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Congruence lattices on a regular semigroup associated with certain operators

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1. Introduction and summary

To each congruence ϱ on a regular semigroup S we may associate a number of congruences on S according to the following scheme. If Φ is a complete $\cap$-congruence on the congruence lattice $\mathscr{C}(S)$ of S, then the Φ-class of ϱ has a least element ϱ_Φ and we may consider an operator on $\mathscr{C}(S)$ whose effect is: $\varrho \to \varrho_\Phi$. For Φ we may take the congruences T_l, T_r, T, U, V and some of their variants, and the $\cap$-congruence K, studied in the authors' papers [7] and [9]. Recall, for example that T_l stands for having the same left trace, T for having the same trace, K for having the same kernel; the congruences U and V have similar interpretations. We call the sublattice of $\mathscr{C}(S)$ generated by the set $\{\varrho_{T_l}, \varrho_{T_r}, \varrho_K, \varrho_U\}$ the lattice *associated* with ϱ. As we shall see, this lattice is always finite. We shall determine the lattice associated with the congruences ω, σ, ν, γ and η on a regular semigroup S. Here ω denotes the universal congruence and σ, ν, γ and η the least group, Clifford, inverse and semilattice congruences, respectively.

In the case of an inverse semigroup S, the sublattice of $\mathscr{C}(S)$ obtained from ω by successively applying the operators sub T and sub K was investigated and dubbed the min-network of S by PETRICH and REILLY in [13]. In contradistinction to this procedure, we apply our operators only once and then form the sublattice of $\mathscr{C}(S)$ generated by the congruences so obtained. In this sense, our scope is narrower than that in [13]. However, it is also wider in two different directions: we start with various congruences, not just with ω, and study the lattice generated by ϱ_{T_l}, ϱ_{T_r}, ϱ_K and ϱ_U, not just by ϱ_T and ϱ_K. Note that the notation $\varrho_{\min}$ and $\varrho^{\min}$ is used in [13] for ϱ_T and ϱ_K, respectively.

We now summarize the contents of the various sections of the paper. Section 2 contains only some terminology and notation, the rest being relegated to the per-

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tinent literature, as well as some preliminary results. The result in Section 3 relates the effect of applying some of our operators to minimal congruences in terms of Malcev products. The lattice associated with a congruence on a regular semigroup is described in Section 4 in terms of an $\cap$-sublattice of the lattice of congruences. The description of the lattices associated with ω and σ forms the content of Section 5, that of ν and γ of Section 6 and that of η of Section 7.

2. Preliminaries

Throughout the paper, S stands for an arbitrary regular semigroup and E for its set of idempotents, unless stated otherwise.

We use the following notation on S:

ω — the universal relation,
σ — the least group congruence,
ν — the least Clifford congruence,
γ — the least inverse semigroup congruence,
η — the least semilattice congruence,
ε — the equality relation (also denoted by I).

For any semigroup T, we denote by $E(T)$ the set of its idempotents, and for $a \in T$, by $V(a)$ the set of inverses of a. For any relation θ on S, θ^* is the congruence generated by θ.

We shall consider classes $\mathscr{C}$ of regular semigroups which satisfy the conditions

(i) all isomorphic copies of members of $\mathscr{C}$ belong to $\mathscr{C}$,

(ii) $\mathscr{C}$ is closed for the formation of subdirect products within the class of regular semigroups.

Remark that a class $\mathscr{C}$ which satisfies the conditions (i) and (ii) is never empty because it contains the trivial semigroup, which is the direct product of the empty system of semigroups from $\mathscr{C}$. The classes $\mathscr{C}$ which satisfy the conditions (i) and (ii) form a lattice $\mathbf{L}$ under inclusion. If $\mathscr{A}$ and $\mathscr{B}$ are classes of regular semigroups satisfying the above conditions (i) and (ii), then the meet of $\mathscr{A}$ and $\mathscr{B}$ in $\mathbf{L}$ is simply $\mathscr{A} \cap \mathscr{B}$ whereas the join $\mathscr{A} \vee \mathscr{B}$ of $\mathscr{A}$ and $\mathscr{B}$ in $\mathbf{L}$ consists of all isomorphic copies of regular semigroups which are subdirect product of members of $\mathscr{A}$ and of members of $\mathscr{B}$.

Let $\mathscr{C}$ be a class of regular semigroups satisfying the above conditions (i) and (ii). Then there exists a least congruence ϱ on S such that $S/\varrho \in \mathscr{C}$. (See [1], exercise 2 of § 11.6.) This congruence will be denoted by $\theta_{\mathscr{C}}$. In order to simplify our statements, when we write $\varrho = \theta_{\mathscr{C}}$, we tacitly imply that $\mathscr{C} \in \mathbf{L}$. The mapping

$$\mathbf{L} \to \mathscr{C}(S), \quad \mathscr{C} \to \theta_{\mathscr{C}}$$

is an antitone mapping of $\mathbf{L}$ into $\mathscr{C}(S)$ such that for $\mathscr{A}, \mathscr{B} \in \mathbf{L}$,

$$\theta_{\mathscr{A} \vee \mathscr{B}} = \theta_{\mathscr{A}} \cap \theta_{\mathscr{B}}, \; \theta_{\mathscr{A} \cap \mathscr{B}} \supseteqq \theta_{\mathscr{A}} \vee \theta_{\mathscr{B}}.$$

If $\mathscr{A}, \mathscr{B} \in \mathbf{L}$ and if $\mathscr{A}$ and $\mathscr{B}$ are closed for taking homomorphic images, then $\theta_{\mathscr{A} \cap \mathscr{B}} = \theta_{\mathscr{A}} \vee \theta_{\mathscr{B}}$. We shall apply these results without further notice.

We now list some of the classes of regular semigroups which belong to $\mathbf{L}$. The abbreviations we introduce here will be used freely throughout the paper. For some of them the defining identities can be found in [12].

$\mathscr{T}$ — trivial semigroups,
$\mathscr{L}\mathscr{Z}$ — left zero semigroups,
$\mathscr{R}\mathscr{Z}$ — right zero semigroups,
$\mathscr{R}e\mathscr{B}$ — rectangular bands,
$\mathscr{L}\mathscr{R}\mathscr{B}$ — left regular bands,
$\mathscr{R}\mathscr{R}\mathscr{B}$ — right regular bands,
$\mathscr{R}\mathscr{B}$ — regular bands,
$\mathscr{B}$ — bands,
$\mathscr{G}$ — groups,
$\mathscr{L}\mathscr{G}$ — left groups,
$\mathscr{R}\mathscr{G}$ — right groups,
$\mathscr{R}e\mathscr{G}$ — rectangular groups,
$\mathscr{S}\mathscr{G}$ — Clifford semigroups,
$\mathscr{L}\mathscr{R}\mathscr{B}\mathscr{G}$ — left regular bands of groups,
$\mathscr{R}\mathscr{R}\mathscr{B}\mathscr{G}$ — right regular bands of groups,
$\mathscr{R}\mathscr{O}\mathscr{B}\mathscr{G}$ — regular orthodox bands of groups,
$\mathscr{U}\mathscr{B}\mathscr{G}$ — E-unitary bands of groups,
$\mathscr{O}\mathscr{B}\mathscr{G}$ — orthodox bands of groups,
$\mathscr{C}\mathscr{S}$ — completely simple semigroups,
$\mathscr{L}\mathscr{U}\mathscr{B}\mathscr{G}$ — locally E-unitary bands of groups,
$\mathscr{B}\mathscr{G}$ — bands of groups,
$\mathscr{L}\mathscr{R}\mathscr{O}\mathscr{G}$ — left regular orthogroups,
$\mathscr{R}\mathscr{R}\mathscr{O}\mathscr{G}$ — right regular orthogroups,
$\mathscr{L}\mathscr{C}\mathscr{R}\mathscr{O}\mathscr{G}$ — left compatible regular orthogroups,
$\mathscr{R}\mathscr{C}\mathscr{R}\mathscr{O}\mathscr{G}$ — right compatible regular orthogroups,
$\mathscr{R}\mathscr{O}\mathscr{G}$ — regular orthogroups,
$\mathscr{O}\mathscr{G}$ — orthogroups,
$\mathscr{C}\mathscr{R}$ — completely regular semigroups,
$\mathscr{I}$ — inverse semigroups,
$\mathscr{L}\mathscr{R}\mathscr{O}$ — left regular orthodox semigroups,
$\mathscr{R}\mathscr{R}\mathscr{O}$ — right regular orthodox semigroups,

$\mathcal{RO}$ — regular orthodox semigroups,
$\mathcal{O}$ — orthodox semigroups,
$\mathcal{QO}$ — quasiorthodox semigroups,
$\mathcal{U}$ — E-unitary regular semigroups,
$\mathcal{R}$ — E-reflexive regular semigroups.

