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A problem of Kátai on sums of additive functions

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1. Introduction

KÁTAI [4] has shown the following result about completely additive functions:

Theorem. *Let F_1, F_2, F_3, F_4 be completely additive functions on the positive integers. Assume that*

$$F_1(n)+F_2(n+1)+F_3(n+2)+F_4(n+3)$$

is an integer for every positive integer n. Then $F_j(n)$, $j=1,2,3,4$, is an integer for every positive integer n.

The theorem can be extended to Gaussian integers, as was done by VAN ROSSUM-WIJSMULLER [9] for four functions and recently has been extended to six functions by KÁTAI and VAN ROSSUM-WIJSMULLER [6].

KÁTAI [5] has shown the analogy of his theorem holds for two *additive* functions by using properties of multiplicative functions. This reference to Kátai's paper may not seem relevant at first glance. But if F and G are additive functions, then $f(n)=\exp(2\pi iF(n))$ and $g(n)=\exp(2\pi iG(n))$ are multiplicative functions; now [5, II, Theorem 2, p. 105] gives the explicit form for f and g and one can then deduce the result.

We wish to extend this to three additive functions. Of course Kátai's theorem as stated is not true for three additive functions. For instance, one can let $F_1(2)=r$, $F_2(2^b)=s$ for all $b\geqq 1$, $F_3(2)=t$, $F_1(2^b)=s-t$, for all $b>1$, $F_3(2^b)=s-r$, for all $b>1$, $F_j(3^b)=-s$, $j=1,2,3$, for all $b\geqq 1$, and $F_j(q)=0$, $j=1,2,3$, for all prime powers q relatively prime to 6. No matter what real numbers r, s, t one chooses, $F_1(n)+F_2(n+1)+F_3(n+2)=0$. We will show, however, that this counterexample is the only way that a sum of three additive functions can be integral without the functions being integral.

More generally, Kátai (personal communication) believes that the following might be true.

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Conjecture. Let $F_0, F_1, \dots, F_{k-1}$ be k additive functions. Assume that

$$(*) \qquad F_0(n)+F_1(n+1)+\dots+F_{k-1}(n+k-1) \equiv 0 \pmod 1$$

for all $n>1$. Then each F_j, $j=0, 1, \dots, k-1$, has finite support.

Here we will say $F_j(n)\equiv 0 \pmod 1$ whenever $F_j(n)$ is an integer. The hypothesis $(*)$ probably need only hold for n sufficiently large. We define finite support to mean.

Definition. An additive function F is of finite support mod 1 if $F(p^a)\equiv \equiv 0 \pmod 1$, $a=1, 2, 3, \dots$, is true for all but finitely many primes p.

This paper has two parts. In the first part we assume Kátai's conjecture and then investigate which primes are within the finite support of the F_j for a fixed arbitrary number of additive functions. The proof is essentially the Chinese remainder theorem. We will see that for k additive functions, only primes p with $p\leqq k$ are in the set of finite support. Indeed, we will explicitly give all the relationships between the nonzero values of the additive functions at these exceptional primes.

The second half of this paper will prove Kátai's conjecture when we have three additive functions. This proof follows closely the proof of Kátai's theorem in [4]. We will, however, find several exponential Diophantine equations arising in our modification of his proof.

2. Primes in the set of finite support

We now begin to investigate the structure of the primes in the set of finite support, assuming Kátai's conjecture. To prepare for this, let k be the number of additive functions. For a prime p, define $\alpha=\alpha(p)$ to be the integer such that $p^\alpha > >k\geqq p^{\alpha-1}$.

First Main Theorem. *Let $F_0, F_1, \dots, F_{k-1}$ be k additive functions on the positive integers. Assume that*

$$(*) \qquad F_0(n)+F_1(n+1)+\dots+F_{k-1}(n+k-1) \equiv 0 \pmod 1$$

for all $n>N$, some integer N. Also assume that each F_j is of finite support mod 1. *Then $F_j(q)\equiv 0 \pmod 1$, $j=0, 1, \dots, k-1$, for every prime power $q=p^b$ with drime $p>k$.*

Now consider only prime powers $q=p^b$ for any prime $p\leqq k$. The number of $F_j(q)$ which may be assigned arbitrary real values is

$$-1+\sum_{\text{primes } p\leqq k} d(p)$$

where

$$d(p)=\begin{cases}(\alpha-1)k-p^{\alpha-1}+1 & \text{if } p^{\alpha}-k>p^{\alpha-1};\\ \alpha k-p^{\alpha}+1 & \text{if } p^{\alpha}-k\leqq p^{\alpha-1}.\end{cases}$$

One can explicitly find the relationship of the remaining $F_j(q)$ in terms of the ones assigned arbitrary real values.

Proof. We will establish a series of lemmas: the first will remove from consideration all prime powers where the prime exceeds k, the second will show the relationship of $F_j(p^b)$ and $F_j(p^{\alpha})$ for $b>\alpha$. Finally, we will see that the rest of the small prime powers lead to a simple linear algebra problem. The proof of each lemma will depend on an application of the Chinese remainder theorem.

The author wishes to thank Professor Kátai for suggesting this problem, and also notes that Professor Kátai independently proved this result.

Lemma 1. *Assume that $F_0, F_1, \ldots, F_{k-1}$ are additive functions of finite support, satisfying $(*)$. Let p be a prime with $p>k$. Then $F_j(q)\equiv 0 \pmod 1), j=0, 1, \ldots, k-1$ for all prime powers $q=p^b$.*

Notation. Number the primes $p_1<p_2<p_3<\ldots<p_s$ where p_s is the largest prime within the finite support. Number the prime powers of these primes by $q_1<<q_2<\ldots$. We say that a prime power $q\|n$ if $q=p^b$ and $p^b|n$ but $p^{b+1}\nmid n$.

