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On the generalized absolute summability of double series

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Dedicated to Professor Béla Csákány on his 60th birthday

1. Introduction. As usual we denote by $\sigma_n^{(\alpha)}$ the n-th Cesaro means of order α of a series $\sum_{n=0}^{\infty} a_n$ and $\tau_n^{(\alpha)}$ the n-th Cesaro means of the sequence $\{na_n\}$. The following definition is due to FLETT [3]: A series $\sum_{u=0}^{\infty} a_n$ is said to be summable $|C, \alpha, u|_\lambda$, $\alpha>-1$, $u\geqq 0$, $\lambda\geqq 1$, if the series

$$\sum_{n=1}^{\alpha} n^{\lambda u+\lambda-1}|\sigma_n^{(\alpha)}-\sigma_{n-1}^{(\alpha)}|^\lambda \equiv \sum_{n=1}^{\alpha} n^{\lambda u-1}|\tau_n^{(\alpha)}|^\lambda$$

converges.

In this note we consider the following definition of the generalized absolute Cesaro summability of double series

$$\sum_{i,k=0}^{\infty} a_{i,k}. \tag{1}$$

Let us denote by $\sigma_{m,n}^{(\alpha,\beta)}$ the (m, n)-th Cesaro mean of order (α, β) of series (1), that is,

$$\sigma_{m,n}^{(\alpha,\beta)} = \frac{1}{A_m^{(\alpha)}}\frac{1}{A_n^{(\beta)}}\sum_{i=0}^{m}\sum_{k=0}^{n} A_{m-i}^{(\alpha)}A_{n-k}^{(\beta)}a_{i,k}, \quad m, n = -1, 0, 1, \ldots, \tag{2}$$

such that in the cases $\min(m, n)=-1$ we define $\sigma_{m,n}^{(\alpha,\beta)}=0$, where $A_m^{(\alpha)}$ denotes the Cesaro numbers, namely, $A_0^{(\alpha)}\equiv 1$ and $A_m^{(\alpha)}=\dfrac{(1+\alpha)(2+\alpha)\ldots(m+\alpha)}{m!}$, $\alpha\neq -1, -2, \ldots$.

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Considering the notations

$$z_{m,n}^{(\alpha,\beta)} = m(\sigma_{m,n}^{(\alpha,\beta)} - \sigma_{m-1,n}^{(\alpha,\beta)}) = \frac{1}{A_m^{(\alpha)} A_n^{(\beta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\alpha-1)} A_{n-k}^{(\beta)} i a_{i,k},$$

$$t_{m,n}^{(\alpha,\beta)} = n(\sigma_{m,n}^{(\alpha,\beta)} - \sigma_{m,n-1}^{(\alpha,\beta)}) = \frac{1}{A_m^{(\alpha)} A_n^{(\beta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\alpha)} A_{n-k}^{(\beta-1)} k a_{i,k}$$

and

$$\tau_{m,n}^{(\alpha,\beta)} = mn(\sigma_{m,n}^{(\alpha,\beta)} - \sigma_{m-1,n}^{(\alpha,\beta)} - \sigma_{m,n-1}^{(\alpha,\beta)} + \sigma_{m-1,n-1}^{(\alpha,\beta)}) =$$

$$= \frac{1}{A_m^{(\alpha)} A_n^{(\beta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\alpha-1)} A_{n-k}^{(\beta-1)} ik a_{i,k}, \quad m, n = 0, 1, 2, \ldots,$$

series (1) is said to be summable $|C, (\alpha, \beta), (u, v)|_\lambda$, where $\alpha, \beta > -1$, $u, v \geqq 0$, $\lambda \geqq 1$ if

$$\sum_{i=1}^{\infty} i^{\lambda u-1} |z_{i,0}^{(\alpha,\beta)}|^\lambda < \infty, \tag{3}$$

$$\sum_{k=1}^{\infty} k^{\lambda v-1} |t_{0,k}^{(\alpha,\beta)}|^\lambda < \infty \tag{4}$$

and

$$\sum_{i,k=1}^{\infty} i^{\lambda u-1} k^{\lambda v-1} |\tau_{i,k}^{(\alpha,\beta)}|^\lambda < \infty. \tag{5}$$

The concept of summability $|C, (\alpha, \beta), (0, 0)|_1$ is well known (see e.g. [1], pp. 209—214). The generalized absolute Cesaro summability of double series was investigated by MÓRICZ [7] and SZALAY [8]. The fundamental theorems of summability $|C, \alpha, u|_\lambda$ were proved by FLETT (see [3], Theorems 1, 3, 4 and 7).

2. Main results. The aim of this paper is to extend the fundamental theorems for the double series (1). The author would like to thank I. Szalay for pointing out this generalization and his valuable hints.

Theorem 1.* *Let $\lambda \geqq 1$, $u, v \geqq 0$, $\alpha > \lambda u - 1$ and $\beta > \lambda v - 1$. If $\gamma, \delta \geqq 0$ then the summability $|C, (\alpha, \beta), (u, v)|_\lambda$ of series (1) implies the summability $|C, (\alpha+\gamma, \beta+\delta), (u, v)|_\lambda$, moreover the inequalities*

$$\sum_{m=1}^{\infty} m^{\lambda u-1} |z_{m,0}^{(\alpha+\gamma,\beta+\delta)}|^\lambda \leqq K \sum_{m=1}^{\infty} m^{\lambda u-1} |z_{m,0}^{(\alpha,\beta)}|^\lambda, \tag{6}$$

$$\sum_{n=1}^{\infty} n^{\lambda v-1} |t_{0,n}^{(\alpha+\gamma,\beta+\delta)}|^\lambda \leqq K \sum_{n=1}^{\infty} n^{\lambda v-1} |t_{0,n}^{(\alpha,\beta)}|^\lambda \tag{7}$$

*) Throughout this article K denotes a positive constant, not necessarily the same at each occurrence which does not depend on addition indices and the formal sum $\sum_{i=0}^{-1}$ means 0.

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1}n^{\lambda v-1}|\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq K \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1}n^{\lambda v-1}|\tau_{m,n}^{(\alpha,\beta)}|^{\lambda} \tag{8}$$

hold.

Theorem 2. *Let* $u, v\geqq 0$, $\alpha>\lambda u-1$ *and* $\beta>\lambda v-1$. *If*

i) $\mu>\lambda>1$ *and* $\delta=1/\lambda-1/\mu$

or

ii) $\mu>\lambda=1$ *and* $\delta>1/\lambda-1/\mu$

then the summability $|C, (\alpha, \beta), (u, v)|_\lambda$ *of series* (1) *implies the summability* $|C, (\alpha+\gamma, \beta+\delta), (u, v)|_\mu$, *moreover the inequalities*

$$\Big\{\sum_{m=1}^{\infty} m^{\mu u-1}|z_{m,0}^{(\alpha+\delta,\beta+\delta)}|^{\mu}\Big\}^{1/\mu} \leqq K\Big\{\sum_{m=1}^{\infty} m^{\lambda u-1}|z_{m,0}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda}, \tag{9}$$

$$\Big\{\sum_{n=1}^{\infty} n^{\mu v-1}|t_{0,n}^{(\alpha+\delta,\beta+\delta)}|^{\mu}\Big\}^{1/\mu} \leqq K\Big\{\sum_{n=1}^{\infty} n^{\lambda v-1}|t_{0,n}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda} \tag{10}$$

and

$$\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\mu u-1}n^{\mu v-1}|\tau_{m,n}^{(\alpha+\delta,\beta+\delta)}|^{\mu}\Big\}^{1/\mu} \leqq K\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1}n^{\lambda v-1}|\tau_{m,n}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda} \tag{11}$$

hold.

We remark that part i) of Theorem 2, together with Theorem 1 is sharper than a former result of SZALAY ([8], Theorem 1).

