Reflexivity and direct sums

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1. Introduction. Let H be a Hilbert space and let $B(H)$ be the set of (bounded linear) operators on H. If $\mathscr{S} \subset B(H)$, then the commutant $\mathscr{S}'$ of $\mathscr{S}$ is the set of all operators that commute with every operator in $\mathscr{S}$. Also Lat $\mathscr{S}$ denotes the set of (closed linear) subspaces of H that are left invariant under every operator in $\mathscr{S}$, and Alg Lat $\mathscr{S}$ denotes the set of all operators T such that Lat $\mathscr{S} \subset$ Lat T. A unital weakly closed subalgebra $\mathscr{A}$ of $B(H)$ is reflexive if $\mathscr{A} =$ Alg Lat $\mathscr{A}$, and an operator T is reflexive if the weakly closed algebra $\mathscr{A}(T)$ generated by 1 and T is reflexive. A commutative subalgebra $\mathscr{A}$ of $B(H)$ is hyporeflexive if $\mathscr{A} = \mathscr{A}' \cap$ Alg Lat $\mathscr{A}$, and an operator T is hyporeflexive if $\mathscr{A}(T)$ is. Much work has been done on reflexivity (see e.g., [1]—[7], [10], [11], [14]—[27], [30]—[32]). Recently W. WOGEN [31], answering a question of P. ROSENTHAL and D. SARASON, has constructed a class of operators that are not hyporeflexive.

This paper contains two main ideas. The first idea deals with very general types of shifts, and we prove, for a large class of operators T, that Alg Lat $T \subset \{T\}'$. For such operators, the problems of reflexivity and hyporeflexivity coincide. In some cases we show that the elements of Alg Lat T correspond to formal power series in T. As a consequence we show that every operator-weighted shift is hyporeflexive and that every operator-weighted shift with 1—1 weights of rank at least 2 is reflexive. In particular, the direct sum of two weighted shifts with nonzero scalar weights is reflexive.

The second idea concerns reflexive graphs. Suppose $\mathscr{A}$ is a reflexive algebra of operators and $\pi: \mathscr{A} \to B(H_\pi)$ is a homomorphism. We deal with the problem of when the graph of π is a reflexive subalgebra of $B(H \oplus H_\pi)$. In particular, we show that if the algebra $\mathscr{A}$ has property D of [17] and if π is continuous in the weak operator topology, then the graph of π is reflexive. We use these results to show that if T is polynomially bounded and S is the unilateral shift operator, then $S \oplus T$ is reflexive.

We also show that if S is a nonreductive subnormal operator, then there is a nonempty open set Ω of complex numbers such that $S \oplus T$ is reflexive for every operator T whose spectrum is contained in Ω.

2. Shifted vectors. The principal technique of this section shows that Alg Lat $T \subset \{T\}'$ for many operators T. Much of what will be done is valid in the context of a complex locally convex topological vector space X; indeed, some of the results hold in an arbitrary vector space. If E is a subset of X, then we will write sp E for the linear span of E and $\overline{\text{sp}}\, E$ for the closed linear span of E. A biorthogonal base for a closed subspace M of X is a finite or infinite sequence $\{e_k\}$ whose closed linear span is M for which there exists a corresponding dual sequence $\{\varphi_k\}$ in X^*, the dual space of all continuous linear functionals on X, such that $\varphi_j(e_k) = \delta_{jk}$, for all j and k, where δ is the Kronecker δ, and such that $M \cap (\bigcap_k \ker \varphi_k) = \{0\}$. Equivalently $\{e_k\}$ is a biorthogonal base for M if and only if $\{e_k\}$ is a spanning set for M, e_j is not in $\overline{\text{sp}}\, \{e_k : k > j\}$ for every j, and $\bigcap_j \overline{\text{sp}}\, \{e_k : k \geq j\} = \{0\}$.

If $\{e_k\}$ is a biorthogonal base for a closed subspace M and $\{\varphi_k\}$ is the corresponding dual base, then, for every x in M, the sequence $\{\varphi_k(x)\}$ determines x. Hence we will write $x \sim \sum_k \varphi_k(x) e_k$ to indicate this relationship. Note that for x in sp $\{e_k : k \geq 0\}$, $x = \sum_k \varphi_k(x) e_k$, but in general the series need not converge. Clearly if $x \sim \sum_k a_k e_k$, $y \sim \sum_k b_k e_k$, and if α and β are scalars, then $\alpha x + \beta y \sim \sum_k (\alpha a_k + \beta b_k) e_k$.