Define F to be the infinite vector

$$F=(F_0(q_1), F_1(q_1), \ldots, F_{k-1}(q_1), F_0(q_2), \ldots, F_{k-1}(q_2), F_0(q_3), \ldots).$$

For a positive integer n, define $R(n)$ by

$$R(n)=(\delta_{0,1}, \delta_{1,1}, \ldots, \delta_{k-1,1}, \delta_{0,2}, \delta_{1,2}, \ldots, \delta_{k-1,2}, \delta_{0,3}, \ldots),$$

where

$$\delta_{i,j}=\begin{cases}1 & \text{if } q_i\|n+j;\\ 0 & \text{otherwise.}\end{cases}$$

$R(n)$ is an infinite sequence of 0 or 1 values. We note that the inner product

$$R(n)\cdot F=F_0(n)+F_1(n+1)+\ldots+F_{k-1}(n+k-1).$$

Thus, the assumption $(*)$ can be written $R(n)\cdot F\equiv 0 \pmod 1)$ for all $n>N$.

Proof of Lemma 1. Fix j. We let p be any prime with $p_s\geqq p>k$. Let $q=p^b$. Recall that we defined α_i for prime p_i by the condition $p_i^{\alpha_i}>k\geqq p_i^{\alpha_i-1}$. By the Chinese remainder theorem, we may choose n_1 and n_2 greater than N such that

$$n_1\equiv 1 \pmod{p_i^{\alpha_i}}, \quad i=1, 2, \ldots, s$$

and

$$n_2 \equiv 1 \pmod{p_i^{\alpha_i}}, \quad i = 1, 2, \ldots, s, \ p_i \neq p;$$
$$n_2 \equiv -j \pmod{p^b};$$
$$n_2 \not\equiv -j \pmod{p^{b+1}}.$$

In other words, n_1 and n_2 (and the next $k-1$ pairs of values) are the same modulo p_i for all the p_i except p. In fact, one can see that

$$[R(n_2)-R(n_1)]\cdot F = F_j(p^b)$$

and so by $(*)$, $F_j(p^b)\equiv 0 \pmod 1$. This proves this lemma.

We now may assume without loss of generality that the primes in the finite support of our additive functions all satisfy $p \leqq k$.

Lemma 2. *Assume that* $F_0, F_1, \ldots, F_{k-1}$ *are additive functions with finite support satisfying* $(*)$. *Let* p *be a prime, and* α *defined as above. Then* $F_j(p^{\alpha+b}) - F_j(p^\alpha) \equiv 0 \pmod 1$, $j=0, 1, \ldots, k-1$, *for all integers* $b \geqq 1$.

Proof. Again we merely apply the Chinese remainder theorem. Fix p, b and j. Choose n_1 and n_2 greater than N such that

$$n_1 \equiv 1 \pmod{p_i^{\alpha_i}}, \quad i = 1, 2, \ldots, s, \ p_i \neq p;$$
$$n_1 \equiv -j \pmod{p^\alpha};$$
$$n_1 \not\equiv -j \pmod{p^{\alpha+1}},$$

and

$$n_2 \equiv 1 \pmod{p_i^{\alpha_i}}, \quad i = 1, 2, \ldots, s, \ p_i \neq p;$$
$$n_2 \equiv -j \pmod{p^{\alpha+b}};$$
$$n_2 \not\equiv -j \pmod{p^{\alpha+b+1}}.$$

Then one can see that

$$[R(n_2)-R(n_1)]\cdot F = F_j(p^{\alpha+b}) - F_j(p^\alpha).$$

This proves the lemma.

We now only have a finite number of prime powers to consider, since any large power will give the same values as a power "close to k". Fix a prime $p \leqq k$, and let r be chosen so that $r+k \equiv 0 \pmod{p^{\alpha-1}}$. The r simply shifts some columns so that we will get an upper triangular matrix.

Define a vector

$$F(p) = \big(F_0(p^\alpha), F_1(p^\alpha), \ldots, F_{k-1}(p^\alpha),$$
$$F_r(p^{\alpha-1}), F_{r+1}(p^{\alpha-1}), \ldots, F_{k-1}(p^{\alpha-1}), F_0(p^{\alpha-1}), F_1(p^{\alpha-1}), F_{r-1}(p^{\alpha-1}),$$
$$F_0(p^{\alpha-2}), F_1(p^{\alpha-2}), \ldots, F_{k-1}(p^{\alpha-2}), \ldots, F_0(p), F_1(p), \ldots, F_{k-1}(p)\big).$$

Also define

$$R(n,p) = \delta_{\alpha,0}, \delta_{\alpha,1}, \ldots, \delta_{\alpha,k-1},$$
$$\delta_{\alpha-1,r}, \delta_{\alpha-1,r+1}, \ldots, \delta_{\alpha-1,k-1}, \delta_{\alpha-1,0}, \delta_{\alpha-1,1}, \ldots, \delta_{\alpha-1,r-1},$$
$$\delta_{\alpha-2,0}, \delta_{\alpha-2,1}, \ldots, \delta_{\alpha-2,k-1}, \ldots, \delta_{1,0}, \delta_{1,1}, \ldots, \delta_{1,k-1}$$

where

$$\delta_{a,j} = \begin{cases} 1 & \text{if } \; a<\alpha \text{ and } p^a \| n+j; \\ 1 & \text{if } \; a=\alpha \text{ and } p^a | n+j; \\ 0 & \text{otherwise.} \end{cases}$$

We note that $F(p)$ and $R(n,p)$ are vectors of length αk. We also note that $R(n)\cdot F= = \sum_{i=1}^{s} R(n,p_i)\cdot F(p_i)$ and that $R(n_1,p)=R(n_2,p)$ whenever $n_1 \equiv n_2 \pmod{p^\alpha}$.

