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Essentially normal composition operators on L^2

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1. Preliminaries. Let (X, Σ, m) be a complete, sigma-finite measure space and let T be a Σ-measurable mapping $(T^{-1}\Sigma\subset\Sigma)$ of X into X. The *composition operator* C induced by T on the set of complex valued, measurable functions on X is defined by $Cf=f\circ T$. Throughout this article $L^2=L^2(X, \Sigma, m)$. For $S\in\Sigma, L^2(S)$ is the L^2 space of functions on S, with the appropriate restrictions of Σ and m. We will regard this space as the subspace of L^2 consisting of those functions with support in S. In general the support of the function f will be denoted S_f. For f in L^∞, M_f will denote the operator of multiplication by f on L^2. We will be concerned with those composition operators C which are bounded linear operators on L^2. A detailed description of the general properties of such operators is given in [3]. In particular, it is shown that C is a bounded operator on L^2 if and only if

(i) $m\circ T^{-1}$ is absolutely continuous with respect to m, and

(ii) $\dfrac{dm\circ T^{-1}}{dm}\in L^\infty$.

Conditions (i) and (ii) are assumed to hold throughout. We set

$$h=\frac{dm\circ T^{-1}}{dm}.$$

We will make use of the following notation. For f in L^2 or measurable and non-negative, $E(f)$ is the conditional expectation $E(f|T^{-1}\Sigma)$. For $f\in L^2$, $E(f)$ is the orthogonal projection of f onto $L^2(X, T^{-1}\Sigma, m)$. Verifications of the following properties are found in [1], [2], and [5].

(iii) $\|C\|^2=\|h\|_\infty$.

(iv) For each f there is a function F such that $E(f)=F\circ T$. If $E(f)=G\circ T$ as well, then $F=G$ on S_h. In particular the function $h\cdot[E(f)]\circ T^{-1}$ is well defined even if T is not invertible. In fact, $C^*f=h\cdot[E(f)]\circ T^{-1}$, $C^*Cf=hf$, and $CC^*f=$ $=h\circ TE(f)$.

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(v) For measurable f and g, $E((f\circ T)\cdot g)=(f\circ T)Eg$. For $f\in L^\infty$ this equation has the operator theoretic form $M_{f\circ T}E=EM_{f\circ T}$.

2. Essential normality. In [5] R. WHITLEY proved that C is normal if and only if T is invertible and bi-measurable, and $h=h\circ T$. Recall that an operator A is *essentially normal* if its image in the Calkin algebra is a normal element. Equivalently A is essentially normal if and only if A^*A-AA^* is compact. R. K. SINGH and T. VELUCHAMY ([4]) have examined the question of essential normality for certain composition operators. Their result in this regard is stated below.

Theorem. *If (X, Σ, m) is completely nonatomic, and if C is essentially normal with dense range, then C is normal.*

In this article we will develop characterizations of essentially normal composition operators. It will be shown that the dense range hypothesis in the above result is unnecessary. We first note that in the atomic case it is possible to have a non-normal, essentially normal composition operator.

2.1. Example. Let $X=\mathbf{N}=\{1, 2, \ldots\}$ and let m be the counting measure. Set $T(1)=1$ and $T(n+1)=n$. Then C is a rank one perturbation of the unilateral shift. In particular, it is an essentially normal operator with index -1, and so is not normal.

For convenience, let $D=C^*C-CC^*=M_h-M_{h\circ T}E$. We will examine D with respect to the orthogonal decomposition of L^2 as $EL^2\oplus(I-E)L^2$. We note that EL^2 consists of those L^2 functions which are $T^{-1}\Sigma$ measurable. The range of C is dense in EL^2 ([1]). Also, $(I-E)L^2$ consists of those L^2 functions f for which $\int_{T^{-1}A} f\,dm=0$ for every Σ-set A.

2.2. Lemma. *D is compact if and only if both $M_h(1-E)$ and $M_{h-h\circ T}E$ are compact.*

Proof. D is compact if and only if both DE and $D(I-E)$ are compact. But $D=M_h-M_{h\circ T}E$, so

$$DE=(M_h-M_{h\circ T}E)E=M_{h-h\circ T}E,$$

and

$$D(I-E)=M_h(I-E).$$

2.3. Corollary. *Suppose that D is compact. Then $M_{h\cdot(h-h\circ T)}$ is compact.*

Proof. $M_h(1-E)$ and $M_{h-h\circ T}E$ are compact. But

$$M_h(M_{h-h\circ T}E)^*+(M_h(I-E))M_{h-h\circ T}=M_hEM_{h-h\circ T}+M_h(I-E)M_{h-h\circ T}=$$
$$=M_hM_{h-h\circ T}=M_{h\cdot(h-h\circ T)}.$$