Let T be a continuous linear operator on X. We will write $M(x) = M_T(x)$ for the smallest invariant linear manifold of T that contains the vector x, i.e. $M(x) = $ sp $\{T^k x : k \geq 0\}$. A vector x is called a shifted vector for T in case the nonzero vectors in $\{T^k x : k \geq 0\}$ form a biorthogonal base for $M(x)^-$. Let the order of x, ord (x), be the dimension of sp $\{T^k x : k \geq 0\}$. The following lemma is easily established, and the proof is left to the reader.

Lemma 2.1. *The following are equivalent:*
(1) *x is a shifted vector for T;*
(2) $\bigcap_j M(T^j x)^- = \{0\}$.

It may be tempting to think that x is a shifted vector for T if $T^n x \notin M(T^{n+1}x)^-$ for $0 \leq n < $ ord (x). However, if one lets $\{e_n : n \geq 0\}$ be an orthonormal basis for a Hilbert space H and lets T be the operator defined by $Te_0 = e_0$, and $Te_n = (n/n+1)e_{n+1}$ for $n = 1, 2, 3, \ldots$, and lets $x = e_0 + e_1$, the temptation quickly fades away.

We remark that if x is a shifted vector for T and $\{T^k x\}$ has a dual sequence $\{\varphi_k\}$, then for each polynomial p, we have $\varphi_k(p(T)x) = \varphi_{k+1}(Tp(T)x)$. Thus if $y \sim \sum_k a_k T^k x$, then $Ty \sim \sum_k a_k T^{k+1}x$. More generally, if $A \in B(X)$, A is invertible, and $AT = TA$, then, for each shifted vector x for T, Ax is a shifted vector for T of the

same order as x, and whenever $y \sim \sum_k a_k T^k x$, we have $Ay \sim \sum_k a_k T^k Ax$. To see this, first note that $\bigcap_j M(T^j Ax)^- = A[\bigcap_j M(T^j x)^-] = \{0\}$; whence, by Lemma 2.1, Ax is a shifted vector for T. If $\{\psi_k\}$ is the corresponding dual base for $\{T^k Ax\}$, then $\{A^* \psi_k\}$ is a dual base for $\{T^k x\}$, i.e., $(A^* \psi_k)(T^n x) = \psi_k(AT^n x) = \delta_{kn}$. Thus $\varphi_k = A^* \psi_k$ on $M(x)$ for $k \geq 0$. If $y \in M(x)$, and $y \sim \sum_k a_k T^k x$, then, for each $k \geq 0$, we have $a_k = \varphi_k(y) = (A^* \psi_k)(y) = \psi_k(Ay)$. Hence, $Ay \sim \sum_k a_k T^k Ax$.

The following notion will be basic for our needs. A pair of vectors x, y is called a shifted pair for T if sp $\{T^k x: k \geq 0\} \cap \mathrm{sp}\{T^k y: k \geq 0\} = \{0\}$ and $(\{T^k x: k \geq 0\} \cup \cup \{T^k y: k \geq 0\}) \setminus \{0\}$ is a biorthogonal base for its closed span $M(x, y)^- = M_T(x, y)^-$. In this case each w in $M(x, y)^-$ is associated with a formal series, $w \sim \sum_k a_k T^k x + + \sum_k b_k T^k y$.

Lemma 2.2. *Suppose x and y are shifted vectors for T and $m = \mathrm{ord}\,(x) < \mathrm{ord}\,(y) = = \infty$. Then $\{x, y\}$ is a shifted pair for T.*

Proof. Note that ord $(y) = \infty$ implies that T is 1—1 on $M(y)^-$. If $v \in M(x)^- \cap \cap M(y)^-$, then $T^m v = 0$ (since $v \in M(x)$) and thus $v = 0$ (since T^m is 1—1 on $M(y)^-$). Hence $M(x)^- \cap M(y)^- = 0$. Since $M(x)$ is finite dimensional, it follows that $M(x, y)^- = M(x)^- + M(y)^-$ is a direct sum and that the projections onto $M(x)^-$ and $M(y)^-$ are continuous. Thus $\{x, y\}$ is a shifted pair for T.