Lemma 3. *Assume that $F_0, F_1, \ldots, F_{k-1}$ are additive functions of finite support satisfying $(*)$. Let p be any prime with $p \leqq k$. Then*

$$R(n_2,p)\cdot F(p) \equiv R(n_1,p)\cdot F(p) \pmod 1$$

for any positive integers n_1 and n_2.

Proof. Again we use the Chinese remainder theorem. Choose integers n_3 and n_4 greater than N such that

$$n_3 \equiv n_1 \pmod{p^\alpha};$$
$$n_3 \equiv 1 \pmod{p_i^{\alpha_i}}, \quad i=1,2,\ldots,s, \; p_i \neq p,$$

and

$$n_4 \equiv n_2 \pmod{p^\alpha};$$
$$n_4 \equiv 1 \pmod{p_i^{\alpha_i}}, \quad i=1,2,\ldots,s, \; p_i \neq p.$$

Then one can see that $R(n_2,p)\cdot F(p) - R(n_1,p)\cdot F(p) = R(n_4,p)\cdot F(p) - R(n_3,p)\times \times F(p) = R(n_4)\cdot F - R(n_3)\cdot F \equiv 0 \pmod 1$. This proves the lemma.

We now prove the first main theorem.

By Lemma 3, we know that for every prime p there is some real number b such that for every n we have $R(n,p)\cdot F(p) \equiv b \pmod 1$. For each prime $p \leqq k$, choose an arbitrary real number $b=b(p)$. Fix a prime p and choose any n with $n \equiv 1 \pmod{p^{\alpha+1}}$. Now define a matrix with p^α rows and αk columns by

$$A = \begin{pmatrix} R(n+p^\alpha, p) \\ R(n+p^\alpha-1, p) \\ \vdots \\ R(n+1,p) \\ R(n,p) \end{pmatrix}.$$

One can verify (because of the way we chose r) that if $p^\alpha - k \leqq p^{\alpha-1}$ then A is of the form

$$A=\begin{pmatrix} I_k & * & * & * \\ 0 & I_{p^\alpha-k} & * & * \end{pmatrix}.$$

If $p^\alpha-k\geqq p^{\alpha-1}$ then A is of the form

$$A=\begin{pmatrix} I_k & * & * & * \\ 0 & I_{p^{\alpha-1}} & * & * \\ 0 & * & * & * \end{pmatrix}.$$

Here I_m is the identity matrix of size m. If we consider this last matrix as having three divisions of the rows, then one can see that every row of the third division is identical to one of the rows of the second division.

Now we note that the matrix equation $AF(p)^t\equiv b(p)(1, 1, ..., 1)^t \pmod 1$ has either $\alpha k-p^\alpha$ or $\alpha k-k-p^{\alpha-1}$ free variables. We also have the free variable $b(p)$ and so this gives us the expression for $d(p)$ stated in the theorem. But now we note that the $b(p)$ are not really free—indeed, since $\sum_{i=1}^{s} b(p)=\sum_{i=1}^{s} R(n,p)\cdot F(p)= =R(n)\cdot F\equiv 0 \pmod 1$, we have one linear relation among the $b(p)$. This explains the -1 in the theorem. (The Chinese remainder theorem again implies that the $b(p)$ have no other relations.) One also sees explicitly in the matrix A the relations between the $F_j(p^b)$ for any given prime $p\leqq k$. This proves the first main theorem.

3. Sums of three additive functions

We now will embark on a proof that when $k=3$, Kátai's conjecture about finite support is indeed true. We will follow the broad outlines of the proof of his theorem quoted at the beginning of this paper. His proof begins by showing that the theorem holds for small prime n, and then he uses induction (with many subcases) to complete the proof. When we attempt to modify his proof, however, we will encounter dozens of exponential Diophantine equations. Fortunately, most of these equations have been studied previously.

Theorem. *Let F_1, F_2, F_3 be additive functions. Assume that*

$$(*)\qquad F_1(n)+F_2(n+1)+F_3(n+2)\equiv 0 \pmod 1,\quad n>1.$$

Then F_1, F_2 and F_3 have finite support.

Indeed, if r, s and t are arbitrary real numbers, and if $F_1(2)\equiv r$, $F_2(2)\equiv s$ and $F_3(2)\equiv t \pmod 1$, then $F_1(2^b)\equiv s-t$, for all $b>1$, $F_2(2^b)\equiv s$ for all $b>1$, $F_3(2^b)\equiv$

$\equiv s-r$, for all $b>1$, $F_j(3^c)\equiv -s$, $j=1,2,3$, for all b, and $F_j(q)\equiv 0 \pmod 1$, $j=1,2,3$, for all prime powers q relatively prime to 2 and 3.