In lieu of a complete explanation of these terms, we offer here only a few basic hints; for the rest we refer to the literature on regular semigroups.

"Left regular" refers to idempotents forming a left regular band (i.e. satisfying the identity $ax=axa$); "right regular" has the corresponding meaning; "regular" means that the idempotents form a regular band (i.e. satisfy the identity $axya= =axaya$). "E-unitary" means that idempotents form a unitary subset. "Locally $\mathcal{P}$" denotes that all subsemigroups of the form eSe, where $e\in E$, have property $\mathcal{P}$. "Left compatible" stands for $\mathcal{L}$ being a congruence; "right compatible" for $\mathcal{R}$ being a congruence. "Orthodox" refers to idempotents forming a subsemigroup; if also the semigroup is completely regular, it is an "orthogroup". Finally "quasi-orthodox" stands for the semigroup generated by the idempotents being completely regular. "E-reflexive regular" means a semilattice of E-unitary regular semigroups.

We now establish some auxiliary statements leading to the lattice of certain quasivarieties of completely regular semigroups which will be useful for later considerations.

Lemma 1. *A regular semigroup S is in $\mathcal{UBG}$ if and only if S is a subdirect product of a band and a group.*

Proof. Let $S\in\mathcal{UBG}$. By ([6], Corollary 6.40), S is a subdirect product of a fundamental regular semigroup T and a group G. Since T is a homomorphic image of S, it must be a band of groups and hence a band.

The converse follows immediately.

Lemma 2. *A regular semigroup S is in $\mathcal{LUBG}$ if and only if S is a subdirect product of a band and a completely simple semigroup.*

Proof. Let $S\in\mathcal{LUBG}$. By ([2], Corollary 5.5(ii)), S is a subdirect product of a band B and a normal band of groups N. According to ([11], IV.4.3), N is a strong semilattice of completely simple semigroups, in notation $N=[Y; S_\alpha, \varphi_{\alpha,\beta}]$. Define a relation ϱ on N by: for $a\in S_\alpha$, $b\in S_\beta$,

$$a\varrho b \Leftrightarrow a\varphi_{\alpha,\gamma} = b\varphi_{\beta,\gamma} \quad \text{for some } \gamma \leqq \alpha\beta.$$

Straightforward verification shows that ϱ is a congruence and that S/ϱ is completely simple.

Now let $a\mathcal{H}\cap\varrho b$. Then $a, b\in S_\alpha$ for some $\alpha\in Y$ and $a\varphi_{\alpha,\gamma}=b\varphi_{\alpha,\gamma}$ for some $\gamma\leqq\alpha$. Letting $e\in E(H_a)$ and $f=e\varphi_{\alpha,\gamma}$, we get

$$(ab^{-1})f = (ab^{-1})\varphi_{\alpha,\gamma} = (a\varphi_{\alpha,\gamma})(b\varphi_{\alpha,\gamma})^{-1} = f.$$

There exits $u, v\in B$ such that $(u, ab^{-1}), (v,f)\in S$. It is easy to see that $(u, ab^{-1})\times\times(u, ab^{-1})^{-1}=(u, e)$ in the band of groups S. Hence $(uvu,f)=(u, e)(v,f)(u, e)$ and (u, ab^{-1}) both belong to $(u, e)S(u, e)$ and

$$(u, ab^{-1})(uvu,f) = (uvu,f).$$

Since $(u, e)S(u, e)$ is E-unitary, the above implies that (u, ab^{-1}) is an idempotent. Consequently $ab^{-1}=e$ and $a=b$. Therefore N is a subdirect product of a normal band and a completely simple semigroup. Thus S itself is a subdirect product of a band and a completely simple semigroup.

Any band and any completely simple semigroup is in $\mathcal{LUBG}$ and thus so is any subdirect product of these since a quasivariety is closed under direct products and subalgebras.

Lemma 3. *Diagram 1 with vertices labelled with script letters depicts the lattice of quasivarieties of completely regular semigroups generated by the set* $\{\mathcal{LG}, \mathcal{RG}, \mathcal{B}, \mathcal{CS}\}$.

Proof. That the meets of any two of these quasivarieties agree with those in the diagram is obvious. The joins $\mathcal{B}\vee\mathcal{G}=\mathcal{UBG}$ and $\mathcal{B}\vee\mathcal{CS}=\mathcal{LUBG}$ follow from Lemma 1 and Lemma 2, respectively; the remaining joins are consequences of well-known properties of these semigroups. The assertion of the lemma now follows by simple inspection.

If $\mathcal{C}\in\mathbf{L}$, then we say that the congruence ϱ on S is *over* $\mathcal{C}$ if the idempotent ϱ-classes belong to $\mathcal{C}$.

We can introduce the relations T_l, T_r, U, K on the congruence lattice $\mathcal{C}(S)$ in the following way. For $\varrho_1, \varrho_2\in\mathcal{C}(S)$ we say that ϱ_1 and ϱ_2 are $T_l-[T_r-, U-, K-]$ related if $\varrho_1/(\varrho_1\cap\varrho_2)$ and $\varrho_2/(\varrho_1\cap\varrho_2)$ are over $\mathcal{LG}[\mathcal{RG}, \mathcal{CS}, \mathcal{B}]$. If we put

$$T=T_l\cap T_r,\ V=U\cap K,\ K_l=T_l\cap K,\ K_r=T_r\cap K,$$

then we obviously have that ϱ_1 and ϱ_2 are $T-[V-, K_l-, K_r-]$ related if and only if $\varrho_1/(\varrho_1\cap\varrho_2)$ and $\varrho_2/(\varrho_1\cap\varrho_2)$ are over $\mathcal{G}[\mathcal{ReB}, \mathcal{LZ}, \mathcal{RZ}]$. We also see that $I=T\cap K$ is the equality relation on $\mathcal{C}(S)$. These relations

$$T_l, T_r, U, K, T, V, K_l, K_r, I \tag{1}$$

were introduced and investigated in [7] and [9]. A survey of the principal results can be found in [5].

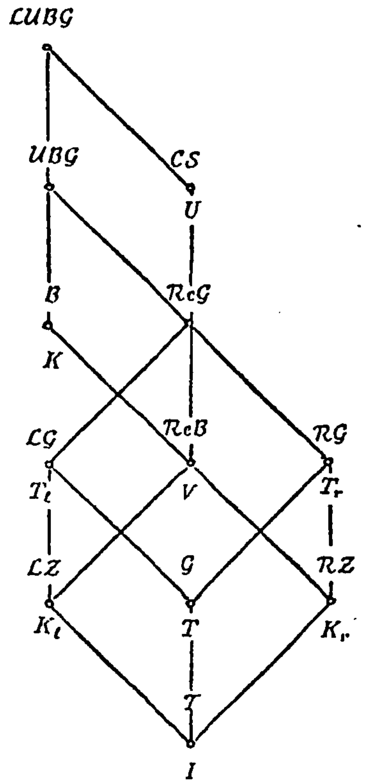

Diagram 1

In [7] and [9] it is proved that all of the relations in (1) except K are complete congruences on $\mathscr{C}(S)$. The relation K is a complete $\cap$-congruence but not necessarily a $\vee$-congruence. If $\varrho\in\mathscr{C}(S)$ and Φ is any of the relations in (1), then the Φ-class of ϱ contains a smallest element which we denote by ϱ_Φ. The sublattice of $\mathscr{C}(S)$ generated by the set $\{\varrho_{T_l}, \varrho_{T_r}, \varrho_U, \varrho_K\}$ will be called the lattice *associated with* ϱ.