Lemma 2.3. *Suppose T is a continuous linear transformation on a locally convex space X and that x and y are shifted vectors of orders m and n respectively. Suppose also that $\{x, y\}$ is a shifted pair for T. Let $S \in \mathrm{Alg}\,\mathrm{Lat}\,T$, and suppose $Sx \sim \sum_k a_k T^k x$ and $Sy \sim \sum_k b_k T^k y$. The following are true.*

(1) *Every nonzero vector in $M(x, y)^-$ is a shifted vector for T.*

(2) *Suppose $z \in M(x, y)^-$, $z \sim \sum_k (c_k T^k x + d_k T^k y)$, $c_i \neq 0$ for some i, and $m = = n = \infty$. Then $\{z, y\}$ is a shifted pair for T.*

(3) *If $m \leq n$, then*
   (a) *$a_i = b_i$ for $0 \leq i < m$,*
   (b) *$Sz \sim \sum_k b_k T^k z$ for every z in $M(x)^-$,*
   (c) *$ST = TS$ on $M(x)^-$.*

Proof. (1). If $z \in M(x, y)^-$, then $\{T^k z: k \geq j\} \subset \overline{\mathrm{sp}}(\{T^k x: k \geq j\} \cup \{T^k y: k \geq j\})$, and since

$$\bigcap_{j=0}^{\infty} \overline{\mathrm{sp}}(\{T^k x: k \geq j\} \cup \{T^k y: k \geq j\}) = \{0\},$$

it follows that z is a shifted vector for T.

(2) Let $z \sim \sum_k (c_k T^k x + d_k T^k y)$ and let i be the smallest index for which $c_i \neq 0$. It will be shown that $\{z, y\}$ is a shifted pair for T. Let $M_j = \overline{sp}(\{T^k z : k \neq j\} \cup \{T^k y : k \geq 0\})$ and $N_j = \overline{sp}(\{T^k z : k \geq 0\} \cup \{T^k y : k \neq j\})$ for every j. If $T^j z \in M_j$ for some j, then there is a smallest p such that

$$T^j z = \alpha_p T^p z + \alpha_{p+1} T^{p+1} z + \ldots + \alpha_{j-1} T^{j-1} z + w,$$

$\alpha_p \neq 0$, and $w \in \overline{sp}(\{T^k z : k > j\} \cup \{T^k y : k \geq 0\})$. It follows that $T^p z \in M_p$, and thus we will suppose that $T^j z \in \overline{sp}(\{T^k z : k > j\} \cup \{T^k y : k \geq 0\})$ and show that this leads to a contradiction. Let φ_k and v_s be continuous linear functionals on X for $k \geq 0$ and $s \geq 0$ such that $\varphi_k(T^p x) = \delta_{kp}$, $\varphi_k(T^p y) = 0$, $v_s(T^p x) = 0$, and $v_s(T^p y) = \delta_{sp}$ for all $p \geq 0$. Then $\varphi_{i+j}(T^j z) = c_i \neq 0$, but $\varphi_{i+j}(T^k z) = 0$ for $k > j$ and $\varphi_{i+j}(T^k y) = 0$ for all k, which is impossible in view of our supposition. Thus $T^j z$ is not in M_j for every j.

A similar argument will show that $T^j y$ is not in N_j for every $j < n$. As above, it will suffice to show that $T^j y$ belonging to $\overline{sp}(\{T^k z : k \geq 0\} \cup \{T^k y : k > j\})$ leads to a contradiction. We have

$$T^j y = \sum_{k \leq j} \alpha_k T^k z + w,$$

where $w \in \overline{sp}(\{T^k z : k > j\} \cup \{T^k y : k > j\})$. Applying $\varphi_i, \varphi_{i+1}, \ldots, \varphi_{i+j}$ successively to $T^j y$, we see that $\alpha_0 = \alpha_1 = \ldots = \alpha_j = 0$. It follows that $T^j y = w$, and hence $1 = v_j(T^j y) = v_j(w) = 0$, which is a contradiction. Thus $T^j y$ is not in N_j for every j, and consequently $\{z, y\}$ is a shifted pair for T.