By our work above, we already know the structure of the nonzero solutions must be the ones stated in the second half of this theorem. Because we could subtract two solutions with $F_1(2)\equiv r$, $F_2(2)\equiv s$ and $F_3(2)\equiv t \pmod 1$, we may assume these values are all zero mod 1. We are then proving

Second Main Theorem. *Let F_1, F_2 and F_3 be additive functions on the positive integers. Assume that*

$$(*)\qquad F_1(n)+F_2(n+1)+F_3(n+2) \equiv 0 \pmod 1$$

for all $n>1$. Also assume that $F_1(2)$, $F_2(2)$ and $F_3(2)$ are $\equiv 0 \pmod 1$. Then $F_j(n)\equiv 0 \pmod 1$ for every n, $j=1,2,3$.

Proof. We first show that our theorem's conclusion holds for small prime powers n, then that it holds for *all* powers of a few small primes, and finally use induction to show the theorem for general n. As in Kátai's proof, we will have many cases depending on the prime power mod low primes. Unlike Kátai's case, however, we find a multitude of exponential Diophantine equations arising.

We first show that the $F_j(n)\equiv 0 \pmod 1$ for small n.

Lemma 4. *Assume that F_1, F_2, and F_3 are additive functions of the positive integers. Assume that*

$$(*)\qquad F_1(n)+F_2(n+1)+F_3(n+2) \equiv 0 \pmod 1$$

for all $n>1$. Then $F_j(n)\equiv 0 \pmod 1$ for all $n<38$, $j=1,2,3$.

Before proving the case of three additive functions, we will illustrate the idea with the case of two additive functions satisfying the analog of $(*)$, namely, $F_1(n)+$ $+F_2(n+1)\equiv 0 \pmod 1$. Consider the set of prime powers $\{2,3,4,5,7,8,9,11\}$. Consider the sixteen values $n=2,3,4,5,6,7,8,9,10,11,14,20,21,35,44,55$. These sixteen n give rise to sixteen equations $F_1(n)+F_2(n+1)\equiv 0 \pmod 1$ which can be expressed in terms of the prime powers in $\{2,3,4,5,7,8,9,11\}$. For instance, $n=55$ gives rise to the equation $F_1(5)+F_1(11)+F_2(7)+F_2(8)\equiv 0 \pmod 1$. We therefore have 16 equations in the 16 variables $F_j(q)$ with $j=1,2$ and $q\in\{2,3,4,5,7,8,9,11\}$. One may set up a matrix equation to represent these, say $AF\equiv 0 \pmod 1$, where

$$F^t = (F_1(2),\ F_2(2),\ F_1(3),\ F_2(3),\ \dots,\ F_1(11),\ F_2(11)),$$

and

$$A=\begin{pmatrix}
1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&1&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&1&0\\
1&0&0&1&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&1&1&0&1&0&0&1&0&0&0&0&0&0\\
0&1&1&0&0&0&0&0&1&0&0&0&0&0&0&1\\
0&0&0&0&0&1&1&0&1&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&1&0&0&1&0&1&0&0&1&0
\end{pmatrix}.$$

Amazingly, this matrix A has determinant -1. Thus we may conclude that it has an inverse with integer entries and therefore that the vector F must have each component $\equiv 0 \pmod 1$. In other words, $F_1(n)$ and $F_2(n)$ are integers for $n=2, 3, 4, 5, 7, 8, 9, 11$.

Indeed, we need not assume that the above matrix A is square; even if A were overdetermined one would "row reduce" with the proviso that one may not divide by integer factors. If one only switches rows or adds integral multiples of one row to another, then one hopes to reduce the matrix A to a diagonal matrix with diagonal entries equal to 1 or -1. If this is possible, then every variable $F_j(q) \equiv 0 \pmod 1$.

We will follow the same ideas when we have three additive functions. One sets up the matrix equation $AF \equiv 0 \pmod 1$ where the vector F contains the variables $F_j(q)$, $j=1, 2, 3$, for the nineteen prime powers q equal to

$$2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23, 25, 27, 29, 31, 32, 37.$$

Recall that we have hypothesized that $F_j(2) \equiv 0 \pmod 1$, $j=1, 2, 3$. This hypothesis eliminates three variables, so we actually have 54 variables. The rows of A come from expanding $F_1(n)+F_2(n+1)+F_3(n+2)$ for the fifty-four values of n

$$2, 3, \ldots, 36, 37, 38, 44, 50, 54, 55, 56, 68, 74, 75, 76, 90, 91, 110, 115, 143, 152, 154, 713.$$

One can verify that the prime factorizations of these fifty-four triples n, $n+1$ and $n+2$ have only prime powers in the set of nineteen powers listed above. Therefore, we get the matrix A to be a 54 by 54 matrix of zeros and ones.

Amazingly, the determinant of A is ± 1 (depending on the ordering of the columns). We conclude that A can be inverted with integer entries and therefore each $F_j(q)\equiv 0 \pmod 1$ for the q listed.

One can change the hypothesis. Instead of $F_j(2)\equiv 0 \pmod 1$, $j=1, 2, 3$, one may assume $F_j(4)\equiv 0 \pmod 1$, $j=1, 2, 3$, or $F_j(8)\equiv 0 \pmod 1$, $j=1, 2, 3$, or $F_1(2)\equiv 0 \pmod 1$, $F_1(3)\equiv 0 \pmod 1$, $F_1(4)\equiv 0 \pmod 1$ or any combination that would lead to $r\equiv s\equiv t\equiv 0 \pmod 1$ in our counterexample.

Also, one need not start the hypothesis on $F_1(n)+F_2(n+1)+F_3(n+2)$ with $n=2$. It seems that one might be able to start at any value of n as long as one has enough rows. For instance, if we begin with $n=17$, (adding 15 new values n to replace the ones we have eliminated) we get a matrix which row reduces to give all the $F_j(q)\equiv 0 \pmod 1$.

