$\int_0^1 J(x, t)^2 dt = \varrho(x)^{-2} \int_x^1 t^{-2} e^{2/t} dt = \varrho(x)^{-2} \left[ -\frac{1}{2} e^2 + \frac{1}{2} e^{2/x} \right]$$

which is a bounded continuous function of  $x$  on  $(0, 1]$ .

Define  $\bar{S}$  on  $L^2$  by

$$\bar{S}(f)(x) = x^{-1} \int_0^x f(t) dt + \int_0^1 J(x, t) f(t) dt.$$

The first summand is the Cesaro operator on  $L^2[0, 1]$ , while the second, as verified in (2), is a Hilbert—Schmidt operator on  $L^2$ . The Cesaro operator is studied in [4] where it is verified that it is bounded on  $L^2$ . Thus,  $\bar{S} \in B(L^2)$ , and by definition  $\bar{S}$  is an extension of  $S$ . Now the Cesaro operator has spectrum a disk [4], and the operator  $f \rightarrow \int_0^1 J(x, t) f(t) dt$  is compact. These two facts imply that  $\bar{S}$  is not compact, and that  $\sigma(\bar{S})$  is not a subset of  $\mathbf{R}$ .

**4. Regularity, hyperinvariant subspaces.** Let  $\mathcal{A}$  be a commutative Banach algebra with unit. We denote the Gelfand space of  $\mathcal{A}$  by  $\Omega_{\mathcal{A}}$  ( $\Omega_{\mathcal{A}}$  is the set of all nonzero multiplicative linear functionals on  $\mathcal{A}$  equipped with the relative weak- $*$  topology). For  $f \in \mathcal{A}$ , let  $\hat{f}$  denote the Gelfand transform of  $f$ , so  $\hat{f}(\psi) = \psi(f)$  for  $\psi \in \Omega_{\mathcal{A}}$ . A subset  $D$  of  $\mathcal{A}$  strongly separates points of  $\Omega_{\mathcal{A}}$  if whenever  $\psi_1, \psi_2 \in \Omega_{\mathcal{A}}, \psi_1 \neq \psi_2$ , then  $\exists f \in D$  with  $\hat{f}(\psi_1) \neq \hat{f}(\psi_2)$ , and whenever  $\psi \in \Omega_{\mathcal{A}}, \exists g \in D$  with  $\hat{g}(\psi) \neq 0$ . The algebra  $\mathcal{A}$  is regular if whenever  $\Gamma$  is a closed subset of  $\Omega_{\mathcal{A}}$  and  $\psi \in \Omega_{\mathcal{A}} \setminus \Gamma$ , then  $\exists f \in \mathcal{A}$  such that  $\hat{f}(\Gamma) = \{0\}$  and  $\hat{f}(\psi) \neq 0$ . Let  $\text{rad}(\mathcal{A})$  denote the Jacobson radical of  $\mathcal{A}$ . A good reference for the theory of Banach algebras is [5].

Now we prove a general result in a Banach algebra setting which applies to operators  $S$  that have selfadjoint factorizations. Some form of this result is certainly known (see [8]), but we include it since the proof is short and elementary.

**Theorem 4.1.** *Let  $\mathcal{B}$  be a regular commutative semisimple Banach algebra with unit. Let  $\mathcal{A}$  be a commutative Banach algebra with unit. Assume  $\varphi: \mathcal{B} \rightarrow \mathcal{A}$  is a unital algebra homomorphism such that  $\varphi(\mathcal{B})$  strongly separates points of  $\Omega_{\mathcal{A}}$ . Then*

(1)  $\mathcal{A}$  is regular;

(2) *Assume  $S \in \mathcal{B}(X)$  and  $\mathcal{A}$  is a closed subalgebra of  $\mathcal{B}(X)$  with  $S$  and  $I$  in  $\mathcal{A}$  such that  $\mathcal{A}$  satisfies the hypotheses of the theorem. Also assume that  $\mathcal{A}$  has the property that when  $R \in \mathcal{B}(X)$  and  $RS = SR$ , then  $RT = TR$  for all  $T \in \mathcal{A}$ . If  $\sigma(S)$  contains more than one number, then  $S$  has a proper closed hyperinvariant subspace.*

**Proof.** Define  $\tau: \Omega_{\mathcal{A}} \rightarrow \Omega_{\mathcal{B}}$  by  $\tau(\psi) = \psi \circ \varphi$ . Then  $\tau$  is one-to-one and continuous. Now assume  $\Gamma$  is a closed subset of  $\Omega_{\mathcal{A}}$  and  $\psi_1 \in \Omega_{\mathcal{A}} \setminus \Gamma$ . Since  $\Gamma$  is compact,  $\tau(\Gamma)$  is compact, and also,  $\tau(\psi_1) \notin \tau(\Gamma)$ . Then  $\exists f \in \mathcal{B}$  such that  $\hat{f}(\tau(\Gamma)) = \{0\}$  and  $\hat{f}(\tau(\psi_1)) \neq 0$ . This proves  $\mathcal{A}$  is regular.