(3a) By (1), $x + y$ is a shifted vector for T. Let φ_k and v_k be functionals as in the proof of part (2). For every polynomial p, it is clear that $\varphi_k(p(T)(x+y)) = v_k(p(T) \cdot (x+y))$ for all $k < m$, and thus $\varphi_k(z) = v_k(z)$ for all $k < m$ and every z in $M(x+y)^- = \overline{sp}\{T^k(x+y) : k \geq 0\}$. Since $M(x+y)^-$ is invariant under S and $S(x+y) = Sx + Sy$, it follows that

$$a_k = \varphi_k(Sx) = \varphi_k(Sx + Sy) = v_k(Sx + Sy) = v_k(Sy) = b_k$$

for $k < m$.

(3b) If $z \in M(x)^-$, then z is a shifted vector for T and $\text{ord}(z) \leq m$. If $m < \infty$, then $\{z, y\}$ is a shifted pair, and if $m = \infty$, then $\{z, y\}$ is a shifted pair by part (2) above. Applying (3a) to the pair $\{z, y\}$, we obtain (3b).

(3c) If $z \in M(x)^-$, then $Tz \in M(x)^-$ and (3b) implies both $Sz \sim \sum_k b_k T^k z$ and $STz \sim \sum_k b_k T^k Tz = \sum_k b_k T^{k+1} z$. Also $Sz \sim \sum_k b_k T^k z$ implies that $TSz \sim \sum_k b_k T^{k+1} z$; thus $STz = TSz$, establishing (3c).

Useful results concerning shifted vectors of finite order can be cast in a purely algebraic setting. A linear transformation T on a vector space V over a field F is locally nilpotent if, for each v in V there is a positive integer $n = n_v$ such that $T^n v = 0$.

Note that if T is locally nilpotent and $\{a_n\}$ is a sequence in F, the sum $\sum_{k \geq 0} a_k T^k$ is finite at each vector, and thus the sum defines a linear transformation that commutes with every transformation commuting with T.

Lemma 2.4. *Suppose S and T are commuting linear transformations on a vector space V over a field F such that*

(1) *T is locally nilpotent,*

(2) *S leaves invariant every T-invariant linear subspace of V.*

Then there is a sequence $\{a_n\}$ in F such that $S = \sum_{k \geq 0} a_k T^k$. Moreover, S commutes with every linear transformation on V that commutes with T.

Proof. It follows from [16, Cor. 5] that there is a net $\{p_\lambda\}$ of polynomials in $F[x]$ such that $p_\lambda(T) \to S$ in the strict topology (i.e., pointwise in the discrete topology on V). If T is nilpotent, then the set of polynomials in T is strictly closed and S is a polynomial in T. We can therefore assume that T is not nilpotent. For each integer $m \geq 1$, choose a vector $v(m)$ in V so that $T^m v(m) \neq 0$. It follows, for sufficiently large λ, that $p_\lambda(T)v(m) = Sv(m)$. Hence, for $0 \leq k \leq m$, the coefficients of x^k in $p_\lambda(x)$ are equal to a constant a_k for sufficiently large λ. It follows that $S = \sum_{k \geq 0} a_k T^k$. It is clear that if A is a linear transformation on V and $AT = TA$, then

$$AS = \sum_{k \geq 0} a_k A T^k = \sum_{k \geq 0} a_k T^k A = SA.$$

The following lemma was proved for matrices by DEDDENS and FILLMORE [11]; it was observed in [17, p. 20] and [14] that the result holds in general. Note that if J is an $n \times n$ Jordan nilpotent matrix and $k \geq 2$ is an integer, then dim (ker J^k/ /ker J^{k-2}) is 2 if $k \leq n$ and is 1 if $k = n+1$, and is 0 if $k \geq n+2$. Hence if T is a nilpotent matrix, $n \geq 2$, $T^n = 0$ and $T^{n-1} \neq 0$, then dim (ker T^n/ker T^{n-2}) is the number of $(n-1) \times (n-1)$ Jordan blocks plus twice the number of $n \times n$ Jordan blocks in the Jordan canonical form for T. Note that this number is greater than 2 if and only if there is one block of size n and another block of size n or $n-1$.