We shall frequently use the following elementary result, the proof of which will be omitted.

Lemma 4. *Let* C *be a complete lattice and* Φ_1, Φ_2 *complete congruences over* C. *For any* $x\in$C *denote by* $x_{\Phi_1}[x_{\Phi_2}, x_{(\Phi_1\cap\Phi_2)}, x_{(\Phi_1\vee\Phi_2)}]$ *the least element in the* $\Phi_1-[\Phi_2-, (\Phi_1\cap\Phi_2)-, (\Phi_1\vee\Phi_2)-]$ *class of* x. *Then*

$$x_{(\Phi_1\cap\Phi_2)} = x_{\Phi_1}\vee x_{\Phi_2},\quad x_{(\Phi_1\vee\Phi_2)}(\Phi_1\vee\Phi_2)\, x_{\Phi_1}\cap x_{\Phi_2}.$$

We remark that

$$I \subseteqq K_l \subseteqq T_l \subseteqq U,\ I \subseteqq T \subseteqq T_l \subseteqq U,\ I \subseteqq K_l \subseteqq V \subseteqq K$$

and their (left-right) duals hold. From this we already have that

$$\varrho_U \subseteqq \varrho_V \subseteqq \varrho_{K_l} \subseteqq \varrho_I = \varrho,\ \varrho_U \subseteqq \varrho_{T_l} \subseteqq \varrho_T \subseteqq \varrho_I = \varrho,\ \varrho_{T_l} \subseteqq \varrho_{K_l},$$

$$\varrho_K \subseteqq \varrho_V \subseteqq \varrho_{K_l} \subseteqq \varrho_I = \varrho$$

and their duals hold. From this we find that $\varrho_V\, U\, \varrho_{K_l}$, $\varrho_T\, T_l\, \varrho$, $\varrho_{K_l}\, T_l\, \varrho$ and so on. Moreover,

Lemma 5. *Let $\varrho \in \mathscr{C}(S)$ and let Φ_1, Φ_2 be any two of the relations in* (1). *Then* $\varrho_{(\Phi_1 \cap \Phi_2)} = \varrho_{\Phi_1} \vee \varrho_{\Phi_2}$.

Proof. If neither Φ_1 nor Φ_2 is K, then we can apply Lemma 4. Let us now consider the case where one of the Φ_i equals K.

We have $\varrho_{U \cap K} = \varrho_V \supseteqq \varrho_U \vee \varrho_K$. Since $\varrho_U \subseteqq \varrho_U \vee \varrho_K \subseteqq \varrho$, we have that $\varrho/(\varrho_U \vee \varrho_K)$ is over $\mathscr{CS}$ and since $\varrho_K \subseteqq \varrho_U \vee \varrho_K \subseteqq \varrho$, we have that $\varrho/(\varrho_U \vee \varrho_K)$ is over $\mathscr{B}$. Therefore $\varrho/(\varrho_U \vee \varrho_K)$ is over $\mathscr{R}e\mathscr{B}$ and we have $\varrho_V \subseteqq \varrho_U \vee \varrho_K$. Consequently the equality $\varrho_V = \varrho_U \vee \varrho_K$ prevails.

The remaining cases involving K can be resolved in a similar way.

From the above we have $K_l = T_l \cap K = T_l \cap V$ and thus $\varrho_{K_l} = \varrho_{T_l} \vee \varrho_K = \varrho_{T_l} \vee \varrho_V$ for every $\varrho \in \mathscr{C}(S)$. Also

$$I = T \cap K = T_l \cap T_r \cap K = T_l \cap T_r \cap V = K_l \cap K_r$$

gives

$$\varrho = \varrho_T \vee \varrho_K = \varrho_{T_l} \vee \varrho_{T_r} \vee \varrho_K = \varrho_{T_l} \vee \varrho_{T_r} \vee \varrho_V = \varrho_{K_l} \vee \varrho_{K_r}$$

for every $\varrho \in \mathscr{C}(S)$.

The results concerning the relations (1) mentioned here will be used without further ado.

3. Malcev products

A class of semigroups is an *isomorphism class* if it is closed for taking isomorphic images. Let $\mathscr{X}$ and $\mathscr{Y}$ be isomorphism classes of regular semigroups. The *Malcev product of $\mathscr{X}$ and $\mathscr{Y}$ (within the class of all regular semigroups)* is the class of regular semigroups

$$\mathscr{X} \circ \mathscr{Y} = \{S \mid \text{there is a congruence } \varrho \text{ on } S \text{ over } \mathscr{X} \text{ such that } S/\varrho \in \mathscr{Y}\}.$$

We are interested here in the case where $\mathscr{X}$ is a variety of completely simple semigroups or a variety of bands and $\mathscr{Y} \in \mathbf{L}$.

For the notation and conventions incorporated in the next result, consult the preceding section.

We now define the following mapping

$$\chi = \begin{pmatrix} \mathcal{T} & \mathcal{LZ} & \mathcal{RZ} & \mathcal{R}e\mathcal{B} & \mathcal{G} & \mathcal{LG} & \mathcal{RG} & \mathcal{B} & \mathcal{CS} \\ I & K_l & K_r & V & T & T_l & T_r & K & U \end{pmatrix}. \tag{2}$$

Note that χ follows the labelling in Diagram 1. Let

$$\Gamma = \{\mathcal{T}, \mathcal{LZ}, \mathcal{RZ}, \mathcal{R}e\mathcal{B}, \mathcal{G}, \mathcal{LG}, \mathcal{RG}, \mathcal{B}, \mathcal{CS}\}, \tag{3}$$

$$\Delta = \{I, K_l, K_r, V, T, T_l, T_r, K, U\}, \tag{4}$$

both ordered by inclusion. Using the information concerning the elements of Δ listed in the preceding section, we see that Δ is an $\cap$-semilattice. Obviously Γ is also an $\cap$-semilattice and χ is an $\cap$-isomorphism of Γ onto Δ.

Theorem 1. *If $\mathcal{C}\in\mathbf{L}$ and $\mathcal{P}\in\Gamma$, then $\mathcal{P}\circ\mathcal{C}\in\mathbf{L}$ and $(\theta_{\mathcal{C}})_{\mathcal{P}\chi}=\theta_{\mathcal{P}\circ\mathcal{C}}$.*

Proof. If $\mathcal{X}, \mathcal{Y}\in\mathbf{L}$, then routine verification shows that $\mathcal{X}\circ\mathcal{Y}\in\mathbf{L}$. In particular, if $\mathcal{P}\in\Gamma$ and $\mathcal{C}\in\mathbf{L}$ we have that $\mathcal{P}\circ\mathcal{C}\in\mathbf{L}$ and $\theta_{\mathcal{P}\circ\mathcal{C}}$ exists.

For $\mathcal{P}=\mathcal{T}$ we have $(\theta_{\mathcal{C}})_I=\theta_{\mathcal{C}}=\theta_{\mathcal{T}\circ\mathcal{C}}$ and the formula holds.

We consider next the case $\mathcal{P}=\mathcal{G}$. We must show that $\theta_T=\theta_{\mathcal{G}\circ\mathcal{G}}$ where $\theta=\theta_{\mathcal{C}}$. To prove that $\theta_T\supseteq\theta_{\mathcal{G}\circ\mathcal{C}}$, we must show that θ_T is a $\mathcal{G}\circ\mathcal{C}$-congruence. By the definition of T we know that θ/θ_T is over $\mathcal{G}$. Further, $(S/\theta_T)/(\theta/\theta_T)\cong S/\theta\in\mathcal{C}$ and therefore $S/\theta_T\in\mathcal{G}\circ\mathcal{C}$. Thus indeed $\theta_{\mathcal{G}\circ\mathcal{C}}\subseteq\theta_T$. In order to establish the opposite inclusion, we consider an arbitrary $\mathcal{G}\circ\mathcal{C}$-congruence ϱ on S. There exists a congruence λ on S/ϱ such that $(S/\varrho)/\lambda\in\mathcal{C}$ and such that all idempotent λ-classes are groups. Lifting λ to S we obtain a congruence τ for which $\tau/\varrho=\lambda$. Since τ/ϱ is over $\mathcal{G}$, we have that $\varrho T\tau$. Since $S/\tau\cong(S/\varrho)/(\tau/\varrho)\in\mathcal{C}$ we have that $\theta\subseteq\tau$. Hence $\theta_T\subseteq\tau_T=\varrho_T\subseteq\varrho$ and we conclude that $\theta_T\subseteq\theta_{\mathcal{G}\circ\mathcal{C}}$.