Now assume  $S$  and  $\mathcal{A}$  are as in (2). By hypothesis  $\sigma(S)$  contains at least two points. Since  $\sigma(S) \subseteq \sigma_{\mathcal{A}}(S)$ ,  $\sigma_{\mathcal{A}}(S)$  contains at least two points. Thus,  $\exists \psi_1, \psi_2 \in \Omega_{\mathcal{A}}$  with  $\psi_1 \neq \psi_2$ , then  $\tau(\psi_1) \neq \tau(\psi_2)$  so we can choose  $f_1, f_2 \in \mathcal{B}$  such that  $\hat{f}_k(\tau(\psi_k)) \neq 0, k=1, 2$ , and  $f_1 f_2 = 0$ . Therefore  $\varphi(f_k) \neq 0, k=1, 2$ , and  $\varphi(f_1)\varphi(f_2) = 0$ . Let  $W$  be the closure of  $\varphi(f_2)X$  in  $X$ .  $W$  is proper since  $\varphi(f_1)W = \{0\}$ . If  $R \in \mathcal{B}(X)$  commutes with  $S$ , then  $R\varphi(f_2) = \varphi(f_2)R$ , so  $R(W) \subseteq W$ .

Theorem 4.1 applies to the situation where  $S$  has a selfadjoint factorization on Hilbert space. The map  $\varphi$  involved is the operational calculus. As part of the proof of this result, we prove a preliminary proposition.

Let  $\Delta$  be a compact subset of  $\mathbb{C}$ . For  $f \in \text{BM}(\Delta)$ , define

$$\|f\|_{\Delta} = \sup \{ |f(\lambda)| : \lambda \in \Delta \}.$$

Also, let  $C(\Delta)$  denote the algebra of all complex-valued continuous functions on  $\Delta$ .

**Proposition 4.2.** *Assume that  $S$  has a selfadjoint factorization through Hilbert space. Also, assume that (A1) and (A2) hold. Set  $\Delta = \sigma(S)$ . Define*

$$\mathcal{I} = \{f \in C(\Delta) : \exists g \in C(\Delta) \text{ with } f(\lambda) = \lambda g(\lambda) \text{ on } \Delta\}.$$

*Let  $\mathcal{B}$  be  $\mathcal{I}$  with a unit adjoined. Let  $\mathcal{A}$  be the closed subalgebra of  $\mathcal{B}(X)$  generated by  $S$  and  $I$ . Then  $\mathcal{B}$  is a regular semisimple Banach algebra, and  $\exists \varphi: \mathcal{B} \rightarrow \mathcal{A}$  with  $\varphi$  a continuous unital algebra homomorphism such that  $\varphi(\mathcal{B})$  separates points of  $\Omega_{\mathcal{A}}$ .*

**Proof.** One easily checks that  $\mathcal{I}$  is an ideal in  $C(\Delta)$  and that  $\mathcal{I}$  is a Banach algebra in the norm  $\|f\| = \max(\|f\|_{\Delta}, \|g\|_{\Delta})$  where  $f \in \mathcal{I}, g \in C(\Delta)$  with  $f(\lambda) = \lambda g(\lambda)$  on  $\Delta$ . It follows that  $\mathcal{I}$ , and hence  $\mathcal{B}$ , is a regular semisimple Banach algebra.