Lemma 2.5. *Suppose T is a nilpotent linear transformation on a vector space V over a field F. The following are equivalent:*

(1) *every linear transformation leaving invariant each T-invariant linear subspace commutes with T,*

(2) *every linear transformation leaving invariant each T-invariant linear subspace is a polynomial in T,*

(3) *for every x in V and every positive integer n such that $T^n x \neq 0$, there is a y in V such that*

$$\text{sp}\,\{T^k x \colon k \geq 0\} \cap \text{sp}\,\{T^k y \colon k \geq 0\} = \{0\} \quad \text{and} \quad T^{n-1} y \neq 0,$$

(4) *if* $n \geqq 2$ *and* $T^{n-1} \neq 0$, *then* $\dim (\ker T^n / \ker T^{n-2}) > 2$.

Theorem 2.6. *Let X be a locally convex space, and let T be a continuous linear transformation of X. Suppose $\bigcup_{k \geqq 1} \ker T^k$ is dense in X and, for every $k \geqq 0$ with $T^{k+1} \neq 0$, $\dim (\ker T^{k+2} / \ker T^k) > 2$. If $S \in \mathrm{Alg\,Lat}\, T$, then $S \in \{T\}''$, and there exists a sequence $\{a_k\}$ of complex numbers such that $Sx \sim \sum_k a_k T^k x$ for every shifted vector x for T.*

Proof. The dimension hypothesis implies, via Lemma 2.5, that $T|\ker T^k$ is reflexive for every $k \geqq 1$. If T is nilpotent, then we are done. Otherwise let S be in Alg Lat T, and let S_k and T_k be the restrictions of S and T respectively to $\ker T^k$. By the reflexivity of T_k, there is a polynomial p_k of degree $k-1$ or less such that $S_k = p_k(T_k)$. If $A \in \{T\}'$ and if $A_k = A|\ker T^k$, then clearly $A_k S_k = S_k A_k$, and since $\bigcup_k \ker T^k$ is dense, it follows that $AS = SA$. Hence $S \in \{T\}''$.

Applying Lemma 2.4 to the restriction of T to $\bigcup_k \ker T^k$, we obtain a sequence $\{a_k\}$ of complex numbers such that $Sx \sim \sum_k a_k T^k x$ for every x in $\bigcup_k \ker T^k$. If y is a shifted vector of infinite order, then $Sy \sim \sum_k b_k T^k y$. There are shifted vectors x of arbitrarily large finite order that may be used in conjunction with Lemma 2.2 and part (3a) of Lemma 2.3 to conclude that $b_k = a_k$ for every k.

Corollary 2.7. *If T is a nonnilpotent operator on a Hilbert space such that T is a direct sum of nilpotent operators, then every vector is a shifted vector for T, and Alg Lat $T \subset \{T\}''$. If $S \in \mathrm{Alg\,Lat}\, T$, then there exists a sequence $\{a_k\}$ of complex numbers such that $Sx \sim \sum_k a_k T^k x$ for every vector x.*

Proof. The hypothesis of the corollary implies that of the theorem, and therefore Alg Lat $T \subset \{T\}''$ and there exists a sequence $\{a_k\}$ such that $Sx \sim \sum_k a_k T^k x$ for every shifted vector x. That every vector is shifted follows easily from the fact that every vector is a direct sum of shifted vectors.

In a Hilbert space, the hypothesis $\bigcap_{k \geqq 1} (\operatorname{ran} T^k)^- = \{0\}$ in the following theorem is equivalent to T having a strictly lower triangular matrix with respect to an orthogonal direct sum decomposition of the space into a sequence of subspaces.

Theorem 2.8. *Suppose X is a locally convex space and $T \in B(X)$ has the property that $\bigcap_{k \geqq 1} (\operatorname{ran} T^k)^- = \{0\}$, and, for each integer $n \geqq 1$, $\dim (X/(T^n)^{-1}[(\operatorname{ran} T^{n+2})^-]) > 2$. Then*

(1) *every vector in X is a shifted vector for T,*

(2) *Alg Lat $T \subset \{T\}''$,*

(3) *for each S in* Alg Lat T, *there is a sequence $\{a_n\}$ of complex numbers such that, for every vector x in X, we have* $Sx \sim \sum_k a_k T^k x$.