Theorem 3. *If* $\lambda\geqq 1$, $u, v\geqq 0$, $\alpha>\lambda u-1$, $\beta>\lambda v-1$, $\xi\leqq u$, $\eta\leqq v$, $\gamma\geqq\xi-u$, $\delta\geqq\eta-v$, $\alpha+\gamma$, $\beta+\delta>-1$, *and series* (1) *is* $|C, (\alpha, \beta), (u, v)|_\lambda$ *summable, then the inequalities*

$$\sum_{m=1}^{\infty} m^{\lambda\xi-1}|z_{m,0}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq K\sum_{m=1}^{\infty} m^{\lambda u-1}|z_{m,0}^{(\alpha,\beta)}|^{\lambda}, \tag{12}$$

$$\sum_{n=1}^{\infty} n^{\lambda\eta-1}|t_{0,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq K\sum_{n=1}^{\infty} n^{\lambda v-1}|t_{0,n}^{(\alpha,\beta)}|^{\lambda} \tag{13}$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda\xi-1}n^{\lambda\eta-1}|\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1}n^{\lambda v-1}|\tau_{m,n}^{(\alpha,\beta)}|^{\lambda} \tag{14}$$

are valid.

Using Theorem 3, in the case of parameters $u=v=0$, $-\alpha\leqq\xi\leqq 0$, $-\beta\leqq\eta\leqq 0$, $\gamma=\xi$ and $\delta=\eta$ and writing $\xi'=-\xi$, $\eta'=-\eta$ we have the following

Corollary 1. *If* $\lambda\geqq 1$, $0\leqq\xi'\leqq\alpha$, $0\leqq\eta'\leqq\beta$ *and series* (1) *is* $|C, (\alpha,\beta), (0,0)|_\lambda$ *summable, then the following inequalities*

$$\sum_{m=1}^{\infty} m^{-1-\lambda\xi'}|z_{m,0}^{(\alpha-\xi',\beta-\eta')}|^\lambda \leqq K\sum_{m=1}^{\infty} m^{-1}|z_{m,0}^{(\alpha,\beta)}|^\lambda,$$

$$\sum_{n=1}^{\infty} n^{-1-\lambda\eta'}|t_{0,n}^{(\alpha-\xi',\beta-\eta')}|^\lambda \leqq K\sum_{n=1}^{\infty} n^{-1}|t_{0,n}^{(\alpha,\beta)}|^\lambda$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1-\lambda\xi'}n^{-1-\lambda\eta'}|\tau_{m,n}^{(\alpha-\xi',\beta-\eta')}|^\lambda \leqq K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1}n^{-1}|\tau_{m,n}^{(\alpha,\beta)}|^\lambda$$

hold.

Considering the case $\alpha=\xi'$, $\beta=\eta'$ a further specialization is the

Corollary 2. *If* $\lambda\geqq 1$, $\alpha,\beta\geqq 0$ *and series* (1) *is* $|C, (\alpha,\beta), (0,0)|_\lambda$ *summable, then*

$$\sum_{m=1}^{\infty} m^{\lambda-1-\lambda\alpha}|a_{m,0}|^\lambda \leqq K\sum_{m=1}^{\infty} m^{-1}|z_{m,0}^{(\alpha,\beta)}|^\lambda,$$

$$\sum_{n=1}^{\infty} n^{\lambda-1-\lambda\beta}|a_{0,n}|^\lambda \leqq K\sum_{n=1}^{\infty} n^{-1}|t_{0,n}^{(\alpha,\beta)}|^\lambda$$

and

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda-1-\lambda\alpha}n^{\lambda-1-\lambda\beta}|a_{m,n}|^\lambda \leqq K\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1}n^{-1}|\tau_{m,n}^{(\alpha,\beta)}|^\lambda$$

are valid.

The Corollary 2 is a useful necessary condition of the generalized absolute Cesaro summability of double series and it is an extension of results of KOGBETLIANZ ([5], Théoréme VI), FLETT ([2], Theorem 3) and ZAK and TIMAN ([11], § 3, Theorem 3).

Theorem 4. *If* $\lambda>\mu\geqq 1$, $u, v\geqq 0$, $\alpha>\mu u-1$, $\beta>\mu v-1$, $\xi\leqq u$, $\eta\leqq v$, $\gamma>\xi-u$, $\delta>\eta-v$ *and series* (1) *is* $|C, (\alpha,\beta), (u,v)|_\lambda$ *summable, then the inequalities*

$$\Big\{\sum_{m=1}^{\infty} m^{\mu\xi-1}|z_{m,0}^{(\alpha+\gamma,\beta+\delta)}|^\mu\Big\}^{1/\mu} \leqq K\Big\{\sum_{m=1}^{\infty} m^{\lambda u-1}|z_{m,0}^{(\alpha,\beta)}|^\lambda\Big\}^{1/\lambda}, \tag{15}$$

$$\Big\{\sum_{n=1}^{\infty} n^{\mu\eta-1}|t_{0,n}^{(\alpha+\gamma,\beta+\delta)}|^\mu\Big\}^{1/\mu} \leqq K\Big\{\sum_{n=1}^{\infty} n^{\lambda v-1}|t_{0,n}^{(\alpha,\beta)}|^\lambda\Big\}^{1/\lambda} \tag{16}$$

and

$$\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\mu\xi-1}n^{\mu\eta-1}|\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^\mu\Big\}^{1/\mu} \leqq K\Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1}n^{\lambda v-1}|\tau_{m,n}^{(\alpha,\beta)}|^\lambda\Big\}^{1/\lambda} \tag{17}$$

hold.

We remark that in cases $\xi, \eta \geqq 0$ Theorems 3 and 4 mean, in other words, that if the suitable conditions are satisfied, the summability $|C, (\alpha, \beta), (u, v)|_\lambda$ of series (1) implies the summability $|C, (\alpha+\gamma, \beta+\delta), (\xi, \eta)|_\lambda$ and $|C, (\alpha+\gamma, \beta+\delta), (\xi, \eta)|_\mu$, respectively.

Series (1) is said to be summable $(C, (\alpha, \beta))$, $\alpha, \beta > -1$ to S, if the double sequence (2) is bounded and converges to S in Pringsheim's sense. Finally we have

Theorem 5. *If* $\lambda > 1, u, v > 0, \alpha > u-1, \beta > v-1, \gamma > \alpha - u - 1/\lambda > 0$ *and* $\delta > \beta - v - 1/\lambda > 0$ *then the summability* $|C, (\alpha, \beta), (u, v)|_\lambda$ *of series* (1) *implies the summability* $(C, (\gamma, \delta))$.

Part 4 of this note contains some negative results. We show that Theorem 2 is the best possible. In relation to Theorems 3 and 4 we show that the parameter u (or v) of summability cannot be increased by no means and parameter λ cannot be decreased if parameters u, v are fixed.

3. Proof of Theorems. If $\tau_n^{(\alpha)}$ denotes the n-th (C, α) mean of the sequence $\{na_n\}$ then it is well known that if $\alpha, \beta, \alpha+\gamma, \beta+\delta \neq -1, -2, \dots$, then

(18)
$$\tau_n^{(\alpha+\delta)} = \frac{1}{A_n^{(\alpha+\delta)}} \sum_{k=0}^{n} A_{n-k}^{(\delta-1)} A_k^{(\alpha)} \tau_k^{(\alpha)},$$

(19)
$$\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)} = \frac{1}{A_m^{(\alpha+\gamma)} A_n^{(\beta+\delta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\gamma-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} \tau_{i,k}^{(\alpha,\beta)},$$

$$A_n^{(\beta+\delta)} = \sum_{k=0}^{n} A_{n-k}^{(\delta-1)} A_k^{(\beta)} \quad (n = 0, 1, 2, \dots),$$

moreover

(20)
$$A_n^{(\alpha)}/n^\alpha \to 1/\Gamma(\alpha+1) \quad (n \to \infty).$$

In order to prove Theorems we require the following lemmas.