We have proved that the above formula holds for $\mathcal{P}=\mathcal{G}$. For the remaining cases we may follow the above argument step by step.

Corollary. *For any $\mathcal{P}, \mathcal{Q}\in\Gamma$ and $\mathcal{C}\in\mathbf{L}$, we have*

$$\theta_{(\mathcal{P}\cap\mathcal{Q})\circ\mathcal{C}} = (\theta_{\mathcal{C}})_{\mathcal{P}\chi\cap\mathcal{Q}\chi} \supseteq (\theta_{\mathcal{C}})_{\mathcal{P}\chi}\vee(\theta_{\mathcal{C}})_{\mathcal{Q}\chi} = \theta_{\mathcal{P}\circ\mathcal{C}}\vee\theta_{\mathcal{Q}\circ\mathcal{C}}.$$

Proof. This follows immediately from the above theorem using the fact that χ is an $\cap$-isomorphism.

4. The lattice associated with a congruence

The main result here describes the lattice associated with any congruence ϱ on a regular semigroup S as the finite $\cap$-sublattice of the congruence lattice of S generated by eight congruences derived from ϱ by the operations introduced earlier.

In the proof of the following theorem we freely use the fact that the relations in (1) different from K are congruences on $\mathscr{C}(S)$, and that for every $\varrho\in\mathscr{C}(S)$,

(5) $$\varrho_V V \varrho_{K_l} V \varrho_{K_r} V \varrho,$$
and
(6) $$\varrho_{T_l} T_l \varrho_{K_l} T_l \varrho_T T_l \varrho$$
hold. The validity of (5) and (6) follows immediately from the definitions of the relations (1).

A glance at Diagram 2 may help visualize the heuristics behind the proof of the following theorem.

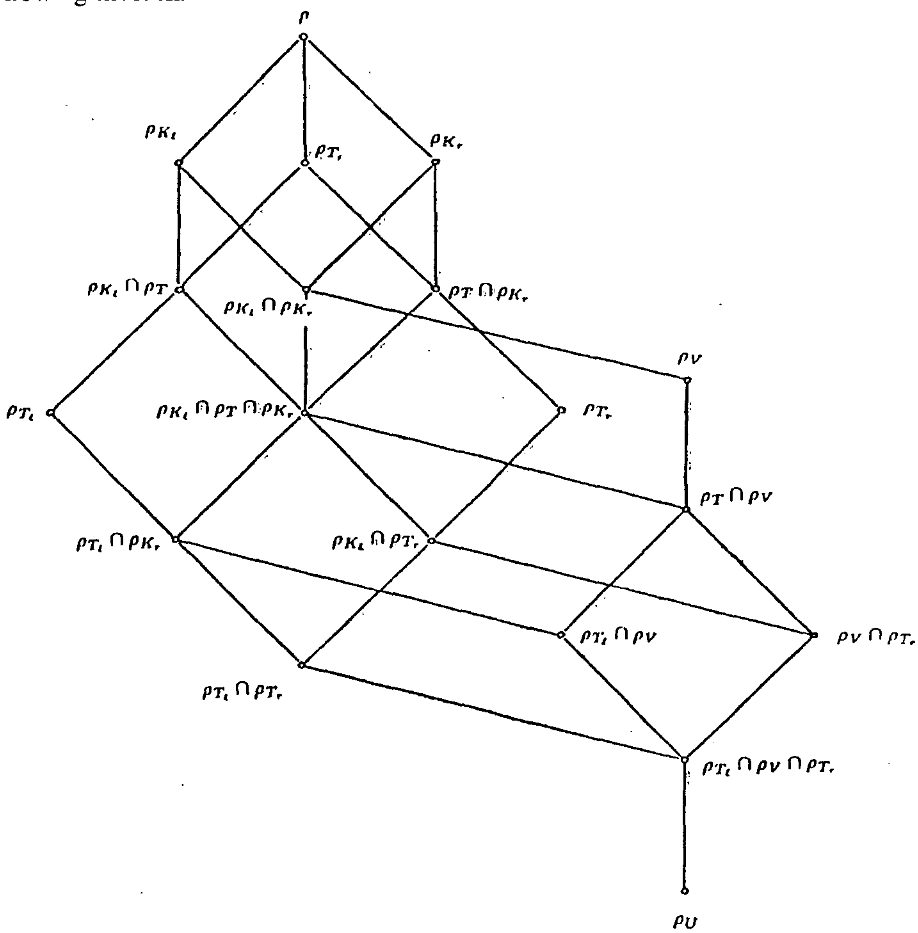

Diagram 2

Theorem 2. *Let ϱ be any congruence on a regular semigroup S. Then the sublattice of the congruence lattice $\mathscr{C}(S)$ generated by $\{\varrho_{T_l}, \varrho_{T_r}, \varrho_V, \varrho_U\}$ is the finite $\cap$-subsemilattice of $\mathscr{C}(S)$ generated by the set*

$$\{\varrho, \varrho_{K_l}, \varrho_{K_r}, \varrho_T, \varrho_{T_l}, \varrho_{T_r}, \varrho_V, \varrho_U\}. \tag{7}$$

Proof. Taking into account that

$$\varrho_U \subseteqq \varrho_{T_l} \subseteqq \varrho_{K_l} \subseteqq \varrho, \quad \varrho_{T_l} \subseteqq \varrho_T \subseteqq \varrho, \quad \varrho_U \subseteqq \varrho_V \subseteqq \varrho_{K_l}$$

and their duals hold, it is easy to see that Diagram 2 gives the $\cap$-semilattice L generated by the set (1). We shall now verify that L is a sublattice of $\mathscr{C}(S)$.

Obviously

$$\varrho_{K_l} \vee \varrho_{K_r} = \varrho, \quad \varrho_{T_l} \vee \varrho_{T_r} = \varrho_T, \quad \varrho_{T_l} \vee \varrho_V = \varrho_{K_l}, \quad \varrho_{T_r} \vee \varrho_V = \varrho_{K_r}$$

in $\mathscr{C}(S)$ and therefore also

$$\varrho_T = \varrho_{T_l} \vee (\varrho_T \cap \varrho_{K_r}) = \varrho_{T_r} \vee (\varrho_{K_l} \cap \varrho_T) = (\varrho_{K_l} \cap \varrho_T) \vee (\varrho_T \cap \varrho_{K_r}). \tag{8}$$

In the following we use the fact that T_l, T_r and V are congruences, that $T_l \cap V \cap T_r$ is the equality on $\mathscr{C}(S)$, and that (5), (6) and the dual of (6) hold. From

$$(\varrho_{K_l} \cap \varrho_T) \vee (\varrho_{K_l} \cap \varrho_{K_r}) T (\varrho_{K_l} \cap \varrho) \vee (\varrho_{K_l} \cap \varrho_{K_r}) = \varrho_{K_l}$$

and

$$(\varrho_{K_l} \cap \varrho_T) \vee (\varrho_{K_l} \cap \varrho_{K_r}) V (\varrho \cap \varrho_T) \vee \varrho = \varrho V \varrho_{K_l}$$

it follows that

$$(\varrho_{K_l} \cap \varrho_T) \vee (\varrho_{K_l} \cap \varrho_{K_r}) = \varrho_{K_l}. \tag{9}$$

From

$$\varrho_{T_l} \vee (\varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}) T_l \varrho_T \cap \varrho_{K_l},$$

$$\varrho_{T_l} \vee (\varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}) T_r \varrho_{T_l} \vee (\varrho_{K_l} \cap \varrho_T \cap \varrho) = \varrho_{K_l} \cap \varrho_T$$

and

$$\varrho_{T_l} \vee (\varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}) V \varrho_{T_l} \vee (\varrho \cap \varrho_T) = \varrho \cap \varrho_T V \varrho_{K_l} \cap \varrho_T,$$

it follows that

$$\varrho_{T_l} \vee (\varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}) = \varrho_{K_l} \cap \varrho_T. \tag{10}$$