Proof. Suppose that $S \in$ Alg Lat T and $A \in \{T\}'$ and $n \geqslant 2$ is a positive integer. Since S, T and A each leave $(\operatorname{ran} T^n)^-$ invariant, they induce operators S_n, T_n and A_n (respectively) on the space $X/(\operatorname{ran} T^n)^-$. Clearly, $S_n \in$ Alg Lat T_n and $A_n \in \{T_n\}'$. Moreover, T_n is nilpotent, and it follows from the hypothesis on dimensions above and Lemma 2.5 that there is a polynomial $p_n(z)$ (unique modulo $z^n \mathbf{C}[z]$) such that $S_n = p_n(T_n)$. Thus $S_n A_n = A_n S_n$.

Translating back in X, we obtain $\operatorname{ran}(S - p_n(T)) \subset (\operatorname{ran} T^n)^-$ and $\operatorname{ran}(AS - SA) \subset (\operatorname{ran} T^n)^-$. Since we have $\operatorname{ran}(S - p_{n+1}(T)) \subset (\operatorname{ran} T^{n+1})^- \subset (\operatorname{ran} T^n)^-$, we have that $p_{n+1}(z) = p_n(z)$ modulo $z^n \mathbf{C}[z]$. Hence there is a sequence $\{a_k\}$ of complex numbers such that we can take $p_n(z) = \sum_{k<n} a_k z^k$ for each n. It follows that $Sx \sim \sum_k a_k T^k x$ for every x in X. Also we have $\operatorname{ran}(AS - SA) \subset \bigcap_{k \geq 1} (\operatorname{ran} T^k)^- = \{0\}$, which implies $AS = SA$. Hence $S \in \{T\}''$.

Corollary 2.9. *If T_1 and T_2 are operators on a Hilbert space that are strictly lower triangular with respect to some infinite direct sum decomposition of the space, and if neither T_1 nor T_2 is nilpotent, then for $T = T_1 \oplus T_2$ on the direct sum H of the space with itself,* Alg Lat $T \subset \{T\}''$, *and for every S in* Alg Lat T *there exists a sequence $\{a_k\}$ such that $Sx \sim \sum_k a_k T^k x$ for every x in H.*

Remark. It follows from the preceding theorem that if $\{x, y\}$ is a shifted pair and $\operatorname{ord}(x) = \operatorname{ord}(y) = \infty$, then, for any S in Alg Lat T there is a sequence $\{a_k\}$ of complex numbers such that $Sz \sim \sum_k a_k T^k z$ for every z in $M(x, y)^-$.

Theorem 2.10. *Suppose that X is locally convex, $T \in B(X)$ and T has shifted vectors of arbitrarily large finite orders and at least one shifted vector of infinite order. If $S \in$ Alg Lat T, then there exists a sequence $\{a_k\}$ of complex numbers such that $Sx \sim \sum_k a_k T^k x$ for every shifted vector x for T. Moreover, if $A \in \{T\}'$ and A is invertible, then $SA - AS = 0$ on the linear span of the set of shifted vectors for T.*

Proof. If y is a shifted vector of infinite order for T, and if S is a linear transformation that leaves invariant all the invariant subspaces of T, then there is a sequence $\{a_k\}$ such that $Sy \sim \sum_k a_k T^k y$. By Lemma 2.2, if x is a shifted vector of finite order, then $\{x, y\}$ is a shifted pair. Thus by part (3a) of Lemma 2.3, $Sx = \sum_{k<m} a_k T^k x$, where $m = \operatorname{ord}(x)$. If z is another shifted vector of infinite order, then $Sz \sim \sum_k b_k T^k z$, and another application of Lemma 2.2 and part (3a) of Lemma 2.3 yields $b_k = a_k$ for $k < m$. Since there are shifted vectors of arbitrarily large finite order, $b_k = a_k$ for every k.

Let A be a continuous invertible linear transformation in $\{T\}'$. If x is a shifted vector for T, then Ax is also a shifted vector for T and

$$SAx \sim \sum_k a_k T^k Ax.$$

On the other hand, $Sx \sim \sum_k a_k T^k x$ implies that $ASx \sim \sum_k a_k T^k Ax$. It follows that $ASx = SAx$, and hence $AS = SA$ on the span of the shifted vectors.