Lemma 1 (Szalay [9]). *If* $\alpha, \beta, \alpha+\gamma, \beta+\delta \neq -1, -2, \dots$, *then for any* $m, n = 0, 1, 2, \dots$

$$z_{m,n}^{(\alpha+\gamma,\beta+\delta)} = \frac{1}{A_m^{(\alpha+\gamma)} A_n^{(\beta+\delta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\gamma-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} z_{i,k}^{(\alpha,\beta)}$$

and

$$t_{m,n}^{(\alpha+\gamma,\beta+\delta)} = \frac{1}{A_m^{(\alpha+\gamma)} A_n^{(\beta+\delta)}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{(\gamma-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} t_{i,k}^{(\alpha,\beta)}.$$

Lemma 2 (Hardy—Littlewood—Pólya [6]). *Let* $\{d_i\}_{i=0}^{\infty}$ *be a non-negative se-*

quence. If $\mu>\lambda>1$ *and* $\delta=1/\lambda-1/\mu$, *then exists a* $K=K(\lambda, \mu)$ *constant, such that*

$$\{\sum_{m=0}^{M}(\sum_{i=0}^{m-1}(m-i)^{\delta-1}d_i)^{\mu}\}^{1/\mu} \leqq K(\sum_{i=0}^{M} d_i^{\lambda})^{1/\lambda}$$

is valid for any $M=0, 1, 2, \ldots$.

Lemma 3 (SZALAY [10]). *Let* $\{d_{i,k}\}_{i,k=0}^{\infty}$ *be a non-negative double sequence. If* $\mu>\lambda>1$ *and* $\delta=1/\lambda-1/\mu$, *then exists a* $K=K(\lambda, \mu)$ *constant, such that*

$$\{\sum_{m=0}^{M}\sum_{n=0}^{N}(\sum_{i=0}^{m-1}\sum_{k=0}^{n-1}(m-i)^{\delta-1}(n-k)^{\delta-1}d_{i,k})^{\mu}\}^{1/\mu} \leqq K(\sum_{i=0}^{M}\sum_{k=0}^{N} d_{i,k}^{\lambda})^{1/\lambda}$$

is valid for any $M, N=0, 1, 2, \ldots$.

Lemma 4 (ZAK—TIMAN [11]). *If series* (1) *is* $|C, (\gamma, \delta), (0, 0)|_1$ *summable, then it is* $(C, (\gamma, \delta))$ *summable, too.*

We remark that if $\lambda>1$ then the summability $|C, (\gamma, \delta), (0, 0)|_{\lambda}$ does not imply the ordinary summability $(C, (\gamma, \delta))$.

Lemma 5. *If* $a_{i,k}=c_i$ *for* $k=0$ *and* $a_{i,k}=0$ *otherwise, then the* $|C, (\alpha, \beta), (u, v)|_{\lambda}$ *summability of series* (1) *and* $|C, \alpha, u|_{\lambda}$ *summability of the series* $\sum_{i=0}^{\infty} c_i$ *are equivalent. Similarly, if* $a_{i,k}=c_k$ *for* $i=0$ *and* $a_{i,k}=0$ *otherwise, then the* $|C, (\alpha, \beta), (u, v)|_{\lambda}$ *summability of series* (1) *and* $|C, \beta, v|_{\lambda}$ *summability of the series* $\sum_{k=0}^{\infty} c_k$ *are equivalent.*

Proof. A fairly trivial calculation gives that for any n, β

$$\sigma_{m,n}^{(\alpha,\beta)}(a_{i,k}) = \frac{1}{A_m^{(\alpha)}A_n^{(\beta)}}\sum_{i=0}^{m}\sum_{k=0}^{n} A_{m-i}^{(\alpha)}A_{n-k}^{(\beta)}a_{i,k} = \frac{1}{A_m^{(\alpha)}}\sum_{i=0}^{m} A_{m-i}^{(\alpha)}a_{i,0} = \sigma_m^{(\alpha)}(c_i)$$

and

$$z_{m,n}^{(\alpha,\beta)}(a_{i,k}) = \frac{1}{A_m^{(\alpha)}}\sum_{i=0}^{m} A_{m-i}^{(\alpha-1)}ic_i \equiv \tau_m^{(\alpha)}(c_i),\ t_{m,n}^{(\alpha\beta)}(a_{i,k}) = \tau_{m,n}^{(\alpha,\beta)}(a_{i,k}) = 0,$$

so by (3)—(5) the statement is obvious.

Proof of Theorem 1. Considering (18) and Lemma 1, it is clear, that $z_{m,0}^{(\alpha,\beta)}$, the m-th τ-mean of order α of the single series $\sum_{i=0}^{\infty} a_{i,0}$, does not depend on β, hence the inequality (6) follows directly from FLETT's result ([3], Theorem 1). The proof of (7) is carried out analogously. In the case $\lambda>1$, to verify (8) we use Hölder's inequality with indices λ and $\lambda/(\lambda-1)$. By (19) and (20) we obtain that for any

$M, N=1, 2, \dots$

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1} n^{\lambda v-1} |\tau_{m,n}^{(\lambda+\gamma,\beta)}|^{\lambda} \leqq$$

$$\leqq \sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1} n^{\lambda v-1} \left(\frac{1}{A_m^{(\alpha+\gamma)}} \sum_{i=0}^{m} A_{m-i}^{(\gamma-1)} A_i^{(\alpha)} |\tau_{i,n}^{(\alpha,\beta)}| \right)^{\lambda} \leqq$$

$$\leqq \sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1} n^{\lambda v-1} \left(\frac{1}{A_m^{(\alpha+\gamma)}} \right)^{\lambda} \Big(\sum_{i=0}^{m} A_{m-i}^{(\gamma-1)} A_i^{(\alpha)} |\tau_{i,n}^{(\alpha,\beta)}|^{\lambda}\Big) \Big(\sum_{i=0}^{m} A_{m-i}^{(\gamma-1)} A_i^{(\alpha)}\Big)^{\lambda/\lambda'} \leqq$$

$$(21)\qquad \leqq K \sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1} n^{\lambda v-1} m^{-\alpha-\gamma} \sum_{i=1}^{m} (m-i+1)^{\gamma-1} i^{\alpha} |\tau_{i,n}^{(\alpha,\beta)}|^{\lambda} =$$

$$= K \sum_{n=1}^{N} n^{\lambda v-1} \sum_{i=1}^{M} i^{\alpha} |\tau_{i,n}^{(\alpha,\beta)}|^{\lambda} \sum_{m=i}^{M} m^{\lambda u-\alpha-\gamma-1} (m-i+1)^{\gamma-1} \leqq$$

$$\leqq K \sum_{n=1}^{N}\sum_{i=1}^{M} i^{\lambda u-1} n^{\lambda v-1} |\tau_{i,n}^{(\alpha,\beta)}|^{\lambda},$$

because a routine calculation gives that if $\gamma>0$, then for any $M=1, 2, \dots$

$$\sum_{m=i}^{M} m^{\lambda u-\alpha-\gamma-1} (m-i+1)^{\gamma-1} \leqq$$

$$\leqq \sum_{m=i}^{2i} m^{\lambda u-\alpha-\gamma-1} (m-i+1)^{\gamma-1} + \sum_{m=2i}^{\infty} m^{\lambda u-\alpha-\gamma-1} (m-i+1)^{\gamma-1} \leqq K i^{\lambda u-1-\alpha}.$$

A similar method can be used if $\delta>0$. In the case $\lambda=1$, we prove (8) in the same way, omitting the last factor in (21).

Proof of Theorem 2. Inequalities (9) and (10) follow directly from FLETT's result ([3], Theorem 1) by similar arguments to the proof of (6) and (7). Turning to the proof of (11), we denote by S the sum of the series on the right side of (11).