At any rate, we have taken care of small values of prime powers q. We must now take care of the case when q is an arbitrary power of 2 or 3. So suppose q is a power of 2.

Lemma 2. *Let $a>5$. Assume that $F_j(n)\equiv 0 \pmod 1$ for all n less than 2^a-3, $j=1, 2, 3$. Then $F_1(2^a)$, $F_2(2^a+1)$, $F_1(2^a-1)$, $F_2(2^a)$, $F_3(2^a+1)$, $F_2(2^a-1)$, and $F_3(2^a)$ are all $\equiv 0 \pmod 1$.*

Remark. The condition $n<2^a-3$ could be replaced by $n<7\cdot 2^{a-3}$ but we only need 2^a-3 (in Case 18).

Proof. We give a case by case analysis depending on what the power a is modulo 12. Each case will state the result obtained, the assumption on a, the exceptions to the proof (invariably Diophantine equations which will be dealt with later), and the synopsis of the proof for the case.

Case 1. $F_1(2^a)\equiv 0 \pmod 1$ for a odd,

unless: $2^a+1=3^b$ for some positive integer b.

$$2^a;\ 3\frac{2^a+1}{3},\ 2(2^{a-1}+1).$$

This last line will be our abbreviated notation for

$$F_1(2^a)+F_2(2^a+1)+F_3(2^a+2)\equiv 0 \pmod 1$$

and the fact that some power of 3 divides 2^a+1 as well as that 2 divides 2^a+2.

Using the fact that 3 divides 2^a+1 when a is odd, we have

$$F_1(2^a)+F_2(3^c)+F_2((2^a+1)/3^c)+F_3(2)+F_3(2^{a-1}+1)\equiv 0 \pmod 1$$

for some positive integer c such that 3^c divides 2^a+1 but 3^{c+1} does not. We exclude the case when $2^a+1=3^b$ for some positive integer b so we may assume that $3^c<2^a-2$. (Fortunately, we will see that this exponential Diophantine equation has no solutions with $a>5$.) By our inductive hypothesis, $F_2(3^c)$, $F_2((2^a+1)/3^c)$, $F_3(2)$, and $F_3(2^{a-1}+1)$ are all $\equiv 0 \pmod 1$. We therefore conclude that $F_1(2^a)\equiv 0 \pmod 1$.

Case 2. $F_1(2^a)\equiv 0 \pmod 1$ and $F_2(2^a+1)\equiv 0 \pmod 1$ for $a\equiv 0 \pmod 4$,

unless: $2^{a+1}+1=3^b$ for some positive integer b, or $2^{a+1}+3=5^b$ for some positive integer b.

$$2^a;\ 2^a+1;\ 2(2^{a-1}+1),$$

$$3\frac{2^{a+1}+1}{3};\ 2(2^a+1);\ 5\frac{2^{a+1}+3}{5}.$$

These lists are shortland for the following argument: starting with the last line of our proof list,

$$F_1\left(3\frac{2^{a+1}+1}{3}\right)+F_2(2(2^a+1))+F_3\left(5\frac{2^{a+1}+3}{5}\right)\equiv 0 \pmod 1$$

or thus

$$F_1(3^c)+F_1\left(\frac{2^{a+1}+1}{3^c}\right)+F_2(2)+F_2(2^a+1)+F_3(5^d)+F_3\left(\frac{2^{a+1}+3}{5^d}\right)\equiv 0 \pmod 1$$

for some c and d with 3^c the highest power dividing $2^{a+1}+1$ and 5^d the highest power dividing $2^{a+1}+3$.

With the inductive hypothesis, noting our exceptions, we have that $F_1(3^c)$, $F_1\left(\frac{2^{a+1}+1}{3^c}\right)$, $F_2(2)$, $F_3(5^d)$, $F_3\left(\frac{2^{a+1}+3}{5^d}\right)$ are all $\equiv 0 \pmod 1$. Thus, $F_2(2^a+1)\equiv$ $\equiv 0 \pmod 1$.

The first line of our proof list says

$$F_1(2^a)+F_2(2^a+1)+F_3(2(2^{a-1}+1))\equiv 0 \pmod 1$$

which says

$$F_1(2^a)+F_2(2^a+1)+F_3(2)+F_3(2^{a-1}+1)\equiv 0 \pmod 1.$$

Using the inductive hypothesis, we have $F_1(2^a)+F_2(2^a+1)\equiv 0 \pmod 1$. Then $F_1(2^a)\equiv 0 \pmod 1$.

As before, the exceptions are exponential Diophantine equations which fortunately will have no solutions with $a>5$.

We will now only give the results without filling in the details.