Remark. If X is a Banach space in the preceding theorem, we can drop the assumption that A is invertible, since there is a scalar λ such that $A - \lambda$ is invertible, and every operator that commutes with $A - \lambda$ also commutes with A.

Theorem 2.11. Suppose that X is a locally convex space, $T \in B(X)$, and $\{M_k : k \in \mathbf{Z}\}$ is a collection of closed linear subspaces with zero intersection such that $T(M_k) \subset M_{k+1} \subset M_k$ for each k in $\mathbf{Z}$. Let $N = \bigcup_{k \in \mathbf{Z}} M_k$ and assume that $S \in \mathrm{Alg\ Lat}\ T$ and $ST - TS = 0$ on N. Then

(1) each vector in N is a shifted vector for T,

(2) there is a sequence $\{a_n\}$ of complex numbers such that, for every vector x in N, we have $Sx \sim \sum_k a_k T^k x$.

Moreover, if, for each integer $n \geq 2$, we have $\dim\left(M_{-n}/[(T^{2n})^{-1}(M_n)] \cap M_{-n}\right) > 2$, then $AT - TA = 0$ on N for every A in $\mathrm{Alg\ Lat}\ T$.

Proof. Statement (1) is clear. Note that S leaves N invariant, since $S \in \mathrm{Alg\ Lat}\ T$. For each positive integer n, we can apply Lemma 2.4 to the operators induced by S and T on M_{-n}/M_n to obtain a polynomial $p_n(z)$ such that $(S - p_n(T))(M_{-n}) \subset M_n$. Moreover, it is clear that there is a single formal power series $f(z) = \sum_k a_k z^k$ such that each $p_n(z)$ is a partial sum of $f(z)$. Since $\bigcap_k M_k = 0$, it follows that $Sx \sim \sum_k a_k T^k x$ for every x in N.

Note that the hypothesis $\dim\left(M_{-n}/[(T^{2n})^{-1}(M_n)] \cap M_{-n}\right) > 2$ implies, via Lemma 2.5, that the operator induced by T on M_{-n}/M_n, is reflexive, which implies that the operator induced by A on M_{-n}/M_n is a polynomial in the operator induced by T. In particular, $(AT - TA)(M_{-n}) \subset M_n$ for all $n \geq 2$. Since $\bigcap_k M_k = 0$, it follows that $AT - TA = 0$ on N.

3. Weighted shifts. The results of the preceding section often yield $Sx \sim \sum_k a_k T^k x$ for every vector x (or for at least a dense set of vectors). This suggests that the operator S is in the weakly closed unital algebra $\mathscr{A}(T)$ generated by T; however, the formal power series $\sum_k a_k T^k$ need not converge, and it is not clear that either the sequence of partial sums (or its Cesaro means) need have a convergent subnet in the weak operator topology. For unilateral weighted shift operators on

Hilbert space with scalar weights, A. L. SHIELDS and L. J. WALLEN [29] proved that the commutant coincides with the generated weakly closed algebra. In the course of the proof they show that the sum of a formal power series in a weighted shift is the strong limit of the sequence of Cesaro means of the sequence of partial sums of the formal power series in the shift. We will show that the Shields—Wallen result holds for shifts of a much more general nature.

Suppose that X is a normed linear space and $\{X_i : i \in I\}$ is a linearly independent family of closed linear subspaces of X whose algebraic sum is M. We say that X is the direct sum of the X_i's if there is a family $\{P_i : i \in I\}$ of idempotents in $B(X)$ such that

(1) $P_i | M$ is the projection onto X_i for each $i \in I$,

(2) M is dense in X,

(3) $\sup \{\| \sum\limits_{i \in F} P_i \| : F \subset I,\ F \text{ finite}\} < \infty$.

It follows that $\sum\limits_{i \in I} P_i = 1$ converges in the strong operator topology, since the net of partial sums is bounded and converges strongly to 1 on the dense subset M. It also follows that the set $\{\sum\limits_{i \in F} P_i : F \subset I,\ F \text{ or } I \setminus F \text{ finite}\}$ is a bounded Boolean algebra of idempotents. Standard results on bounded Boolean algebras of projections (see [13]) imply that there is a constant K such that, for every function $\alpha : I \to \mathbf{C}$ with finite support, we have

$$(*) \qquad\qquad \left\| \sum\limits_{i \in I} \alpha(i) P_i \right\| \leq K \sup_i |\alpha(i)|.$$

Moreover, if X is a Banach space, the preceding inequality holds for every bounded $\alpha : I \to \mathbf{C}$ and $\sum\limits_{i \in I} \alpha(i) P_i$ converges in the strong operator topology.

