Acta Sci. Math.; 56 (1992), 47—49

On rings satisfying $(x, y, z)=(y, z, x)$

YASH PAUL

Let R be an arbitrary nonassociative ring having a Peirce decomposition, into a direct sum of Z modules, relative to nonzero idempotent e in the nucleus of R. If R satisfies the identity $(x, y, z)=(y, z, x)$ then (i) under certain conditions on the Z modules, R is associative and (ii) if R is prime then e is the identity element of R.

As usual, the associator (x, y, z) denotes $(xy)z-x(yz)$ and the commutator $(x, y)=xy-yx$. STERLING [3] has shown that semiprime rings satisfying

(1) $$(x, y, z) = (y, z, x)$$

are alternative. The nucleus $N(R)$ of an arbitrary nonassociative ring R consists of all elements n in R such that

$$(n, r, s) = 0 = (r, s, n) = (s, n, r) \quad \text{for all} \quad r, s \quad \text{in} \quad R.$$

It is well known, see SCHAFER [2] p. 18, that $N(R)$ is an associative subring of R. We need an identity valid in all rings, the so-called Teichmüller identity

(2) $$(wx, y, z)-(w, xy, z)+(w, x, yz) = w(x, y, z)+(w, x, y)z.$$

We now take turns letting one of the four elements in (2) be in the nucleus. Thus

(3) $$(nx, y, z) = n(x, y, z)$$

(4) $$(wn, y, z) = (w, ny, z)$$

(5) $$(w, xn, z) = (w, x, nz)$$

(6) $$(w, x, yn) = (w, x, y)n$$

for any element n in $N(R)$ and the rest of the elements arbitrary in R.

Received June 7, 1990 and in revised forms January 16 and July 24, 1991.

Lemma. *Let R be a ring satisfying* (1). *Then*

$$(R, N(R)) \subseteqq N(R).$$

Proof. Let x, y, z be arbitrary elements of R and n be an element of $N(R)$. Using (1), (3), (5) and (6) we get

$$n(x, y, z) = (nx, y, z) = (y, z, nx) = (y, zn, x) = (x, y, zn) = (x, y, z)n.$$

Thus

$$n(x, y, z) = (x, y, z)n. \tag{7}$$

Again using (1), (3), (6) and (7) we get

$$(nx, y, z) = n(x, y, z) = (x, y, z)n = (y, z, x)n = (y, z, xn) = (xn, y, z).$$

Thus $((x, n), y, z)=0$. This implies that $(R, N(R))\subseteqq N(R)$.

A ring R is said to have a Peirce decomposition relative to the idempotent $e\in R$ if R can be decomposed into a direct sum of Z modules R_{ij} $(i, j=0, 1)$ where

$$R_{ij} = \{x\in R\colon\ xe = jx \text{ and } ex = ix\}.$$

It is known, see Jacobson [1], that if R is an associative ring and if e is an idempotent in R then R has a Peirce decomposition relative to e. Also, if R has an identity element 1 and if we write $e_1=e$ and $e_0=1-e$ then $R_{ij}=e_iRe_j$.

Let $e\in N(R)$. Embed R into the ring $R'=Z+R$ which contains an identity element 1. Clearly, e and $1-e$ are in $N(R')$. It follows that $R=\oplus R_{ij}$ and $R_{ij}=e_iRe_j$ for $i, j=0, 1$. Thus

$$R_{ij}R_{kl} = (e_iRe_j)(e_kRe_l) = e_iR(e_ie_k)Re_l \subseteqq \delta_{jk}e_iRe_l = \delta_{jk}R_{il}$$

for $i, j, k, l=0, 1$ (δ denotes the Kronecker delta).

Theorem. *Let R be a ring satisfying* (1) *with an idempotent $e\neq 0$ in $N(R)$.*

(i) *If R satisfies the condition*

$$R_{ij}R_{ji} = R_{ii} \quad \textit{when} \quad i\neq j$$

then R is associative.

(ii) *If R is prime then e is the identity element of R.*

Proof. (i) $R_{10}=(e, R_{10})=-(R_{10}, e)$ and $R_{01}=(R_{01}, e)$. Since $e\in N(R)$, by the above Lemma, R_{10} and $R_{01}\subseteqq N(R)$. $N(R$ is an associative subring of R. So $R_{10}R_{01}$ and $R_{01}R_{10}\subseteqq N(R)$. By the given condition R_{11} and $R_{00}\subseteqq N(R)$. It follows that

$$R = R_{11}+R_{10}+R_{01}+R_{00} \subseteqq N(R).$$

Hence R is associative.

(ii) $R_{10}R_{01}+R_{01}R_{10}\subseteq N(R)$. This, together with the property $R_{ij}R_{kl}\subseteq\delta_{jk}R_{il}$, allows us to conclude that $B=R_{10}R_{01}+R_{10}+R_{01}+R_{01}R_{10}$ is an ideal of R contained in $N(R)$. All rings R have an ideal A, called the associater ideal. It is defined as the smallest ideal which contains all associators. It actually consists of all finite sums of associators and right multiples of associators. The associator ideal is never zero, except when R is associative. We shall show that $BA=(0)$. Let $b\in B$. Then using Teichmüller identity (2) we get

$$(bx, y, z)-(b, xy, z)+(b, x, yz) = b(x, y, z)+(b, x, y)z.$$

Since B is an ideal contained in $N(R)$ and $b\in B$ we have

$$(bx, y, z) = (b, xy, z) = (b, x, yz) = (b, x, y) = 0.$$

Thus, from the above equation, we get

$$b(x, y, z) = 0.$$

Also, since $b\in N(R)$,

$$b\big((x, y, z)w\big) = \big(b(x, y, z)\big)w = 0.$$

Thus we have proved that $bA=(0)$ for all b in B. Hence $BA=(0)$. But R is prime and nonassociative. This implies that $B=(0)$. So we have

$$R = R_{11}\oplus R_{00}.$$

Thus, R_{11} and R_{00} are ideals of R such that

$$R_{11}R_{00} = (0).$$

From the primeness of R again $R_{11}=(0)$ or $R_{00}=(0)$. But $0\neq e\in R_{11}$ so that $R_{11}\neq(0)$. We must have $R_{00}=(0)$. This implies that e is the identity element of R.

References

[1] N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloquium Publications 37, Amer. Math. Soc. (Providence, R.I. 1964).

[2] R. D. Schafer, *An introduction to nonassociative algebras*, Pure and Appl. Math. 22, Academic Press (New York—London, 1966).

[3] N. J. Sterling, Rings satisfying $(x, y, z)=(y, z, x)$, *Can. J. Math.*, **20** (1968), 913—918.

INDIAN INSTITUTE OF TECHNOLOGY
NEW DELHI-110016
INDIA