Now  $\mathcal{I} \subseteq \mathfrak{M}(\Delta)$ . For  $f \in \mathcal{I}$ , let  $\varphi(f) = f(S)$ , and extend  $\varphi$  to  $\mathcal{B}$  by setting  $\varphi(1) = I$ . Note that  $\varphi$  is continuous on  $\mathcal{I}$  by Theorem 2.4 (3). We still must check that  $\varphi(\mathcal{I}) \subseteq \mathcal{A}$ . Assume  $f \in \mathcal{I}$  with  $g \in C(\Delta), f(\lambda) = \lambda g(\lambda)$  on  $\Delta$ . Choose a sequence of polynomials  $\{q_n(\lambda)\}$  such that  $\|q_n - g_n\|_{\Delta} \rightarrow 0$ . Set  $p_n(\lambda) = \lambda q_n(\lambda)$ , so  $\{p_n\} \subseteq \mathcal{I}$ . Then  $\|p_n - f\|_{\Delta} \rightarrow 0$ , so  $p_n \rightarrow f$  in the norm on  $\mathcal{I}$ . Therefore  $\{p_n(S)\} \subseteq \mathcal{A}$  and  $p_n(S) \rightarrow f(S)$  in  $\mathcal{B}(X)$ . Thus,  $f(S) \in \mathcal{A}$ . Finally,  $\varphi(\mathcal{B})$  separates points of  $\Omega_{\mathcal{A}}$  since  $I, S \in \varphi(\mathcal{B})$ .

**Theorem 4.3.** *Assume  $S \in \mathcal{B}(X)$  has a selfadjoint factorization through Hilbert space.*

(1) *If  $\sigma(S)$  contains at least two numbers, then  $S$  has a proper closed hyperinvariant subspace.*

*Let  $\mathcal{A}$  be the closed subalgebra of  $\mathcal{B}(X)$  generated by  $S$  and  $I$ . Assume (A1) and (A2) hold. Then*

(2)  *$\mathcal{A}$  is a regular Banach algebra,  $\text{rad}(\mathcal{A})^2 = \{0\}$ , and  $SR = RS = 0$  for all  $R \in \text{rad}(\mathcal{A})$ ;*

(3) *If  $\mathcal{R}(S)$  is dense in  $X$  or  $\mathfrak{N}(S) = \{0\}$ , then  $\mathcal{A}$  is semisimple.*

**Proof.** If  $S \neq 0$  and  $\mathfrak{N}(S) \neq \{0\}$ , then  $\mathfrak{N}(S)$  is a proper closed hyperinvariant subspace of  $S$ . Thus we may assume  $\mathfrak{N}(S) = \{0\}$ . Let  $N$  be the nilpotent part of  $S$ . Since  $SN = 0$ , in this case  $N = 0$ . Then by Propositions 1.2 and 1.3 we may assume that  $S$  has a factorization  $S = AT$  with  $TA$  selfadjoint such that (A1) and (A2) hold. If  $TA$  is invertible, then by Proposition 1.3  $S$  is invertible. In this case  $S$  is similar to the selfadjoint operator  $TA$ . It follows easily that  $S$  has a proper closed hyperinvariant subspace (assuming  $\sigma(S)$  has more than one point). Thus, to establish (1) we may assume (A1) and (A2) hold.

Assuming (A1) and (A2) hold, Proposition 4.2 applies. Then Theorem 4.1 proves (1) and that  $\mathcal{A}$  is regular.

Now assume  $R \in \text{rad}(\mathcal{A})$ . Choose a sequence of polynomials  $\{p_n(\lambda)\}$  such that  $p_n(S) \rightarrow R$  in  $\mathcal{A}$ . Since  $\mathcal{A}$  is a closed subalgebra of  $\mathcal{B}(X)$ , the spectral radius of  $V \in \mathcal{A}$  relative to  $\mathcal{A}$  is the same as  $r(V)$ , the spectral radius of  $V$  in  $B(X)$ . Now  $r(p_n(S)) \rightarrow r(R) = 0$ . Since  $\sigma(p_n(S)) = \{p_n(\mu) : \mu \in \sigma(S)\}$ , it follows that  $p_n(\lambda) \rightarrow 0$  uniformly on  $\Delta$ . Therefore by Theorem 2.4 (3)  $Sp_n(S) \rightarrow 0$ , so  $SR = 0$ . Also, it now follows that  $Rp_n(S) = p_n(0)R$ , and since  $p_n(0) \rightarrow 0$ , we have  $R^2 = 0$ . This completes the proof of (2).

(3) follows easily from the fact derived in (2) that for  $R \in \text{rad}(\mathcal{A})$ ,  $SR = RS = 0$ .

In our final result, we show that when  $S$  has a selfadjoint factorization through Hilbert space, then  $S$  can be approximated by operators which are similar to selfadjoint operators.