In the case i), $\delta=1/\lambda-1/\mu$, by (19) we have

$$|\tau_{m,n}^{(\alpha+\delta,\beta+\delta)}| \leqq \frac{1}{A_m^{(\alpha+\delta)} A_n^{(\beta+\delta)}} \sum_{i=0}^{m}\sum_{k=0}^{n} A_{m-i}^{(\delta-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} |\tau_{i,k}^{(\alpha,\beta)}| \leqq$$

$$\leqq \frac{1}{A_m^{(\alpha+\delta)}} \frac{1}{A_n^{(\beta+\delta)}} \Big(\sum_{i=0}^{m/2}\sum_{k=0}^{n/2} + \sum_{i=0}^{m/2}\sum_{k=n/2}^{n} + \sum_{i=m/2}^{m}\sum_{k=0}^{n/2} + \sum_{i=m/2}^{m}\sum_{k=n/2}^{n} \Big) \equiv$$

$$\equiv T_{m,n}^{(1)} + T_{m,n}^{(2)} + T_{m,n}^{(3)} + T_{m,n}^{(4)}.$$

By (20) we have

$$T_{m,n}^{(1)} = \frac{1}{A_m^{(\alpha+\delta)}} \frac{1}{A_n^{(\beta+\delta)}} \sum_{i=0}^{m/2} \sum_{k=0}^{n/2} A_{m-i}^{(\delta-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} |\tau_{i,k}^{(\alpha,\beta)}| \leqq$$

$$\leqq K \frac{1}{A_m^{(\alpha+1)}} \frac{1}{A_n^{(\beta+1)}} \sum_{i=1}^{m} \sum_{k=1}^{n} A_i^{(\alpha)} A_k^{(\beta)} |\tau_{i,k}^{(\alpha,\beta)}|.$$

Let ω be a number such that

$$\max(-\alpha - 1/\lambda + u,\ -\beta - 1/\lambda + v) < \omega < (\lambda - 1)/\lambda.$$

A routine calculation gives that

$$\text{(22)} \qquad \Big\{\sum_{i=1}^{m} \sum_{k=1}^{n} i^{-\omega\lambda/(\lambda-1)} k^{-\omega\lambda/(\lambda-1)}\Big\}^{(\lambda-1)/\lambda} \leqq K m^{-\omega+(\lambda-1)/\lambda} n^{-\omega+(\lambda-1)/\lambda}.$$

Applying the Hölder inequality with indices μ, $\lambda/(\lambda-1)$, $\mu\lambda/(\mu-\lambda)$, we obtain that

$$T_{m,n}^{(1)} \leqq K m^{-\alpha-1} n^{-\beta-1} \sum_{i=1}^{m} \sum_{k=1}^{n} \{i^{\alpha+\omega-(\lambda u-1)(\mu-\lambda)/\lambda\mu} k^{\beta+\omega-(\lambda v-1)(\mu-\lambda)/\lambda\mu} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda/\mu}\} \times$$

$$\times \{i^{-\omega} k^{-\omega}\} \{i^{(\lambda u-1)(\mu-\lambda)/\lambda\mu} k^{(\lambda v-1)(\mu-\lambda)/\lambda\mu} |\tau_{i,k}^{(\alpha,\beta)}|^{(\mu-\lambda)/\mu}\} \leqq$$

$$\leqq K m^{-\alpha-1} n^{-\beta-1} \Big\{\sum_{i=1}^{m} \sum_{k=1}^{n} i^{\alpha\mu+\omega\mu-(\lambda u-1)(\mu-\lambda)/\lambda} k^{\beta\mu+\omega\mu-(\lambda v-1)(\mu-\lambda)/\lambda} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda} \times$$

$$\times \Big\{\sum_{i=1}^{m} \sum_{k=1}^{n} i^{-\omega\lambda/(\lambda-1)} k^{-\omega\lambda/(\lambda-1)}\Big\}^{(\lambda-1)/\lambda} \Big\{\sum_{i=1}^{m} \sum_{k=1}^{n} i^{\lambda u-1} k^{\lambda v-1} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{(\mu-\lambda)/\lambda\mu} \leqq$$

$$\leqq K S^{(\mu-\lambda)/\lambda\mu} m^{-\alpha-\omega-1/\lambda} n^{-\beta-\omega-1/\lambda} \Big\{\sum_{i=1}^{m} \sum_{k=1}^{n} i^{\alpha\mu+\omega\mu-(\lambda u-1)(\mu-\lambda)/\lambda} \times$$

$$\times k^{\beta\mu+\omega\mu-(\lambda v-1)(\mu-\lambda)/\lambda} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\mu},$$

whence

$$(T_{m,n}^{(1)})^{\mu} \leqq K S^{(\mu-\lambda)/\lambda} m^{-\alpha\mu-\omega\mu-\mu/\lambda} n^{-\beta\mu-\omega\mu-\mu/\lambda} \sum_{i=1}^{m} \sum_{k=1}^{n} i^{\lambda u-1} k^{\lambda v-1} \times$$

$$\times |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda} i^{(\alpha\lambda\mu+\omega\lambda\mu-u\lambda\mu+\mu)/\lambda} k^{(\beta\lambda\mu+\omega\lambda\mu-v\lambda\mu+\mu)/\lambda},$$

and for any $M, N = 1, 2, \ldots$

$$\sum_{m=1}^{M} \sum_{n=1}^{N} m^{\mu u-1} n^{\mu v-1} (T_{m,n}^{(1)})^{\mu} \leqq K S^{(\mu-\lambda)/\mu} \sum_{i=1}^{M} \sum_{k=1}^{N} i^{\lambda u-1} k^{\lambda v-1} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda} \times$$

$$\times i^{(\alpha\lambda\mu+\omega\lambda\mu-u\lambda\mu+\mu)/\lambda} k^{(\beta\lambda\mu+\omega\lambda\mu-v\lambda\mu+\mu)/\lambda} \times$$

$$\times \sum_{m=i}^{M} \sum_{n=k}^{N} m^{-\alpha\mu-\omega\mu-\mu/\lambda+\mu u-1} n^{-\beta\mu-\omega\mu-\mu/\lambda+\mu v-1} \leqq K S^{\mu/\lambda},$$

because, with standard computation,

$$i^{(\alpha\lambda\mu+\omega\lambda\mu-u\lambda\mu+\mu)/\lambda}\sum_{m=i}^{M} m^{-\alpha\mu-\omega\mu-\mu/\lambda+\mu u-1} \leqq K. \tag{23}$$

$$T_{m,n}^{(2)} = \frac{1}{A_m^{(\alpha+\delta)}}\frac{1}{A_n^{(\beta+\delta)}}\sum_{i=0}^{m/2}\sum_{k=n/2}^{n} A_{m-i}^{(\delta-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} |\tau_{i,k}^{(\alpha,\beta)}| \leqq$$

$$\leqq K\frac{1}{A_m^{(\alpha+1)}}\sum_{i=0}^{m/2}\sum_{k=n/2}^{n} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} k^{-\delta} |\tau_{i,k}^{(\alpha,\beta)}|$$

and

$$n^{v-1/\mu}T_{m,n}^{(2)} \leqq K\frac{1}{A_m^{(\alpha+1)}}\sum_{i=0}^{m/2}\sum_{k=n/2}^{n} (n-k+1)^{\delta-1} A_i^{(\alpha)} k^{v-1/\lambda} |\tau_{i,k}^{(\alpha,\beta)}| \leqq$$

$$\leqq K\frac{1}{A_m^{(\alpha+1)}}\sum_{i=0}^{m}\sum_{k=0}^{n-1} (n-k)^{\delta-1} A_i^{(\alpha)} (k+1)^{v-1/\lambda} |\tau_{i,k}^{(\alpha,\beta)}| +$$

$$+K\frac{1}{A_m^{(\alpha+1)}}\sum_{i=0}^{m} A_i^{(\alpha)} n^{v-1/\lambda} |\tau_{i,n}^{(\alpha,\beta)}| \equiv T_{m,n}^{(2,1)} + T_{m,n}^{(2,2)}.$$