Case 3. $F_1(2^a)\equiv 0 \pmod 1$ for $a\equiv 2 \pmod 4$,

unless: $2^a+1=5^b$ for some positive integer b.

$$2^a;\ 5\frac{2^a+1}{5};\ 2(2^{a-1}+1).$$

Case 4. $F_2(2^a)\equiv 0 \pmod 1$ and $F_3(2^a+1)\equiv 0 \pmod 1$ for $a\equiv 0 \pmod{12}$,

unless: $2^a-1=3^b$ for some positive integer b, or $2^{a+1}+1=3^b$ for some positive integer b, or $2^{a+2}+3=7^b$ for some positive integer b.

$$3\frac{2^a-1}{3};\ 2^a;\ 2^a+1,$$

$$6\frac{2^{a+1}+1}{3};\ 7\frac{2^{a+2}+3}{7};\ 4(2^a+1).$$

Case 5. $F_1(2^a-1)\equiv 0 \pmod 1$ and $F_2(2^a)\equiv 0 \pmod 1$ for $a\equiv 1 \pmod 4$,

unless: $2^a+1=3^b$ for some positive integer b, or $2^{a+1}-1=3^b$ for some positive integer b, or $2^{a+2}-3=5^b$ for some positive integer b.

$$2^a-1;\ 2^a;\ 3\frac{2^a+1}{3},$$

$$4(2^a-1);\ 5\frac{2^{a+2}-3}{5},\ 6\frac{2^{a+1}-1}{3}.$$

A minor note: the Diophantine equation $2^\alpha-3=5^\beta$ has a rather large solution, namely $2^7-3=5^3$. We are fortunate that $\alpha=7$ corresponds to $a=5$.

Case 6.. $F_2(2^b)\equiv 0 \pmod 1$ for $a\equiv 2 \pmod 4$,

unless: $2^a-1=3^b$ for some positive integer b, or $2^a+1=5^b$ for some positive integer b.

$$3\frac{2^a-1}{3};\ 2^a;\ 5\frac{2^a+1}{5}.$$

Case 7. $F_1(2^a-1)\equiv 0 \pmod 1$ and $F_2(2^a)\equiv 0 \pmod 1$ for $a\equiv 3 \pmod 4$,

unless: $2^a+1=3^b$ for some positive integer b, or $2^a-1=7^b$ for some positive integer b, or $7\cdot 2^{a-1}-3=5^b$ for some positive integer b, or $7\cdot 2^a-5=3^b$ for some positive integer b, or $7\cdot 2^a-11=5\cdot 3^b$ for some positive integer b, or $7\cdot 2^a-11=$ $=3\cdot 5^b$ for some positive integer b.

$$2^a-1;\ 2^a;\ 3\frac{2^a+1}{3},$$

$$7(2^a-1);\ 10\frac{7\cdot 2^{a-1}-3}{5};\ 3\frac{7\cdot 2^a-5}{3},$$

$$5\frac{7\cdot 2^a-11}{15};\ 8\frac{7\cdot 2^{a-3}-1}{3};\ \frac{7\cdot 2^a-5}{3}.$$

Case 8. $F_2(2^a)\equiv 0 \pmod 1$ and $F_3(2^a+1)\equiv 0 \pmod 1$ for $a\equiv 4 \pmod{12}$,

unless: $2^a-1=3^b$ for some positive integer b, or $3\cdot 2^a+1=7^b$ for some positive integer b, or $3\cdot 2^{a-1}+1=5^b$ for some positive integer b.

$$3\,\frac{2^a-1}{3};\ 2^a;\ 2^a+1,$$

$$7\,\frac{3\cdot 2^a+1}{7};\ 10\,\frac{3\cdot 2^{a-1}+1}{5};\ 3(2^a+1).$$

Case 9. $F_2(2^a)\equiv 0 \pmod 1$ and $F_3(2^a+1)\equiv 0 \pmod 1$ for $a\equiv 8 \pmod{12}$,

unless: $2^a-1=3^b$ for some positive integer b, or $13\cdot 2^a+11=3\cdot 7^b$ for some positive integer b, or $13\cdot 2^a+11=7\cdot 3^b$ for some positive integer b, or $13\cdot 2^{a-2}+3==5^b$ for some positive integer b, or $2^a+1=13^b$ for some positive integer b.

$$3\,\frac{2^a-1}{3};\ 2^a;\ 2^a+1,$$

$$21\,\frac{13\cdot 2^a+11}{21};\ 20\,\frac{13\cdot 2^{a-2}+3}{5};\ 13(2^a+1).$$

Case 10. $F_3(2^a)\equiv 0 \pmod 1$ for a even,

unless: $2^a-1=3^b$ for some positive integer b.

$$2(2^a-1);\ 3\,\frac{2^a-1}{3};\ 2^a.$$

Case 11. $F_2(2^a-1)\equiv 0 \pmod 1$ and $F_3(2^a)\equiv 0 \pmod 1$ for $a\equiv 1 \pmod 4$,

unless: $11\cdot 2^{a-2}-3=5^b$ for some positive integer b, or $2^a-1=11^b$ for some positive integer b, or $11\cdot 2^{a-1}-5=3^b$ for some positive integer b, or $2^{a-1}-1=3\cdot 11^b$ for some positive integer b.

$$2(2^{a-1}-1);\ 2^a-1;\ 2^a,$$

$$20\,\frac{11\cdot 2^{a-2}-3}{5};\ 11(2^a-1);\ 6\,\frac{11\cdot 2^{a-1}-5}{3},$$

$$11\,\frac{2^{a-1}-1}{3};\ 8\,\frac{11\cdot 2^{a-4}-1}{3};\ \frac{11\cdot 2^{a-1}-5}{3}.$$

Case 12. $F_2(2^a-1)\equiv 0 \pmod 1$ and $F_3(2^a)\equiv 0 \pmod 1$ for $a\equiv 3 \pmod 4$, *unless:* $2^a-1=7^b$ for some positive integer b, or $7\cdot 2^{a-1}-3=5^b$ for some positive integer b.

$$2(2^{a-1}-1);\ 2^a-1;\ 2^a,$$

$$8(7\cdot 2^{a-3}-1);\ 7(2^a-1);\ 10\frac{7\cdot 2^{a-1}-3}{5}.$$

We also need to consider all powers of three. Fortunately, the powers of 3 are much easier.

