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On the additive groups of m-rings

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Notation.

$Z(n)$ a cyclic group of order n.
R a ring.
R^+ the additive group of R.
R_p the p-primary component of R^+, p a prime.
P_m $\{p \text{ a prime} | m \equiv 1 \pmod{p-1}, m > 1\}$, m a positive integer.

Definition. Let $m>1$ be a positive integer. A ring R is said to be an m-ring if $a^m = a$ for all $a \in R$.

Pierce [2, Corollary 12.5, and following comments] showed that an m-ring R with unity satisfies $R = \bigoplus_{p \in P_m} R_p$, with R_p of characteristic p for each $p \in P_m$. The existence of a unity in R is essential to Pierce's proof, as is the sheaf representation theory for commutative regular rings. In this note m-rings are not assumed to possess a unity. A complete description of the additive groups of m-rings will be obtained by elementary means. This classification contains the Pierce result.

Using Fermat's little theorem, and the existence of a primitive root of unity modulo p for every prime p, (see [1]), one can prove:

Lemma 1. *Let $m>1$ be a positive integer. A prime p satisfies $p | q^m - q$ for all positive integers q and m if and only if $p \in P_m$.*

Lemma 2. *Let R be a ring which does not possess non-zero nilpotent elements. Then $R_p = \bigoplus_{\alpha_p} Z(p)$ with α_p a cardinal, for every prime p.*

Proof. Let $a \in R_p$ with $|a| = p^k$. Then $(pa)^k = (p^k a) a^{k-1} = 0$, and so $k=1$.

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Theorem 3. *Let $m>1$ be a positive integer, and let G be an additive abelian group. There exists an m-ring R with $R^+=G$ if and only if*

$$G = \bigoplus_{p\in P_m} \bigoplus_{\alpha_p} Z(p)$$

with α_p an arbitrary cardinal for each $p\in P_m$.

Proof. Let R be an m-ring, $a\in R$, and $q>1$ be an arbitrary integer. Then $q^m a=q^m a^m=(qa)^m=qa$, i.e., $(q^m-q)a=0$. Therefore R^+ is a torsion group, and by Lemma 1 it follows that $R=\bigoplus_{p\in P_m} R_p$. Clearly R does not possess non-zero nilpotent elements, and so Lemma 2 yields the result.

Conversely, let $G=\bigoplus_{p\in P_m}\bigoplus_{\alpha_p} Z(p)$ with α_p an arbitrary cardinal for each $p\in P_m$. Let F_p be a field of order p. Every non-zero element $a\in F_p$ satisfies $a^{p-1}=1$. If $p\in P_m$, then $a^{m-1}=1$, and so $a^m=a$. Clearly $R=\bigoplus_{p\in P_m}\bigoplus_{\alpha_p} F_p$ is an m-ring with $R^+\cong G$.

The m-ring R with additive group $G=\bigoplus_{p\in P_m}\bigoplus_{\alpha_p} Z(p)$ is not unital if α_p is an infinite cardinal for some prime p. To construct a unital m-ring with additive group G, it clearly suffices to consider $G=\bigoplus_{\alpha} Z(p)$, with p a prime.

R. S. Pierce communicated to us the following example:

View F_p as a topological space with the discrete topology, and let X_p be the one point compactification of a discretely topologized set of cardinality α. Then $C(X_p, F_p)$, the ring of F_p-valued continuous functions, is a unital m-ring with additive group isomorphic to G.

Another example of a unital m-ring with additive group $\bigoplus_{\alpha} Z(p)$ is the following:

Let I be an index set, $|I|=\alpha$, and let $S=\prod_{|I|} F_p$, with elements of S regarded as generalized sequences $(a_i)_{i\in I}$. Let R be the subring of S consisting of $a\in S$ for which there exists a finite subset $J\subseteq I$ such that $a_i=a_j$ for all $i,j\in I\setminus J$. Clearly R is a unital m-ring, with $R^+=\bigoplus_{\alpha} Z(p)$.

An argument similar to that used in proving Theorem 3 yields:

Theorem 4. *Let R be a ring such that for every $a\in R$ there exists a positive integer $m(a)>1$, depending on a, with $a^{m(a)}=a$. Then $R^+=\bigoplus_{p\in P}\bigoplus_{\alpha_p} Z(p)$ with P a set of primes. Conversely, every such group is the additive group of a ring with the above property.*

For a different elementary approach to m-rings see [3].

References

[1] W. J. LeVeque, *Topics in Number Theory, vol. I,* Addison-Wesley (Reading, Mass., 1956).

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[3] T. Chinburg and M. Henriksen, Multiplicatively periodic rings, *Amer. Math. Monthly,* **83** (1976), 547—549.

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