Applying Lemma 2, with the single sequence

$$d_k^{(m)} = (k+1)^{v-1/\lambda}\sum_{i=1}^{m} A_i^{(\alpha)} |\tau_{i,k}^{(\alpha,\beta)}|,$$

we obtain that for any $N=1, 2, \dots$

$$\sum_{n=1}^{N} (T_{m,n}^{(2,1)})^{\mu} \leqq Km^{-\mu\alpha-\mu}\sum_{n=0}^{N}\Big(\sum_{k=0}^{n-1} (n-k)^{\delta-1} d_k^{(m)}\Big)^{\mu} \leqq$$

$$\leqq Km^{-\mu\alpha-\mu}\Big\{\sum_{k=0}^{N} (k+1)^{\lambda v-1}\Big(\sum_{i=1}^{m} A_i^{(\alpha)} |\tau_{i,k}^{(\alpha,\beta)}|\Big)^{\lambda}\Big\}^{\mu/\lambda}.$$

Applying the Hölder inequality with indices λ and $\lambda/(\lambda-1)$, by (22) we have

$$\sum_{i=1}^{m} A_i^{(\alpha)} |\tau_{i,k}^{(\alpha,\beta)}| \leqq K\sum_{i=0}^{m} \{i^{\alpha+\omega} |\tau_{i,k}^{(\alpha,\beta)}|\}\{i^{-\omega}\} \leqq$$

$$\leqq K\Big\{\sum_{i=1}^{m} i^{\lambda\alpha+\lambda\omega} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda}\Big\{\sum_{i=1}^{m} i^{-\omega\lambda/(\lambda-1)}\Big\}^{(\lambda-1)/\lambda} \leqq \tag{24}$$

$$\leqq Km^{-\omega+(\lambda-1)/\lambda}\Big\{\sum_{i=1}^{m} i^{\lambda\alpha+\lambda\omega} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{1/\lambda},$$

and, by Hölder's inequality with indices μ/λ and $\mu/(\mu-\lambda)$ we have

$$\sum_{n=1}^{N}(T_{m,n}^{(2,1)})^{\mu} \leqq Km^{-\mu\alpha-\mu-\omega\mu+(\lambda-1)\mu/\lambda}\Big\{\sum_{k=0}^{N}\sum_{i=1}^{m}(k+1)^{\lambda v-1}i^{\lambda\alpha+\lambda\omega}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}^{\mu/\lambda} =$$

$$(25) \qquad = Km^{-\mu\alpha-\mu-\omega\mu+(\lambda-1)\mu/\lambda}\Big[\sum_{k=0}^{N}\sum_{i=1}^{m}\{(k+1)^{(\lambda v-1)\lambda/\mu}i^{\lambda\alpha+\lambda\omega-(\lambda u-1)(\mu-\lambda)/\mu}\times$$

$$\times|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda^2/\mu}\}\{(k+1)^{(\lambda v-1)(\mu-\lambda)/\mu}i^{(\lambda u-1)(\mu-\lambda)/\mu}|\tau_{i,k}^{(\alpha,\beta)}|^{(\mu-\lambda)\lambda/\mu}\}\Big]^{\mu/\lambda} \leqq$$

$$\leqq Km^{-\mu\alpha-\mu-\omega\mu+(\lambda-1)\mu/\lambda}S^{(\mu-\lambda)/\mu}\sum_{k=0}^{N}\sum_{i=1}^{m}(k+1)^{\lambda v-1}i^{\alpha\mu+\omega\mu-(\lambda u-1)(\mu-\lambda)/\lambda}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}.$$

Finally, by using (23), for any $M=1,2,\ldots$

$$\sum_{m=1}^{M}\sum_{n=1}^{N}m^{\mu u-1}(T_{m,n}^{(2,1)})^{\mu} \leqq KS^{(\mu-\lambda)/\lambda}\sum_{k=0}^{N}\sum_{i=1}^{M}i^{\lambda u-1}(k+1)^{\lambda v-1}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\times$$

$$\times i^{\alpha\mu+\omega\mu-u\mu+\mu/\lambda}\sum_{m=i}^{M}m^{-\alpha\mu-\omega\mu-\mu/\lambda+\mu u-1} \leqq KS^{\mu/\lambda}.$$

By (24) we obtain that for any $N=1,2,\ldots$

$$\sum_{n=1}^{N}(T_{m,n}^{(2,2)})^{\mu} = Km^{-\mu\alpha-\mu}\sum_{n=1}^{N}n^{(v-1/\lambda)\mu}\Big(\sum_{i=1}^{m}A_i^{(\alpha)}|\tau_{i,n}^{(\alpha,\beta)}|\Big)^{\mu} \leqq$$

$$\leqq Km^{-\mu\alpha-\mu-\omega\mu+(\lambda-1)\mu/\lambda}\sum_{n=1}^{N}n^{(v-1/\lambda)\mu}\Big(\sum_{i=1}^{m}i^{\lambda\alpha+\lambda\omega}|\tau_{i,n}^{(\alpha,\beta)}|^{\lambda}\Big)^{\mu/\lambda}.$$

It is known that if $a_i \geqq 0$ and $0<p\leqq 1$ then

$$(26) \qquad (a_1+a_2+\ldots+a_k)^p \leqq a_1^p+a_2^p+\ldots+a_k^p,$$

whence

$$\sum_{n=1}^{N}(T_{m,n}^{(2,2)})^{\mu} \leqq Km^{-\mu\alpha-\mu-\omega\mu+(\lambda-1)\mu/\lambda}\Big\{\sum_{n=1}^{N}\sum_{i=1}^{m}n^{\lambda v-1}i^{\lambda\alpha+\lambda\omega}|\tau_{i,n}^{(\alpha,\beta)}|^{\lambda}\Big\}^{\mu/\lambda},$$

and we may finish the estimate as in (25). This completes the proof of

$$\sum_{m=1}^{M}\sum_{n=1}^{N}m^{\mu u-1}n^{\mu v-1}(T_{m,n}^{(2)})^{\mu} \leqq KS^{\mu/\lambda},$$

and, by similar arguments, we have that

$$\sum_{m=1}^{M}\sum_{n=1}^{N}m^{\mu u-1}n^{\mu v-1}(T_{m,n}^{(3)})^{\mu} \leqq KS^{\mu/\lambda}.$$

Now let us consider $T_{m,n}^{(4)}$.

$$T_{m,n}^{(4)} = \frac{1}{A_m^{(\alpha+\delta)} A_n^{(\beta+\delta)}} \sum_{i=m/2}^{m} \sum_{k=n/2}^{n} A_{m-i}^{(\delta-1)} A_{n-k}^{(\delta-1)} A_i^{(\alpha)} A_k^{(\beta)} |\tau_{i,k}^{(\alpha,\beta)}| \leqq$$

$$\leqq K \sum_{i=m/2}^{m} \sum_{k=n/2}^{n} i^{-\delta} k^{-\delta} A_{m-i}^{(\delta-1)} A_{n-k}^{(\delta-1)} |\tau_{i,k}^{(\alpha,\beta)}|,$$

and

$$m^{u-1/\mu} n^{v-1/\mu} T_{m,n}^{(4)} \leqq K \sum_{i=m/2}^{m-1} \sum_{k=n/2}^{n-1} (m-i)^{\delta-1}(n-k)^{\delta-1} i^{u-1/\lambda} k^{v-1/\lambda} |\tau_{i,k}^{(\alpha,\beta)}| +$$

$$+ K n^{v-1/\lambda} \sum_{i=m/2}^{m-1} (m-i)^{\delta-1} i^{u-1/\lambda} |\tau_{i,n}^{(\alpha,\beta)}| + K m^{u-1/\lambda} \sum_{k=n/2}^{n-1} (n-k)^{\delta-1} k^{v-1/\lambda} |\tau_{m,k}^{(\alpha,\beta)}| +$$

$$+ K m^{u-1/\lambda} n^{v-1/\lambda} |\tau_{m,n}^{(\alpha,\beta)}| \equiv T_{m,n}^{(4,1)} + T_{m,n}^{(4,2)} + T_{m,n}^{(4,3)} + T_{m,n}^{(4,4)}.$$