Lemma 6. *Let* $a>3$. *Assume that* $F_j(n)\equiv 0 \pmod 1$ *for all* n *less than* 3^a-2, $j=1,2,3$. *Then* $F_1(3^a)$, $F_3(3^a+2)$, $F_2(3^a)$, $F_1(3^a-2)$, *and* $F_3(3^a)$ *are* all $\equiv 0 \pmod 1$.

Remark. The condition $n<3^a-2$ could be replaced by $n<2(3^{a-1}+1)$ but we only need the stated condition (for Case 16).

Proof. As with the powers of 2, we will do a case analysis, only this time each case will have arbitrary powers a. We will again find several exponential Diophantine equations which we deal with in a later section.

Case 13. $F_1(3^a)\equiv 0 \pmod 1$ and $F_3(3^a+2)\equiv 0 \pmod 1$,

unless: $3^a+1=2^b$ for some positive integer b.

$$3^a;\ 2\frac{3^a+1}{2};\ 3^a+2,$$

$$4\frac{3^a+1}{2};\ 3(2\cdot 3^{a-1}+1);\ 2(3^a+2).$$

Case 14. $F_2(3^a)\equiv 0 \pmod 1$,

unless: $3^a+1=2^b$ for some positive integer b, or $3^a-1=2^b$ for some positive integer b.

$$2\frac{3^a-1}{2};\ 3^a;\ 2\frac{3^a+1}{2}.$$

Case 15. $F_1(3^a-2)\equiv 0 \pmod 1$ and $F_3(3^a)\equiv 0 \pmod 1$,

unless: $3^a-1=2^b$ for some positive integer b.

$$3^a-2;\ 2\frac{3^a-1}{2};\ 3^a,$$

$$2(3^a-2);\ 3(2\cdot 3^{a-1}-1);\ 4\frac{3^a-1}{2}.$$

Now we can do the general prime power q case.

Lemma 7. *Let* $q > 37$ *be a prime power. Assume that* $F_j(n) \equiv 0 \pmod 1$ *for all* n *less than* q, $j = 1, 2, 3$. *Then* $F_1(q)$, $F_2(q)$, *and* $F_3(q)$ *are all* $\equiv 0 \pmod 1$.

Proof. Suppose q is even; then q is a power of 2 which we have already completed above in the fifth lemma. If q is divisible by 3, we see that the sixth lemma completed the proof. Therefore, we may assume q is not divisible by 2 or 3.

Since $F_1(q-2) + F_2(q-1) + F_3(q) \equiv 0 \pmod 1$, the induction hypothesis immediately gives that $F_3(q) \equiv 0 \pmod 1$.

Case 16. $F_1(q) \equiv 0 \pmod 1$ for $q \equiv 1 \pmod 3$,

unless: $q+1 = 2^b$ for some positive integer b, or $q+2 = 3^b$ for some positive integer b.

$$q;\ 2\frac{q+1}{2};\ 3\frac{q+2}{3}.$$

Fortunately, if $q = 2^b - 1$, then we have already shown that $F_1(q) \equiv 0 \pmod 1$ (Cases 5 and 7 above). If $q = 3^b - 2$ then $F_1(q) \equiv 0 \pmod 1$ from Case 15.

When $q \equiv 2 \pmod 3$, we will give two different ways to achieve the desired result.

Case 17. $F_1(q) \equiv 0 \pmod 1$ for $q \equiv 2 \pmod 3$,

unless: $4q+1 = 3^b$ for some positive integer b, or $2q-1 = 3^b$ for some positive integer b, or $q+1 = 2^b$ for some integer b.

$$4q;\ 3\frac{4q+1}{3};\ 2(2q+1),$$

$$2\frac{2q-1}{3};\ \frac{4q+1}{3};\ 4\frac{q+1}{3},$$

$$3\frac{2q-1}{3};\ 2q;\ 2q+1,$$

$$2\frac{q-1}{2};\ q;\ 2\frac{q+1}{2}.$$

Fortunately, $q+1 = 2^b$ has already been covered by Lemma 5.

Case 18. $F_1(q) \equiv 0 \pmod 1$ for $q \equiv 2 \pmod 3$,

unless: $q+1 = 2^b$ for some positive integer b,

$4q+7=3^b$ for some positive integer b, or $2q+5=3^b$ for some positive integer b, or $q+3=2^b$ for some positive integer b.

$$q;\ 2\frac{q+1}{2};\ q+2,$$

$$2(2q+3);\ 3\frac{4q+7}{3};\ 4(q+2),$$

$$2q+3;\ 2(q+2);\ 3\frac{2q+5}{3},$$

$$2\frac{q+1}{2};\ q+2;\ 2\frac{q+3}{2},$$

$$4\frac{q+1}{3};\ \frac{4q+7}{2};\ 2\frac{2q+5}{3}.$$

Fortunately, when $q=2^b-3$, Lemma 5 tells us that $F_2(2^b-1)\equiv 0 \pmod 1$ and. $F_3(2^b)\equiv 0 \pmod 1$ so the fourth line of this proof list is still valid even when $q+3=2^b$ We therefore only have two exceptions to consider.

Cases 17 and 18 give us a choice; we will choose the one which avoids the exceptions listed whenever possible. In particular, we can avoid the exceptions listed unless we have one of the following:

$4q+1=3^b$ for some positive integer b and $4q+7=3^c$ for some positive integer c,
$4q+1=3^b$ for some positive integer b and $2q+5=3^c$ for some positive integer c,
$2q-1=3^b$ for some positive integer b and $4q+7=3^c$ for some positive integer c,
$2q-1=3^b$ for some positive integer b and $2q+5=3^c$ for some positive integer c.