From

$$(\varrho_{T_l} \cap \varrho_{K_r}) \vee (\varrho_{K_l} \cap \varrho_{T_r}) T_l (\varrho \cap \varrho_{K_r}) \vee (\varrho \cap \varrho_{T_r}) =$$
$$= \varrho_{K_r} = \varrho \cap \varrho \cap \varrho_{K_r} T_l \varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r},$$

the dual

$$(\varrho_{T_l} \cap \varrho_{K_r}) \vee (\varrho_{K_l} \cap \varrho_{T_r}) T_r \varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}$$

and

$$(\varrho_{T_l} \cap \varrho_{K_r}) \vee (\varrho_{K_l} \cap \varrho_{T_r}) V (\varrho_{T_l} \cap \varrho) \vee (\varrho \cap \varrho_{T_r}) =$$
$$= \varrho_T = \varrho \cap \varrho_T \cap \varrho V \varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r},$$

it follows that

$$(\varrho_{T_l} \cap \varrho_{K_r}) \vee (\varrho_{K_l} \cap \varrho_{T_r}) = \varrho_{K_l} \cap \varrho_T \cap \varrho_{K_r}. \tag{11}$$

From

$$(\varrho_{T_l}\cap\varrho_V)\vee(\varrho_V\cap\varrho_{T_r})T_l(\varrho\cap\varrho_V)\vee(\varrho_V\cap\varrho_{T_r}) = \varrho_V = \varrho\cap\varrho_V T_l\varrho_T\cap\varrho_V,$$

the dual

$$(\varrho_{T_l}\cap\varrho_V)\vee(\varrho_V\cap\varrho_{T_r})T_r\varrho_T\cap\varrho_V$$

and

$$(\varrho_{T_l}\cap\varrho_V)\vee(\varrho_V\cap\varrho_{T_r})V(\varrho_{T_l}\cap\varrho)\vee(\varrho\cap\varrho_{T_r}) = \varrho_T = \varrho_T\cap\varrho V\varrho_T\cap\varrho_V,$$

it follows that

(12) $$(\varrho_{T_l}\cap\varrho_V)\vee(\varrho_V\cap\varrho_{T_r}) = \varrho_T\cap\varrho_V.$$

From

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee(\varrho_{T_l}\cap\varrho_V)T_l(\varrho\cap\varrho_{T_r})\vee(\varrho\cap\varrho_V) = \varrho_{K_r} = \varrho\cap\varrho_{K_r}T_l\varrho_{T_l}\cap\varrho_{K_r},$$

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee(\varrho_{T_l}\cap\varrho_V)T_r(\varrho_{T_l}\cap\varrho)\vee(\varrho_{T_l}\cap\varrho_V) = \varrho_{T_l} = \varrho_{T_l}\cap\varrho T_r\varrho_{T_l}\cap\varrho_{K_r}$$

and

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee(\varrho_{T_l}\cap\varrho_V)V(\varrho_{T_l}\cap\varrho_{T_r})\vee(\varrho_{T_l}\cap\varrho) = \varrho_{T_l} = \varrho_{T_l}\cap\varrho V\varrho_{T_l}\cap\varrho_{K_r},$$

it follows that

(13) $$(\varrho_{T_l}\cap\varrho_{T_r})\vee(\varrho_{T_l}\cap\varrho_V) = \varrho_{T_l}\cap\varrho_{K_r}.$$

From

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee\varrho_V T_l(\varrho\cap\varrho_{T_r})\vee\varrho_V = \varrho_{K_r} = \varrho\cap\varrho_{K_r}T_l\varrho_{K_l}\cap\varrho_{K_r},$$

the dual

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee\varrho_V T_r\varrho_{K_l}\cap\varrho_{K_r},$$

and

$$(\varrho_{T_l}\cap\varrho_{T_r})\vee\varrho_V V(\varrho_{T_l}\cap\varrho_{T_r})\vee\varrho = \varrho V\varrho_{K_l}\cap\varrho_{K_r}$$

it follows that

(14) $$(\varrho_{T_l}\cap\varrho_{T_r})\vee\varrho_V = \varrho_{K_l}\cap\varrho_{K_r}.$$

The remaining cases now follow easily from the above equalities (8)—(14) and their duals.

Depending on the special nature of S and ϱ, some of the elements of the lattice L occurring in Diagram 2 may coincide. Therefore, we have the following result.

Corollary. *Let ϱ be a congruence on the regular semigroup S. Then the sublattice of $\mathscr{C}(S)$ generated by $\{\varrho_{T_l}, \varrho_{T_r}, \varrho_V, \varrho_U\}$ is a homomorphic image of the lattice of Diagram 2.*

In the following we shall show that the lattice in Diagram 2 *can be* the lattice generated by $\{\varrho_{T_l}, \varrho_{T_l}, \varrho_V, \varrho_U\}$ for a suitable ϱ. For this we shall consider $\varrho=\eta$. Therefore we can say that, *in general,* Diagram 2 depicts the lattice generated by $\{\varrho_{T_l}, \varrho_{T_r}, \varrho_V, \varrho_U\}$.

5. The lattices associated with ω and σ

As we shall see, the situation here is very simple.

Theorem 3. *Diagrams* 3a *and* 3b *depict the lattices associated with the universal relation* ω *and the least group congruence* σ, *respectively.*

Proof. All the equalities in Diagram 3a follow from Theorem 1 since for any class $\mathcal{P}$ of regular semigroups we have $\mathcal{P}\circ\mathcal{T}=\mathcal{P}$ and $\mathcal{T}\chi=I$. As in the proof of Lemma 3, the only assertions about meets and joins which are not well-known are $\theta_{\mathcal{G}}\cap\theta_{\mathcal{B}}=\theta_{\mathcal{U}\mathcal{B}\mathcal{G}}$ and $\theta_{\mathcal{C}\mathcal{S}}\cap\theta_{\mathcal{B}}=\theta_{\mathcal{C}\mathcal{U}\mathcal{B}\mathcal{G}}$. These follow directly from Lemma 1 and Lemma 2, respectively.

The equalities at the vertices of Diagram 3b follow from Theorem 1 in view of the equalities

$$\mathcal{G}\circ\mathcal{G}=\mathcal{G},\ \mathcal{L}\mathcal{G}\circ\mathcal{G}=\mathcal{L}\mathcal{Z}\circ\mathcal{G}=\mathcal{L}\mathcal{G},$$

$$\mathcal{R}\mathcal{G}\circ\mathcal{G}=\mathcal{R}\mathcal{Z}\circ\mathcal{G}=\mathcal{R}\mathcal{G},\ \mathcal{B}\circ\mathcal{G}=\mathcal{U},\ \mathcal{C}\mathcal{S}\circ\mathcal{G}=\mathcal{C}\mathcal{S}.$$