Write $X=X_c\cup\{a_i\colon i\in J)$ where m is completely nonatomic on X_c and $\{a_i\colon i\in J)$ consists of the atoms for m. Let $A=T^{-1}X_c$ and $A_i=T^{-1}a_i$, $i\in J$. These sets are pairwise disjoint, so that the corresponding subspaces of L^2 are orthogonal. Note that for any measurable set S, $L^2(T^{-1}S)$ is a reducing subspace for D, because if $S_f\subset T^{-1}S$, then

$$hf-h\circ TEf = hf-h\circ TE(f\chi_{T^{-1}S}) = hf-h\circ T(Ef)\chi_{T^{-1}S} = 0 \quad \text{off} \quad T^{-1}S.$$

We have established the following result.

2.4. Theorem. *C is essentially normal if and only if $D|_{L^2(A)}$ and $D|_{L^2(A_i)}$ $(i\in J)$ are compact, and*

$$\lim_{i\to\infty}\|D|_{L^2(A_i)}\| = 0.$$

This result is strengthened somewhat by Lemma 2.6 below. Its proof depends on the following fact.

2.5. Lemma. *If S is a subset of X_c with $0<m(S)<\infty$, then there is a subset A of S with*

$$\frac{1}{4}m(S) < m(A) < \frac{3}{4}m(S).$$

Proof. Suppose no such set A exists. Then for every measurable subset E of S, either $m(E)<\frac{1}{4}m(S)$ or $m(E)>\frac{3}{4}m(S)$. Let $\mathscr{E}=\left\{E\subset S\colon m(E)>\frac{3}{4}m(S)\right\}$. If E and F are in $\mathscr{E}$, then

$$m(E\cap F) = m(E)+m(F)-m(E\cup F) > \frac{1}{2}m(S).$$

Thus $E\cap F\in\mathscr{E}$. Let $\alpha=\inf\{m(E)\colon E\in\mathscr{E}\}$, and let $\{E_n\}$ be a decreasing sequence of sets in $\mathscr{E}$ whose measures converge to α. Let $G=\bigcap E_n$. Then $m(G)=\alpha\geqq\frac{3}{4}m(S)$. Now, there is a measurable subset B of G with $0<m(B)<m(G)$. But then neither B nor $G-B$ are in $\mathscr{E}$. It then follows that both B and $G-B$ must have measures less than $\frac{3}{4}m(S)$, which implies that the measure of G is less than $\frac{1}{2}m(S)$. This contradicts the location of G in $\mathscr{E}$.

2.6. Lemma. *If $D|_{L^2(A)}$ is compact then it is* 0.

Proof. Assume $D_0=D|_{L^2(A)}$ is compact. Since D is selfadjoint and reduced by $L^2(A)$, D_0 is selfadjoint. In particular, if D_0 is not 0 then it has a nonzero eigenvalue r. Let $\mathscr{E}_r$ be the corresponding finite dimensional eigenspace, and let φ be any L^∞ function with $S_\varphi\subset X_c$. Then $S_{\varphi\circ T}=T^{-1}S_\varphi\subset A$. Now, $M_{\varphi\circ T}L^2(A)\subset L^2(A)$

and for any f in $L^2(X)$,

$$M_{\varphi\circ T}Df = (\varphi\circ T)(hf - h\circ TEf) = (h)\cdot(\varphi\circ T)\cdot f - (h\circ T)\cdot E((\varphi\circ T)\cdot f) = DM_{\varphi\circ T}f.$$

It follows that $M_{\varphi\circ T}$ leaves $\mathscr{E}_r$ invariant. But $\mathscr{E}_r$ is finite dimensional and so there is a function $f\in\mathscr{E}_r$ other than 0, and a scalar λ such that $(\varphi\circ T)f=\lambda f$ a.e. dm. In particular, $\varphi\circ T=\lambda$ on a set of positive measure. This shows that every $L^\infty(X_c)$ function is constant on a set of positive measure. But by definition X_c is completely nonatomic. Let S be a set of finite, positive measure in X_c. Via Lemma 2.5 we partition S into two measurable sets, each of measure no more than 3/4 that of S. Define the function f_1 to take the values 1/2, 1 respectively on the sets. Repeat this procedure by replacing S by each of the sets of constancy of f_1 and defining f_2 to take the value of f_1 on one part of each of the original two subsets and to be 1/4, 3/4 respectively on the remaining two sets. Continuation of this procedure gives rise to a monotonically decreasing sequence of functions whose pointwise limit is bounded and not constant on any set of positive measure in X_c. Indeed, we have for each x,

$$0 \leqq f_n(x) - f_{n+1}(x) \leqq \frac{1}{2^{n+1}},$$

so that

$$f_n(x) - f(x) \leqq \frac{1}{2^n}.$$

Thus, for any $r>0$ and any positive integer n,

$$\{x\colon f(x) = r\} \subset \left\{x\colon r \leqq f_n(x) \leqq r + \frac{1}{2^n}\right\}.$$