Applying Lemma 3, with the double sequence

$$d_{i,k} = (i+1)^{u-1/\lambda}(k+1)^{v-1/\lambda} |\tau_{i,k}^{(\alpha,\beta)}|$$

we obtain that for any $M, N = 1, 2, \ldots$

$$\sum_{m=1}^{M} \sum_{n=1}^{N} (T_{m,n}^{(4,1)})^\mu \leqq K \sum_{m=1}^{M} \sum_{n=1}^{N} \Big(\sum_{i=0}^{m-1} \sum_{k=0}^{n-1} (m-i)^{\delta-1}(n-k)^{\delta-1} d_{i,k} \Big)^\mu \leqq$$

$$\leqq K \Big(\sum_{i=1}^{M} \sum_{k=1}^{N} (i+1)^{\lambda u-1}(k+1)^{\lambda v-1} |\tau_{i,k}^{(\alpha,\beta)}|^\lambda \Big)^{\mu/\lambda} \leqq K S^{\mu/\lambda}.$$

Applying Lemma 2, with the single sequence

$$d_i^{(n)} = (i+1)^{u-1/\lambda} |\tau_{i,n}^{(\alpha,\beta)}|$$

we obtain that for any $M = 1, 2, \ldots$

$$\sum_{m=1}^{M} (T_{m,n}^{(4,2)})^\mu \leqq K n^{v\mu-\mu/\lambda} \sum_{m=1}^{M} \Big(\sum_{i=0}^{m-1} (m-i)^{\delta-1} d_i^{(n)} \Big)^\mu \leqq$$

$$\leqq K n^{v\mu-\mu/\lambda} \Big(\sum_{i=1}^{M} (i+1)^{\lambda u-1} |\tau_{i,n}^{(\alpha,\beta)}|^\lambda \Big)^{\mu/\lambda},$$

and, by (26), for any $N = 1, 2, \ldots$

$$\sum_{m=1}^{M} \sum_{n=1}^{N} (T_{m,n}^{(4,2)})^\mu \leqq K \Big\{ \sum_{i=1}^{M} \sum_{n=1}^{N} (i+1)^{\lambda u-1} n^{\lambda v-1} |\tau_{i,n}^{(\alpha,\beta)}|^\lambda \Big\}^{\mu/\lambda} \leqq K S^{\mu/\lambda}.$$

By similar arguments we have

$$\sum_{m=1}^{M} \sum_{n=1}^{N} (T_{m,n}^{(4,3)})^\mu \leqq K S^{\mu/\lambda}$$

and, finally, using (26) again

$$\sum_{m=1}^{M}\sum_{n=1}^{N}(T_{m,n}^{(4,4)})^{\mu} \leqq K\sum_{m=1}^{M}\sum_{n=1}^{N} m^{u\mu-\mu/\lambda} n^{v\mu-\mu/\lambda}|\tau_{m,n}^{(\alpha,\beta)}|^{\mu} \leqq$$

$$\leqq K\Big(\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1} n^{\lambda v-1}|\tau_{m,n}^{(\alpha,\beta)}|^{\lambda}\Big)^{\mu/\lambda} \leqq KS^{\mu/\lambda}.$$

Estimates for $i=1, 2, 3, 4$

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\mu u-1} n^{\mu v-1}(T_{m,n}^{(i)})^{\mu} \leqq KS^{\mu/\lambda}$$

complete the proof of (11) in the case i), for $\delta=1/\lambda-1/\mu$.

In the case ii) for $\delta>1/\lambda-1/\mu=1-1/\mu$, by (19), we have that for any $M, N=$ $=1, 2, \ldots$

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\mu u-1} n^{\mu v-1}|\tau_{m,n}^{(\alpha+\delta,\beta+\delta)}|^{\mu} \leqq K\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\mu u-1-\mu\alpha-\mu\delta} n^{\mu v-1-\mu\beta-\mu\delta}\times$$

$$\times\Big\{\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{\delta-1}(n-k+1)^{\delta-1} i^{\alpha} k^{\beta}|\tau_{i,k}^{(\alpha,\beta)}|\Big\}^{\mu}.$$

Applying Hölder's inequality with indices μ and $\mu/(\mu-1)$ we obtain that

$$\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{\delta-1}(n-k+1)^{\delta-1} i^{\alpha} k^{\beta}|\tau_{i,k}^{(\alpha,\beta)}| =$$

$$=\sum_{i=1}^{m}\sum_{k=1}^{n}\{(m-i+1)^{\delta-1}(n-k+1)^{\delta-1} i^{\alpha-(u-1)(\mu-1)/\mu} k^{\beta-(v-1)(\mu-1)/\mu}|\tau_{i,k}^{(\alpha,\beta)}|^{1/\mu}\}\times$$

$$\times\{i^{(u-1)(\mu-1)/\mu} k^{(v-1)(\mu-1)/\mu}|\tau_{i,k}^{(\alpha,\beta)}|^{(\mu-1)/\mu}\} \leqq$$

$$\leqq\Big\{\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{(\delta-1)\mu}(n-k+1)^{(\delta-1)\mu} i^{\alpha\mu-(u-1)(\mu-1)} k^{\beta\mu-(v-1)(\mu-1)}|\tau_{i,k}^{(\alpha,\beta)}|\Big\}^{1/\mu}\times$$

$$\times\Big\{\sum_{i=1}^{m}\sum_{k=1}^{n} i^{u-1} k^{v-1}|\tau_{i,k}^{(\alpha,\beta)}|\Big\}^{(\mu-1)/\mu},$$

and

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\mu u-1} n^{\mu v-1}|\tau_{m,n}^{(\alpha+\delta,\beta+\delta)}|^{\mu} \leqq$$

$$KS^{\mu-1}\sum_{i=1}^{M}\sum_{k=1}^{N} i^{u-1} k^{v-1}|\tau_{i,k}^{(\alpha,\beta)}|\, i^{\alpha\mu-u\mu+\mu} k^{\beta\mu-v\mu+\mu}\times$$

$$\times\sum_{m=i}^{M}\sum_{n=k}^{N}(m-i+1)^{\delta\mu-\mu}(n-k+1)^{\delta\mu-\mu} m^{\mu u-1-\mu\alpha-\mu\delta} n^{\mu v-1-\mu\beta-\mu\delta} \leqq KS^{\mu},$$

since

$$(27)\qquad i^{\alpha\mu-u\mu+\mu}\Big(\sum_{m=i}^{2i}+\sum_{m=2i}^{\infty}\Big)(m-i+1)^{\delta\mu-\mu}m^{\mu u-1-\mu\alpha-\mu\delta}\leqq$$

$$\leqq Ki^{\mu-\delta\mu-1}\sum_{m=1}^{i}m^{\delta\mu-\mu}+Ki^{\alpha\mu-u\mu+\mu}\sum_{m=i}^{\infty}m^{-\mu+\mu u-1-\mu\alpha}\leqq K.$$

Proof of Theorem 3. Inequalities (12) and (13) follow directly from Flett's result ([3], Theorem 3), by similar arguments to the proof of (6) and (7). In the proof of (14), considering Theorem 1, we may assume that $\gamma=\xi-u<0$ and $\delta=\eta-v<0$.