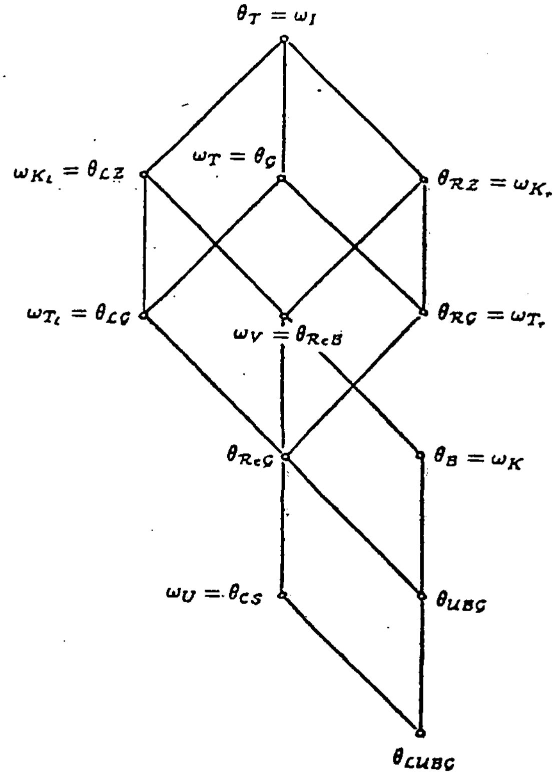

Diagram 3a

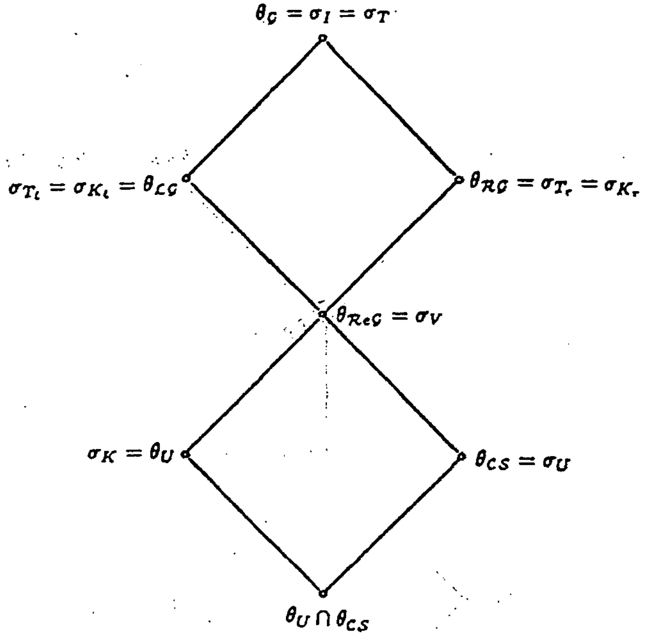

Diagram 3b

Here $\mathcal{B}\circ\mathcal{G}=\mathcal{U}$ follows from ([6], Theorem 6.37); the remaining equalities can be easily verified. Joins and meets follow from well-known characterizations of semigroups in the respective classes.

The above proof also indicates that Diagram 3a is just the inverted Diagram 1. To see that the meets in Diagram 3a are the correct ones, we observe that the joins in Diagram 1 for quasivarieties amount to taking subdirect products which corresponds to taking intersection of minimal congruences $\theta_{\mathcal{C}}$ in Diagram 1. We leave the structural description of $\mathcal{U}\vee\mathcal{CS}$ open.

6. The lattices associated with ν and γ

Recall that Clifford semigroup is a synonym for semilattice of groups. The situation here is somewhat more complex.

Theorem 4. *Diagrams* 4a *and* 4b *depict the lattices associated with the least Clifford congruence ν and the least inverse congruence γ, respectively.*

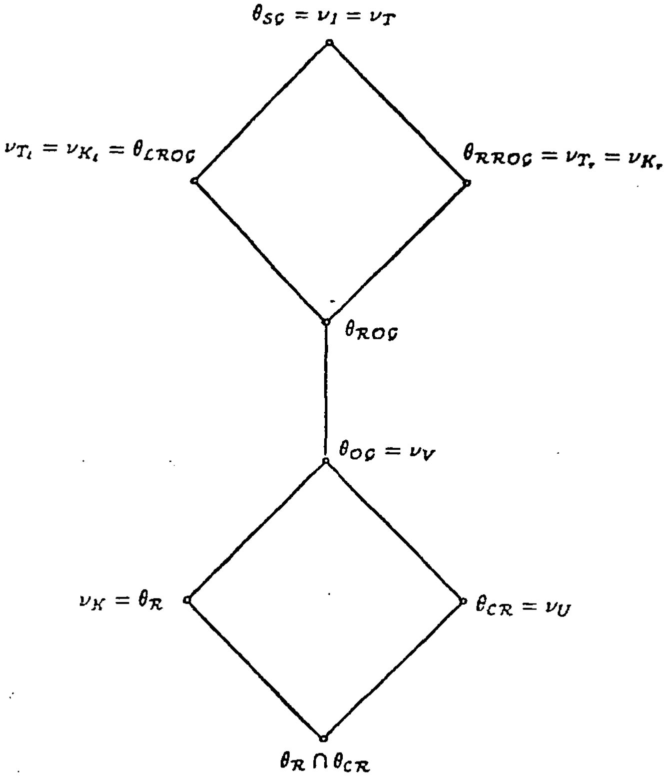

Diagram 4a

Proof. The equalities at the vertices of Diagrams 4a and 4b will follow from Theorem 1 if we establish the corresponding statements about Malcev products, which we proceed to do.

1. $\mathscr{G}\circ\mathscr{S}\mathscr{G}=\mathscr{S}\mathscr{G}$. Let $S\in\mathscr{G}\circ\mathscr{S}\mathscr{G}$ so that there exists a congruence θ on S such that θ is over $\mathscr{G}$ and $S/\theta\in\mathscr{S}\mathscr{G}$. Hence S/θ is a semilattice Y of groups G_α. For each $\alpha\in Y$, let $S_\alpha=G_\alpha\theta$ so that S is a semilattice Y of semigroups S_α. But $S_\alpha\in\mathscr{G}\circ\mathscr{G}$ whence $S_\alpha\in\mathscr{G}$. Thus S is a semilattice of groups S_α whence $S\in\mathscr{S}\mathscr{G}$. Therefore $\mathscr{G}\circ\mathscr{S}\mathscr{G}\subseteqq\mathscr{S}\mathscr{G}$; the oppositive inclusion is trivial.

2. $\mathscr{L}\mathscr{Z}\circ\mathscr{S}\mathscr{G}=\mathscr{L}\mathscr{G}\circ\mathscr{S}\mathscr{G}=\mathscr{L}\mathscr{R}\mathscr{O}\mathscr{G}$. Trivially $\mathscr{L}\mathscr{Z}\circ\mathscr{S}\mathscr{G}\subseteqq\mathscr{L}\mathscr{G}\circ\mathscr{S}\mathscr{G}$. Next let $S\in\mathscr{L}\mathscr{G}\circ\mathscr{S}\mathscr{G}$ so that there exists a congruence θ on S such that θ is over $\mathscr{L}\mathscr{G}$ and $S/\theta\in\mathscr{S}\mathscr{G}$. Let $a\in S$ and $b\in V(a)$. Then $b\theta\in V(a\theta)$ so $(a\theta)(b\theta)=(b\theta)(a\theta)$ since $S/\theta\in\mathscr{S}\mathscr{G}$. Thus $ab\;\theta\;ba$ and $(ab)\theta\in\mathscr{L}\mathscr{G}$ so there exists $x\in S$ such that $ab=xba$ whence $a=aba=xba^2$. It follows that $a\in aSa^2$ for every $a\in S$ and thus $S\in\mathscr{C}\mathscr{R}$ by ([11], IV. 1.6). Now let $e,f\in E$. Then $ef\,\theta\,fe$ and $(ef)\theta=(e\theta)(f\theta)\in E(S/\theta)$ since

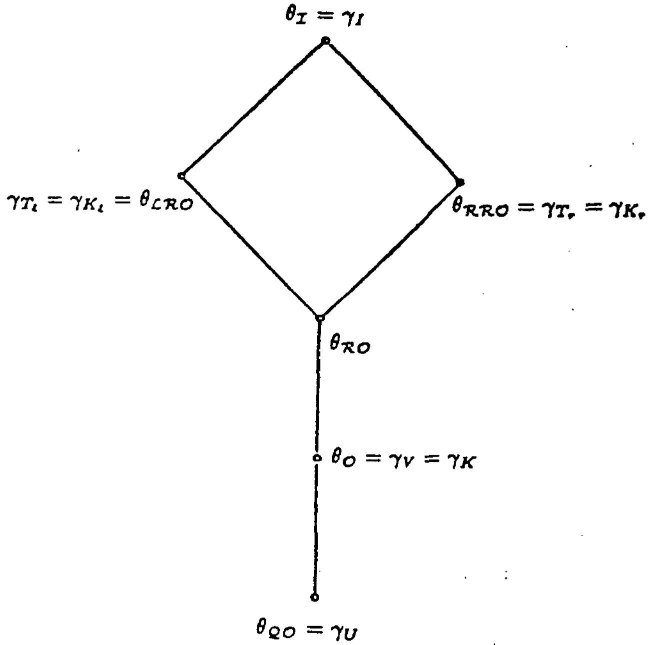

Diagram 4b

$S/\theta\in\mathscr{S}\mathscr{G}$. Hence $(ef)\theta\in\mathscr{L}\mathscr{G}$ and there exists $x\in(ef)\theta$ such that $ef=xfe$. It follows that $efe=ef$ which proves that $S\in\mathscr{L}\mathscr{R}\mathscr{O}\mathscr{G}$.