But this latter set contains at most two sets of constancy for f_n, so

$$m\left\{x\colon r \leqq f_n(x) \leqq r + \frac{1}{2^n}\right\} \leqq 2\cdot\left(\frac{3}{4}\right)^n m(S).$$

It then follows that $f\neq r$ a.e. dm. This contradiction completes the proof of the lemma.

Note that the result of Singh and Veluchamy as stated in Section 1 of this paper follows as a special case of Lemma 2.6, for in the completely nonatomic case $A=X$. But then $D=0$, i.e. C is normal. It is interesting to see that one basic property from Whitley's characterization of normality carries over to the general essentially normal setting.

2.7. Corollary. *If C is essentially normal then $h=h\circ T$ a.e. on $T^{-1}X_c$.*

Proof. Assume that C is essentially normal. Then $D|_{L^2(T^{-1}X_c)}=0$. Let Y be a subset of X_c of finite measure. Since $h\in L^\infty$ we have $m(T^{-1}Y)=\int\limits_Y h\,dm<\infty$ and in particular $\chi_{T^{-1}Y}\in L^2(T^{-1}X_c)$. But then we see that

$$0 = D\chi_{T^{-1}Y} = h\cdot\chi_{T^{-1}Y}-(h\circ T)\cdot E(\chi_{T^{-1}Y}) = (h-h\circ T)\chi_{T^{-1}Y}.$$

It then follows that $h=h\circ T$ a.e. on X_c.

We will conclude this paper with an example establishing the existence of an essentially normal composition operator for which $h>0$ a.e. and for which there is an atom a with $T^{-1}a$ infinite. First we investigate the structure of the sets $T^{-1}a_i$, $i\in J$, when C is essentially normal. Let a be an atom for m and let $B=T^{-1}a$. Then $D|_{L^2(B)}$ is compact. Let $f\in L^2(B)$. Since m is sigma-finite and h is essentially bounded, B is a set of finite measure. Noting that Ef is constant on $B=T^{-1}a$, we see that

$$\int\limits_B f dm = \int\limits_{T^{-1}a} f dm = \int\limits_a h\cdot(Ef)\circ T^{-1}dm = m(a)h(a)(Ef)\circ T^{-1}(a) =$$

$$= m(a)\frac{m(T^{-1}a)}{m(a)}(Ef)\circ T^{-1}(a) = m(B)(Ef)\circ T^{-1}(a).$$

It then follows that $Ef\equiv\frac{1}{m(B)}\int\limits_B f\,dm$ on B.

Also, for x in B, $h\circ T(x)=h(a)=\frac{m(B)}{m(a)}$. In particular $(M_{h\circ T}E)|_{L^2(B)}$ is the rank one operator $f\to\frac{1}{m(a)}\int\limits_B f\,dm$. But then the compactness of $D|_{L^2(B)}$ implies $M_h|_{L^2(B)}$ is compact. This in turn shows that

$$B\cap S_h = \{b_k\colon k\in K\}$$

where each b_k is an atom.

2.8. Example. Let O be the origin in the plane and let $X=\{O\}\cup(\mathbf{N}\times\mathbf{N})$. Define m by $m(O)=1$; $m(i,j)=1/2^{ij}$. Finally, define T on X by

$$T(O) = O;\ T(i,1) = O;\ T(i,j) = (i,j-1)\quad\text{for}\quad j>1.$$

Then

$$T^{-1}(O) = \{O\}\cup(\mathbf{N}x\{1\}),$$

and

$$h(O) = \frac{m(T^{-1}O)}{m(O)} = 1+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = 2,$$

while

$$h(i,j) = \frac{m(T^{-1}(i,j))}{m(i,j)} = \frac{m(i,j+1)}{m(i,j)} = \frac{2^{-i(j+1)}}{2^{-ij}} = 2^{-i}.$$

For f supported on $T^{-1}O$, $Df=hf-2\int_{T^{-1}O} f\,dm$. Since $\lim_{n\to\infty} h(n,1)=0$, $D|_{L^2(T^{-1}O)}$ is compact. On the other hand,

$$(Df)(i,j+1) = 2^{-i}f(i,j+1)-2^{-i}f(i,j+1) = 0.$$

Thus C is essentially normal.

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