Let $\lambda>1$. Using (19), we have that

$$m^{\gamma}n^{\delta}|\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|\leqq Km^{-\alpha}n^{-\beta}\sum_{i=1}^{m}\sum_{k=1}^{n}|A_{m-i}^{(\gamma-1)}||A_{n-k}^{(\delta-1)}|i^{\alpha}k^{\beta}|\tau_{i,k}^{(\alpha,\beta)}|.$$

Let ω be a number such that

$$\max(-\alpha-1/\lambda+u,\ -\beta-1/\lambda+v)<\omega<(\lambda-1)/\lambda.$$

Applying Hölder's inequality with indices λ, $\lambda/(\lambda-1)$ we obtain that

$$m^{\lambda\gamma}n^{\lambda\delta}|\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda}\leqq Km^{-\alpha\lambda}n^{-\beta\lambda}\Big\{\sum_{i=1}^{m}\sum_{k=1}^{n}|A_{m-i}^{(\gamma-1)}||A_{n-k}^{(\delta-1)}|i^{\lambda\alpha+\lambda\omega}k^{\lambda\beta+\lambda\omega}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda}\Big\}\times$$

$$\times\Big\{\sum_{i=1}^{m}\sum_{k=1}^{n}|A_{m-i}^{(\gamma-1)}||A_{n-k}^{(\delta-1)}|i^{-\omega\lambda/(\lambda-1)}k^{-\omega\lambda/(\lambda-1)}\Big\}^{(\lambda-1)}.$$

A routine calculation gives that

$$\sum_{i=1}^{m}|A_{m-i}^{(\gamma-1)}|i^{-\omega\lambda/(\lambda-1)}\leqq K\sum_{i=1}^{m}(m-i+1)^{\gamma-1}i^{-\omega\lambda/(\lambda-1)}\leqq K\Big(\sum_{i=1}^{m/2}+\sum_{i=m/2}^{m}\Big\}\leqq$$

$$\leqq Km^{\gamma-1}\sum_{i=1}^{m/2}i^{-\omega\lambda/(\lambda-1)}+Km^{-\omega\lambda/(\lambda-1)}\sum_{i=m/2}^{\infty}(m-i+1)^{\gamma-1}\leqq Km^{\gamma-\omega\lambda/(\lambda-1)},$$

whence

$$(28)\qquad m^{\lambda\gamma}n^{\lambda\delta}|\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda}\leqq Km^{-\alpha\lambda-\omega\lambda}n^{-\beta\lambda-\omega\lambda}\times$$

$$\times\sum_{i=1}^{m}\sum_{k=1}^{n}(m-i+1)^{\gamma-1}(n-k+1)^{\delta-1}i^{\lambda\alpha+\lambda\omega}k^{\lambda\beta+\lambda\omega}|\tau_{i,k}^{(\alpha,\beta)}|^{\lambda},$$

and for any $M, N=1, 2, \ldots$

$$\sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda(u+\gamma)-1} n^{\lambda(v+\delta)-1} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq$$

$$\leqq K \sum_{m=1}^{M}\sum_{n=1}^{N} m^{\lambda u-1-\alpha\lambda-\omega\lambda} n^{\lambda v-1-\beta\lambda-\omega\lambda} \times$$

$$\times \sum_{i=1}^{m}\sum_{k=1}^{n} (m-i+1)^{\gamma-1}(n-k+1)^{\delta-1} i^{\lambda\alpha+\lambda\omega} k^{\lambda\beta+\lambda\omega} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda} =$$

$$= K \sum_{i=1}^{M}\sum_{k=1}^{N} i^{\lambda u-1} k^{\lambda v-1} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda} i^{\lambda\alpha+\lambda\omega-\lambda u+1} k^{\lambda\beta+\lambda\omega-\lambda v+1} \times$$

$$\times \sum_{m=i}^{M}\sum_{n=k}^{N} (m-i+1)^{\gamma-1}(n-k+1)^{\delta-1} n^{\lambda u-1-\alpha\lambda-\omega\lambda} n^{\lambda v-1-\alpha\lambda-\omega\lambda} \leqq$$

$$\leqq K \sum_{i=1}^{M}\sum_{k=1}^{N} i^{\lambda u-1} k^{\lambda v-1} |\tau_{i,k}^{(\alpha,\beta)}|^{\lambda},$$

because

$$i^{\lambda\alpha+\lambda\omega-\lambda u+1}\Big(\sum_{m=i}^{2i-1} + \sum_{m=2i}^{M}\Big)\big((m-i+1)^{\gamma-1} m^{\lambda u-1-\alpha\lambda-\omega\lambda}\big) \leqq$$

$$\leqq K \sum_{m=i}^{2i-1} (m-i+1)^{\gamma-1} + K i^{\lambda\alpha+\lambda\omega-\lambda u+1} \sum_{m=2i}^{\infty} (m-i+1)^{\lambda u-2-\alpha\lambda-\omega\lambda+\gamma} \leqq \tag{29}$$

$$\leqq K \sum_{m=1}^{i} m^{\gamma-1} + K i^{\lambda\alpha+\lambda\omega-\lambda u+1} \sum_{m=i+1}^{\infty} m^{\lambda u-2-\alpha\lambda-\omega\lambda+\gamma} \leqq K.$$

In the case of $\lambda=1$ we set $\omega=0$ and the inequality (28) remains valid and we obtain (29) in this case, too, so our proof is complete.

Proof of Theorem 4. Inequalities (15) and (16) follow directly from Flett's result ([3], Theorem 4), by similar arguments to the proof of (6) and (7). In the proof of (17), thinking about Theorem 1, we may assume that $\gamma, \delta<0$. Using (14) we have that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda(\gamma+u)-1} n^{\lambda(\delta+v)-1} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda} \leqq K \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda u-1} n^{\lambda v-1} |\tau_{m,n}^{(\alpha,\beta)}|^{\lambda}. \tag{30}$$

Applying Hölder's inequality with indices λ/μ and $\lambda/(\lambda-\mu)$, we obtain that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\mu\xi-1} n^{\mu\eta-1} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\mu} =$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \{m^{\mu(\gamma+u-1/\lambda)} n^{\mu(\delta+v-1/\lambda)} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\mu}\} \times$$

$$\times \{m^{-1-\mu(\gamma+u-\xi-1/\lambda)} n^{-1-\mu(\delta+v-\eta-1/\lambda)}\} \leqq$$

$$\leqq \Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{\lambda(\gamma+u)-1} n^{\lambda(\delta+v)-1} |\tau_{m,n}^{(\alpha+\gamma,\beta+\delta)}|^{\lambda}\Big\}^{\mu/\lambda} \times$$

$$\times \Big\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} m^{-1-\lambda\mu(\gamma+u-\xi)/(\lambda-\mu)} n^{-1-\lambda\mu(\delta+v-\eta)/(\lambda-u)}\Big\}^{(\lambda-\mu)/\lambda}.$$

The last factor is bounded, because $\gamma+u-\xi$, $\delta+v-\eta$ are positive, and inequality (17) follows from this and (30).

Proof of Theorem 5. We can observe that if the conditions are satisfied, applying Theorem 4, we obtain that the summability $|C, (\alpha, \beta), (u, v)|_{\lambda}$ of series (1) implies the summability $|C, (\gamma, \delta), (0, 0)|_1$. Now Theorem 5 follows from Lemma 4.

4. Negative results. First we show that Theorem 2 is the best possible in the following sense:

a) If $\mu>\lambda>1$ and $\min(\gamma, \delta)<1/\lambda-1/\mu$, then for any $\xi, \eta\geqq 0$ the summability $|C, (\alpha, \beta), (u, v)|_{\lambda}$ does not imply the summability $|C, (\alpha+\gamma, \beta+\delta), (\xi, \eta)|_{\mu}$.