We now let $S\in\mathscr{L}\mathscr{R}\mathscr{O}\mathscr{G}$. Then ([6], Theorem 6.20) gives the structure of S in terms of $L=(Y; L_\alpha)\in\mathscr{L}\mathscr{R}\mathscr{B}$, $T=(Y; G_\alpha)\in\mathscr{S}\mathscr{G}$ and $R=Y\in\mathscr{R}\mathscr{R}\mathscr{B}$. We identify S with the construction in the above reference. Let $\varphi\colon S\to T$ be the homomorphism $(i, g, \lambda)\to g$. Let $e,f\in E(T)$ be such that $(i, e, \lambda)\varphi=(j, f, \mu)\varphi$. Then $e=f$ and thus $\lambda=\mu$ which implies that $(i, e, \lambda)=(i, e, \lambda)(j, e, \lambda)$. Therefore θ is over $\mathscr{L}\mathscr{Z}$ and $S/\theta\in\mathscr{S}\mathscr{G}$ and thus $S\in\mathscr{L}\mathscr{Z}\circ\mathscr{S}\mathscr{G}$.

3. $\mathscr{R}e\mathscr{B}\circ\mathscr{S}\mathscr{G}=\mathscr{O}\mathscr{G}$. Let $S\in\mathscr{R}e\mathscr{B}\circ\mathscr{S}\mathscr{G}$ so that there exists a congruence θ on S such that θ is over $\mathscr{R}e\mathscr{B}$ and $S/\theta\in\mathscr{S}\mathscr{G}$. Let $a\in S$ and $b\in V(a)$. Then $(ab)\theta= =(ba)\theta\in E(S/\theta)$ and so $ab=abbaab$ since θ is over $\mathscr{R}e\mathscr{B}$. Therefore

$$a = aba = abbaaba = abba^2\in aSa^2$$

which in view of ([11], IV. 1.6) gives that $S\in\mathscr{C}\mathscr{R}$. Next let $e,f\in E$. We have $(ef)\theta= =(e\theta)(f\theta)\in E(S/\theta)$ since $S/\theta\in\mathscr{S}\mathscr{G}$. Since θ is over $\mathscr{R}e\mathscr{B}$, it follows that $ef\in E$. Therefore $S\in\mathscr{O}\mathscr{G}$.

Conversely, let $S\in\mathscr{O}\mathscr{G}$. Then the relation γ defined by

$$a\gamma b \Leftrightarrow V(a) = V(b)$$

is the least inverse congruence on S, see ([3], VI. 1.12). Hence $S/\gamma\in\mathcal{S}\mathcal{G}$. Let $e\in E$ and $a\,\gamma\, e$. Then $e\in V(e)=V(a)$ so that $a\in V(e)$. By ([14], Lemma 1.3), we must have $a\in E$. In addition $a=aea$ and since e and a are arbitrary γ-related elements, we conclude that γ is over $\mathcal{R}e\mathcal{B}$. Consequently $S\in\mathcal{R}e\mathcal{B}\circ\mathcal{S}\mathcal{G}$.

4. $\mathcal{B}\circ\mathcal{S}\mathcal{G}=\mathcal{R}$. This forms a part of ([4], Theorem 2 and [6], Theorem 6.43).

5. $\mathcal{C}\mathcal{S}\circ\mathcal{S}\mathcal{G}=\mathcal{C}\mathcal{R}$. Let $S\in\mathcal{C}\mathcal{S}\circ\mathcal{S}\mathcal{G}$ so that there exists a congruence θ on S such that θ is over $\mathcal{C}\mathcal{S}$ and $S/\theta\in\mathcal{S}\mathcal{G}$. Then S/θ is a semilattice Y of groups G_α, say. Letting $S_\alpha=G_\alpha\theta$ for every $\alpha\in Y$, we get that S is a semilattice Y of semigroups S_α, where $S_\alpha\in\mathcal{C}\mathcal{S}\circ\mathcal{G}$. It follows easily that $\mathcal{C}\mathcal{S}\circ\mathcal{G}=\mathcal{C}\mathcal{S}$. Hence S is a semilattice of completely simple semigroups and therefore $S\in\mathcal{C}\mathcal{R}$. Thus $\mathcal{C}\mathcal{S}\circ\mathcal{S}\mathcal{G}\subseteq\mathcal{C}\mathcal{R}$; the opposite inclusion follows from $\mathcal{C}\mathcal{S}\circ\mathcal{S}=\mathcal{C}\mathcal{R}$.

That $\theta_{\mathcal{L}\mathcal{R}\mathcal{O}\mathcal{G}}\cap\theta_{\mathcal{R}\mathcal{R}\mathcal{O}\mathcal{G}}=\theta_{\mathcal{R}\mathcal{O}\mathcal{G}}$ follows directly from ([15], Theorem 3).

6. $\mathcal{L}\mathcal{Z}\circ\mathcal{I}=\mathcal{L}\mathcal{G}\circ\mathcal{I}=\mathcal{L}\mathcal{R}\mathcal{O}$. The argument here amounts to a simplification of that in part 2 above.

7. $\mathcal{R}e\mathcal{B}\circ\mathcal{I}=\mathcal{B}\circ\mathcal{I}=\mathcal{O}$. Trivially $\mathcal{R}e\mathcal{B}\circ\mathcal{I}\subseteq\mathcal{B}\circ\mathcal{I}$. Let $S\in\mathcal{B}\circ\mathcal{I}$ so that there exists a congruence θ on S such that θ is over $\mathcal{B}$ and $S/\theta\in\mathcal{I}$. Let $e,f\in E$. Then $ef\,\theta\, fe$ since $S/\theta\in\mathcal{I}$ and hence $ef\,\theta\, fe\,\theta\, efef$. It follows that $(ef)\theta$ is an idempotent θ-class so that $(ef)\theta\in\mathcal{B}$. But then $ef\in E$ which proves that $S\in\mathcal{O}$. The argument for $\mathcal{O}\subseteq\mathcal{R}e\mathcal{B}\circ\mathcal{I}$ is virtually identical to the proof of the converse of Part 2 above.

8. $\mathcal{C}\mathcal{S}\circ\mathcal{I}=\mathcal{Q}\mathcal{O}$. This is the content of ([17], Theorem 7.1).

The relation $\theta_{\mathcal{L}\mathcal{R}\mathcal{O}}\cap\theta_{\mathcal{R}\mathcal{R}\mathcal{O}}=\theta_{\mathcal{R}\mathcal{O}}$ follows directly from ([15], Theorem 3).

7. The lattice associated with η

In order to treat this case, we need some preparation.

Lemma 6. *A regular semigroup is in $\mathcal{L}\mathcal{R}\mathcal{B}\mathcal{G}$ if and only if it is a subdirect product of a Clifford semigroup and a left regular band.*

Proof. Let $S\in\mathcal{L}\mathcal{R}\mathcal{B}\mathcal{G}$. By ([10], Theorem 3.2), S is a subdirect product of $S/\theta_{\mathcal{S}\mathcal{G}}$ and $S/\theta_{\mathcal{B}}$. Since $S\in\mathcal{L}\mathcal{R}\mathcal{B}\mathcal{G}$, we have that $\theta_{\mathcal{B}}=\theta_{\mathcal{L}\mathcal{R}\mathcal{B}}$ so that $\theta_{\mathcal{S}\mathcal{G}}\cap\theta_{\mathcal{L}\mathcal{R}\mathcal{B}}=\varepsilon$ and the assertion follows. The converse is trivial.