These give rise to the exponential Diophantine equations:

$$6 = 3^c-3^d,\quad 9 = 2\cdot 3^c-3^b,\quad 9 = 3^c-2\cdot 3^b,\quad \text{and}\quad 6 = 3^c-3^b.$$

Of course, these are rather trivial and one sees that these have no solutions with c or d exceeding 3.

Putting Lemmas 4—7 together, we find an inductive proof for our main theorem provided that we can remove the exceptions from each case. In other words, we have reduced the entire problem to solving several two variable exponential Diophantine equations. Most of these have been solved (in much greater generality) by TRYGVE NAGELL [8] and later TOSHIRO HADANO [3]. Nagell solved all equations of the form $a^x+b^y=c^z$ for distinct a, b and c primes less than or equal to seven. Hadano extended this to a, b and c primes up to seventeen. In particular, their results take care of

$$2^a+1 = 3^b,$$
$$2^{a+1}+1 = 3^b,$$
$$2^{a+1}+3 = 5^b,$$
$$2^a+1 = 5^b,$$
$$2^a-1 = 3^b,$$
$$2^{a+2}+3 = 7^b,$$
$$2^{a+1}-1 = 3^b,$$
$$2^{a+2}-3 = 5^b,$$
$$2^a-1 = 7^b,$$
$$2^a+1 = 13^b,$$
$$2^a-1 = 11^b.$$

D. H. LEHMER [7] solved a host of exponential Diophantine equations of the form $S+1=T$ where S and T have prime factors in some small set. His calculations take care of our equations

$$3\cdot 2^a+1 = 7^b;$$
$$3\cdot 2^{a-1}+1 = 5^b,$$
$$2^{a-1}-1 = 3\cdot 11^b.$$

LEO ALEX [1], when looking at possible indices for simple groups, has solved equations of the form $x+y=z$ where x, y, and z are of the form $2^r3^s5^t7^u$. His work takes care of the equations

$$7\cdot 2^{a-1}-3 = 5^b,$$
$$7\cdot 2^a-5 = 3^b.$$

The rest of the exponential Diophantine equations are

$$7\cdot 2^a-11 = 5\cdot 3^b,$$
$$7\cdot 2^a-11 = 3\cdot 5^b,$$
$$13\cdot 2^a+11 = 3\cdot 7^b,$$
$$13\cdot 2^a+11 = 7\cdot 3^b,$$
$$13\cdot 2^{a-2}+3 = 5^b,$$
$$11\cdot 2^{a-2}-3 = 5^b,$$
$$11\cdot 2^{a-1}-5 = 3^b.$$

We are only interested in solutions when $a>5$; indeed, one can easily compute that these have no solutions for $a=6$ so we may certainly view these equations modulo 16. Fortunately, the equations $7\cdot 2^a-11=3\cdot 5^b$, $13\cdot 2^a+11=3\cdot 7^b$, $13\cdot 2^a+11=7\cdot 3^b$, and $13\cdot 2^{a-2}+3=5^b$ are all impossible modulo 16.

$11\cdot 2^{a-2}-3=5^b$ and $11\cdot 2^{a-1}-5=3^b$ are impossible modulo 11.

This leaves $7\cdot 2^a-11=5\cdot 3^b$ which has a solution $7\cdot 2^3-11=5\cdot 3^2$. Then $7\cdot 2^3(2^\alpha-1)=5\cdot 3^2(3^\beta-1)$. Viewing this modulo 16 gives $\beta=2\gamma$ with γ odd, unless $\beta=0$. Now 3^2 divides $2^\alpha-1$ and one can verify that this implies $\alpha\equiv 0 \pmod 6$ Then $7=2^3-1$ divides $2^\alpha-1$, so 7^2 divides $3^\beta-1$. One verifies that this gives $\beta\equiv 0 \pmod{42}$. Then 1093 divides 3^7-1 which divides $3^\beta-1$, so 1093 must divide $2^\alpha-1$. One can verify that this implies $\alpha\equiv 0 \pmod{364}$. Then 113 divides $2^{14}+1$ which divides $2^\alpha-1$, so 113 divides $3^\beta-1$. One verifies that this implies $\beta\equiv 0 \pmod{112}$. But then 4 divides β, a contradiction, unless $\beta=0$, that is, unless $7\cdot 2^3-11=5\cdot 3^2$ is the largest solution to this exponential Diophantine equation.

The procedure used to solve this last equation is exactly the same that Guy, Lacampagne, and Selfridge [2] use to solve equations such as $5=2^a-3^b$.

This finishes the solution to all of the Diophantine equations, which removes the exceptions from the cases analyzed above, and so one can now use the lemmas to prove the main theorem by induction.

Similar ideas surely work when one considers the analogy of $(*)$ with four additive functions. One can easily find a matrix A involving all prime powers up to 89 which will give the analogy of Lemma 1 for the small prime powers. Instead of dealing with powers of 2 and 3, one must now deal with all powers of 2, 3, 5, 7, and 13. One can then find the necessary cases to deal with the general prime power. But by now one has over a hundred cases, each with many exceptions. Even the task of listing all of the relevant Diophantine equations would be formidable. To attempt this approach with five additive functions seems untenable. Our method is clearly not appropriate for large numbers of additive functions, and we hope that someone will find a better approach which proves the problem in its deserved generality.

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