Without loss of generality, we can assume that $\gamma<1/\lambda-1/\mu$ and $0\leqq u, v\leqq 1/\lambda$. Applying Lemma 5, let $\sum_{i=0}^{\infty} c_i$ be a single series, such that $\tau_m^{(\alpha)}=m^{1/p}$ if $m=2^{\nu}$ and $\tau_m^{(\alpha)}=0$ otherwise, where $\lambda/(1-u\lambda)<p<\lambda$ and $u>0$. The series $\sum_{i=0}^{\infty} c_i$ is summable $|C, \alpha, u|_{\lambda}$, since

$$\sum_{m=1}^{\infty} m^{\lambda u-1}|\tau_m^{(\alpha)}|^{\lambda} = \sum_{\nu=0}^{\infty} 2^{\nu(u\lambda-1-\lambda/p)} < \infty,$$

but not summable $|C, \alpha+\gamma, 0|_{\mu}$, since

$$\sum_{m=1}^{\infty} m^{-1-\mu\gamma}|\tau_m^{(\alpha)}|^{\mu} = \sum_{\nu=0}^{\infty} 2^{\nu(-1-\mu\gamma+\mu/p)} = \infty,$$

and we may use Corollary 1. Thus the assertion is proved, because it is clear that the summability $|C, (\alpha+\gamma, \beta+\delta), (\xi, \eta)|_{\mu}$ implies the summability $|C, (\alpha+\gamma, \beta+\delta), (0, 0)|_{\mu}$. In the case $u=0$, the assertion was proved by Flett ([2], part 2.7).

b) If $\mu>\lambda=1$ and $\min(\gamma, \delta)\leqq 1-1/\mu$ then for any $\xi, \eta\geqq 0$ the summability $|C, (\alpha, \beta), (u, v)|_1$ does not imply the summability $|C, (\alpha, \beta), (\xi, \eta)|_\mu$.

Without loss of generality, we can assume that $\gamma=1-1/\mu$ and $\xi=0$. If $u>0$, the proof is carried out analogously to the proof of preceding assertion. In the case $u=0$, by using Lemma 5, let $\sum_{i=0}^{\infty} c_i$ be a single series such that $\tau_m^{(\alpha)}=p^{-2}l_p$ if $m=l_p=2^{2^p}$ and $\tau_m^{(\alpha)}=0$ otherwise. It is clear, that

$$\sum_{m=1}^{\infty} m^{-1}|\tau_m^{(\alpha)}| = \sum_{p=1}^{\infty} p^{-2} < \infty,$$

so the series $\sum_{i=0}^{\infty} c_i$ is summable $|C, \alpha, 0|_1$. On the other hand, from (18), with the notation $n=l_p+t$, $0\leqq t\leqq l_p$

$$\tau_n^{(\alpha+\gamma)} \geqq \frac{1}{A_n^{(\alpha+\gamma)}} A_{n-l_p}^{(-1/\mu)} A_{l_p}^{(\alpha)} \tau_{l_p}^{(\alpha)} \geqq K(t+1)^{-1/\mu} l_p^{1/\mu} p^{-2}$$

and

$$\sum_{n=1}^{\infty} n^{-1}|\tau_n^{(\alpha+\gamma)}|^\mu \geqq \sum_{p=1}^{\infty} \sum_{n=l_p}^{2l_p} n^{-1}\tau_n^{(\alpha+\gamma)} \geqq K \sum_{p=1}^{\infty} \sum_{t=0}^{l_p} (l_p+t)^{-1}(t+1)^{-1} l_p p^{-2\mu} \geqq$$

$$\geqq K \sum_{p=1}^{\infty} p^{-2\mu} \sum_{t=0}^{l_p} (t+1)^{-1} \leqq K \sum_{p=1}^{\infty} 2^p p^{-2\mu} = \infty,$$

and therefore this series is not summable $|C, \alpha+\gamma, 0|_\mu$. We remark that this example is due to Flett [4], in connection with strong summability $[C, \alpha]_\lambda$.

Now we investigate the parameters u, v. The following result shows that the parameters u, v cannot be increased. (It is clear that ones can be decreased.)

c) If $\lambda, \mu\geqq 1$, $u, v\geqq 0$, $\alpha>\mu-1$, $\beta>v-1$ and $\xi>u$ or $\eta>v$ then for any $\alpha_1, \beta_1 (\alpha_1>\xi-1, \beta_1>\eta-1)$ the summability $|C, (\alpha, \beta), (u, v)|_\lambda$ does not imply the summability $|C, (\alpha_1, \beta_1), (\xi, \eta)|_\mu$.

We can assume that $\xi>u$. Applying Lemma 5, with $c_i=i^{-p}$, where $u+1<p<\xi+1$, we obtain that the series $\sum_{i=0}^{\infty} c_i$ is summable $|C, \alpha, u|_\lambda$, since

$$\sum_{m=1}^{\infty} m^{\lambda u-1}|\tau_m^{(\alpha)}|^\lambda \leqq K \sum_{m=1}^{\infty} m^{\lambda(u+1-p)-1} < \infty,$$

but is not summable $|C, \alpha_1, \xi|_\mu$, since

$$\sum_{m=1}^{\infty} m^{\lambda\xi-1}|\tau_m^{(\alpha_1)}|^\mu \geqq K \sum_{m=1}^{\infty} m^{\mu(\xi+1-p)-1} = \infty.$$

Finally we prove that the parameter λ cannot be decreased if parameters u, v are fixed.

d) If $\lambda > \mu \geqq 1$ and $u, v \geqq 0$, then for any $\alpha, \alpha_1, \beta, \beta_1 (\alpha, \alpha_1 > u-1, \beta, \beta_1 > v-1)$ the summability $|C, (\alpha, \beta), (u, v)|_\lambda$ does not imply the summability $|C, (\alpha_1, \beta_1), (u, v)|_\mu$.

We apply Lemma 5, with a single series $\sum_{i=0}^{\infty} c_i$ such that $\tau_m^{(\alpha)} = (\log m)^{-1/p} m^{-u}$, $\tau_0^{(\alpha)} = 0$, where $\mu < p < \lambda$. Since

$$\sum_{m=1}^{\infty} m^{\lambda u-1} |\tau_m^{(\alpha)}|^\lambda = \sum_{m=1}^{\infty} m^{-1} (\log m)^{-\lambda/p} < \infty,$$

the series $\sum_{i=0}^{\infty} c_i$ is summable $|C, \alpha, u|_\lambda$. On the other hand, using (18), a routine calculation gives that

$$\tau_m^{(\alpha_1)} = \frac{1}{A_m^{(\alpha_1)}} A_{m-i}^{(\alpha_1-\alpha-1)} A_i^{(\alpha)} \tau_i^{(\alpha)} \geqq K m^{-\alpha_1} \sum_{i=0}^{m} (m-i+1)^{\alpha_1-\alpha-1} \times$$

$$\times i^\alpha (\log i)^{-1/p} i^{-u} = \sum_{i=1}^{m/2} + \sum_{i=m/2}^{m} \equiv T_m^{(1)} + T_m^{(2)},$$

and therefore

$$T_m^{(1)} \geqq K m^{-\alpha-1} (\log m)^{-1/p} \sum_{i=1}^{m/2} i^{\alpha-u} \geqq K m^{-u} (\log m)^{-1/p}$$

and

$$T_m^{(2)} \geqq K m^{-\alpha_1+\alpha-u} (\log m)^{-1/p} \sum_{i=1}^{m/2} i^{\alpha_1-\alpha-1} \geqq K m^{-u} (\log m)^{-1/p},$$

furthermore

$$\sum_{m=1}^{\infty} m^{\mu u-1} |\tau_m^{(\alpha_1)}|^\mu \geqq K \sum_{m=1}^{\infty} m^{-1} (\log m)^{-\mu/p} = \infty,$$

so the series $\sum_{i=0}^{\infty} c_i$ is not summable $|C, \alpha_1, u|_\mu$.

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