Lemma 7. *A regular semigroup is in $\mathcal{L}\mathcal{C}\mathcal{R}\mathcal{O}\mathcal{G}$ if and only if it is a subdirect product of a left regular orthogroup and a right regular band.*

Proof. Let S be in $\mathcal{L}\mathcal{C}\mathcal{R}\mathcal{O}\mathcal{G}$. By ([16], Theorem 2), we have $\gamma=(\mathcal{L}|_E)^*\vee(\mathcal{R}|_E)^*$, the least inverse congruence. The argument in Part 3 of the proof of Theorem 5 shows that γ is over $\mathcal{R}\mathcal{B}$. Since $(\mathcal{R}|_E)^*\subseteq\gamma$, it follows that $\ker(\mathcal{R}|_E)^*\subseteq\ker\gamma=E$

and equality prevails so that

$$\ker\left((\mathscr{R}|_E)^* \cap \mathscr{L}^*\right) = \ker(\mathscr{R}|_E)^* \cap \ker \mathscr{L}^* = E. \tag{1}$$

By ([16], Theorem 2), $(\mathscr{L}|_E)^* \cap (\mathscr{R}|_E)^* = \theta_{\mathscr{RO}}$ and since $S \in \mathscr{LCROG}$, we get $(\mathscr{L}|_E)^* \cap (\mathscr{R}|_E)^* = \varepsilon$, the equality relation on S. Hence no distinct $\mathscr{L}$-related idempotents of S can be $(\mathscr{R}|_E)^*$-related and we conclude that $\operatorname{tr}(\mathscr{R}|_E)^* = \mathscr{R}|_E$. Since $S \in \mathscr{LCROG}$, we also have that $\mathscr{L} = \mathscr{L}^*$ so that $\operatorname{tr}\mathscr{L}^* = \mathscr{L}|_E$ which gives

$$tr\left((\mathscr{R}|_E)^* \cap \mathscr{L}^*\right) = tr(\mathscr{R}|_E)^* \cap tr\mathscr{L}^* = \mathscr{R}|_E \cap \mathscr{L}|_E = \varepsilon, \tag{2}$$

the equality relation on E. It is well-known that relations (1) and (2) imply that $(\mathscr{R}|_E)^* \cap \mathscr{L}^* = \varepsilon$. It now follows from ([16], Theorem 2) and ([8], Theorem 1(i)), that $\theta_{\mathscr{LRO}} \cap \theta_{\mathscr{RRB}} = \varepsilon$ whence $\theta_{\mathscr{LROG}} \cap \theta_{\mathscr{RRB}} = \varepsilon$ and thus S is a subdirect product of a left regular orthogroup and a right regular band.

Conversely, let S be a subdirect product of a left regular orthogroup T and a right regular band B. Then S is a regular orthogroup since $\mathscr{ROG}$ is closed under direct products and regular subsemigroups. Since $\mathscr{L}$ is a congruence in both T and B, it follows easily that the same holds for $T \times B$ and hence also for S. Therefore $S \in \mathscr{LCROG}$.

Theorem 5. *Diagram 5 depicts the lattice associated with the least semilattice congruence η.*

Proof. Equalities at the vertices of Diagram 5 follow directly from Theorem 1 in view of the well-known equalities:

$$\mathscr{LZ} \circ \mathscr{S} = \mathscr{LRB}, \quad \mathscr{G} \circ \mathscr{S} = \mathscr{SG}, \quad \mathscr{LG} \circ \mathscr{S} = \mathscr{LROG}, \quad \mathscr{CS} \circ \mathscr{S} = \mathscr{CR},$$

$$\mathscr{RB} \circ \mathscr{S} = \mathscr{B} \circ \mathscr{S} = \mathscr{B}$$

and their duals.

The relation $\theta_{\mathscr{LRB}} \cap \theta_{\mathscr{SG}} = \theta_{\mathscr{LRBG}}$ follows from Lemma 6 and $\theta_{\mathscr{LROG}} \cap \theta_{\mathscr{RRB}} = \theta_{\mathscr{LCROG}}$ from Lemma 7. The relations

$$\theta_{\mathscr{LRB}} \cap \theta_{\mathscr{RRB}} = \theta_{\mathscr{RB}}, \quad \theta_{\mathscr{LRBG}} \cap \theta_{\mathscr{RRBG}} = \theta_{\mathscr{ROBG}}, \quad \theta_{\mathscr{LROG}} \cap \theta_{\mathscr{RROG}} = \theta_{\mathscr{ROG}}$$

follow easily from ([15], Theorem 3). Also, the relation $\theta_{\mathscr{SG}} \cap \theta_{\mathscr{B}} = \theta_{\mathscr{OBG}}$ follows from ([10], Theorem 3.4).

One can convine oneself on examples of regular semigroups that the classes $\mathscr{OBG}$, $\mathscr{LROG} \vee \mathscr{B}$, $\mathscr{RROG} \vee \mathscr{B}$, $\mathscr{ROG} \vee \mathscr{B}$ and $\mathscr{CR}$ are distinct. Hence the lower right part of Diagram 5 does not collapse in general. The assertion of the theorem now follows by Theorem 2, see Diagram 2.

We leave the structural description of the semigroups in $\mathscr{LROG} \vee \mathscr{B}$, $\mathscr{RROG} \vee \mathscr{B}$ and $\mathscr{ROG} \vee \mathscr{B}$ open.

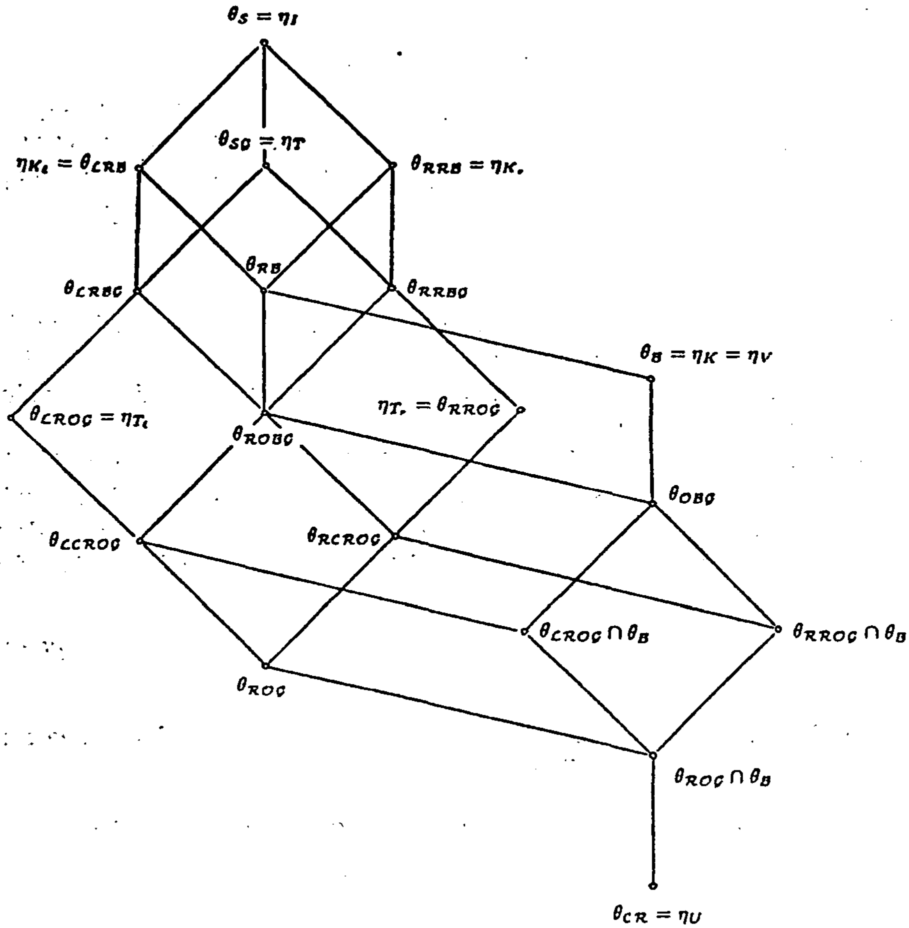

Diagram 5

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