**Theorem 4.4.** *Assume  $S \in \mathcal{B}(X)$  has a selfadjoint factorization through a Hilbert space, and that (A1) and (A2) hold. Then there exists a collection of projection operators  $\{P_\varepsilon\}_{\varepsilon > 0} \subseteq \mathcal{B}(X)$  such that  $P_\varepsilon S = SP_\varepsilon$  for  $\varepsilon > 0$ , and*

(i)  $P_\varepsilon S$  considered as an operator on  $P_\varepsilon(X)$  is similar to a selfadjoint operator for each  $\varepsilon > 0$ ; and

(ii)  $S$  is the strong limit as  $\varepsilon \rightarrow 0^+$  of  $SP_\varepsilon$  on  $X$ .

**Proof.** Assume  $S = AT$  is factorization of  $S$  through  $H$  with  $TA$  selfadjoint and that (A1) and (A2) hold. Let  $\Delta = \sigma(TA)$ . For  $t \in \mathbf{R}$ , let  $\chi_{(-\infty, t]}$  be the characteristic function of the specified interval and set  $E_t = \chi_{(-\infty, t]}(TA)$ . Thus,  $\{E_t\}_{t \in \mathbf{R}}$  is the usual spectral resolution of the identity for  $TA$ . In this situation the strong limit of  $E_t - E_0$  as  $t \rightarrow 0^-$  is the projection on  $\mathfrak{N}(TA)$  which is 0 by (A1) [11, p. 361]. Also,  $E_t$  is strongly continuous from the right on  $\mathbf{R}$ , so the strong limit of  $E_\varepsilon - E_{-\varepsilon} = 0$  as  $\varepsilon \rightarrow 0^+$ . Let  $\chi_\varepsilon$  be the characteristic function of  $(-\infty, \varepsilon] \cup (\varepsilon, +\infty)$ . Then  $Q_\varepsilon = \chi_\varepsilon(TA) = I - (E_\varepsilon - E_{-\varepsilon})$  has strong limit  $I$  as  $\varepsilon \rightarrow 0^+$ . Let  $P_\varepsilon = \chi_\varepsilon(S)$ . Consider the operator  $SP_\varepsilon$  on the space  $X_\varepsilon = P_\varepsilon X$ . Let  $H_\varepsilon = Q_\varepsilon H$ . Applying the operational calculus to the function  $\lambda \chi_\varepsilon(\lambda)$ , we have

$$(1) \quad SP_\varepsilon = AQ_\varepsilon T.$$

Then  $TA(TP_\varepsilon A) = T(SP_\varepsilon)A = T(AQ_\varepsilon T)A$  by (1). Then  $TA(TP_\varepsilon A - Q_\varepsilon TA) = 0$ , so

$$(2) \quad (TA)Q_\varepsilon = TP_\varepsilon A.$$

Let

$$T_\varepsilon = Q_\varepsilon TP_\varepsilon : X \rightarrow H_\varepsilon, \quad \text{and} \quad A_\varepsilon = P_\varepsilon AQ_\varepsilon : H_\varepsilon \rightarrow X_\varepsilon.$$

Then using (1) and (2) we have

$$A_\varepsilon T_\varepsilon = P_\varepsilon AQ_\varepsilon TP_\varepsilon = SP_\varepsilon, \quad \text{and} \quad T_\varepsilon A_\varepsilon = Q_\varepsilon TP_\varepsilon AQ_\varepsilon = (TA)Q_\varepsilon.$$

Now let  $f(\lambda) = \lambda^{-1} \chi_\varepsilon(\lambda)$ . Then  $TAf(TA) = Q_\varepsilon$ , while  $Sf(S) = P_\varepsilon$ . Therefore  $TAQ_\varepsilon$

is invertible on  $H_\varepsilon$  and  $SP_\varepsilon$  is invertible on  $X_\varepsilon$ . By Proposition 1.3  $SP_\varepsilon$  as an operator on  $X_\varepsilon$  is similar to the selfadjoint operator  $TAQ_\varepsilon$  on  $H_\varepsilon$ . This proves (i).

To prove (ii), recall that we have shown that  $I$  is the strong limit of  $Q_\varepsilon$  as  $\varepsilon \rightarrow 0^+$ . Then for  $x \in X$ ,  $Q_\varepsilon Tx \rightarrow Tx$ , and therefore by (1),

$$SP_\varepsilon x = AQ_\varepsilon Tx \rightarrow ATx = Sx.$$

Thus, (ii) holds.

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