Note that a c_0-sum or an l^p-sum $(1 \leq p < \infty)$ of subspaces is a direct sum in the above sense; however, an l^∞-sum is not a direct sum since M fails to be dense.

An operator T is a (forward) unilateral operator-weighted shift on a normed space X if there is a sequence $\{X_n : n \in \mathbf{Z}^+\}$ of subspaces of X such that

(1) X is the direct sum of the X_n's,

(2) $T(X_n) \subset X_{n+1}$ for $n \in \mathbf{Z}^+$.

Here $\mathbf{Z}^+$ denotes the set of non-negative integers. If $\mathbf{Z}^+$ is replaced by the set $\mathbf{Z}$ of integers, then T is called a bilateral operator-weighted shift. The restriction operators $T | X_n$ are the weights of the shift. If all of the X_n's are 1-dimensional, then the weighted shift is called a scalar-weighted shift. If, on the other hand, condition (2) above is replaced by

$$(2)' \qquad\qquad T(X_0) = 0, \quad T(X_{n+1}) \subset X_n \quad \text{for} \quad n \in \mathbf{Z}^+,$$

then T is a backwards unilateral operator-weighted shift. If T is an operator of any of the three types defined above we say that T is an operator-weighted shift.

The following is a generalization of the SHIELDS—WALLEN theorem [29].

Theorem 3.1. *Suppose T is an operator-weighted shift on a normed space X that is a direct sum of subspaces $\{X_n\}$. Suppose $A \in B(X)$ and $\{a_n\}$ is a sequence of scalars such that, for each x in $\cup X_n$, we have $Ax \sim \sum_k a_k T^k x$. Let $\{A_n\}$ be the sequence of Cesaro means of the sequence of partial sums of the series $\sum_k a_k T^k$. Then*

(1) $\sup_n \|A_n\| < \infty$,

(2) $A_n \to A$ *in the strong operator topology.*

Proof. First note that since X is the direct sum of the X_n's, it follows, for each x in $\cup X_n$, that the sum $\sum_k a_k T^k x$ converges in norm to Ax. It follows that $\|A_n x - Ax\| \to 0$ for every x in $\cup X_n$. If (1) is true, then $\{x \in X : \|A_n x - Ax\| \to 0\}$ is a closed linear subspace of X containing $\cup X_n$, which implies (2). Hence it suffices to prove (1).

Let P_n denote the projection of X onto X_n, and, for each finite set F of indices, let $Q_F = \sum_{i \in F} P_i$. Let K be as in ($*$) above. It follows that, for each index n, we have $\{Q_F A_n Q_F\}$ converges in the strong operator topology to A_n; whence, $\|A_n\| \leq \leq \lim\sup_F \|Q_F A_n Q_F\|$. We will show that if m is any index and k and n are positive integers, and if $F = \{j : m \leq j \leq m+k\}$, then $\|Q_F A_n Q_F\| \leq K^2 \|A\|$. From this it follows that $\|A_n\| \leq K^2 \|A\|$ for $n = 1, 2, 3, \ldots$, which proves (1).

Define continuous functions $f, g : [-\pi, \pi] \to B(X)$ by

$$f(t) = \sum_{0 \leq j \leq k} e^{ijt} P_{m+j} \quad \text{and} \quad g(t) = \sum_{0 \leq j \leq k} e^{-ijt} P_{m+j}.$$

Let

$$K_n(t) = \frac{1}{n+1} \left[\frac{\sin (n+1) t/2}{\sin (t/2)} \right]^2$$

be the nth Fejér kernel. A simple computation shows that

$$Q_F A_n Q_F = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) A g(t) K_n(t) \, dt,$$

and it follows that

$$\|Q_F A_n Q_F\| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(t) A g(t) K_n(t)\| \, dt \leq K^2 \|A\| \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t) \, dt = K^2 \|A\|.$$

This completes the proof.

Corollary 3.2. *Suppose that T is scalar-weighted unilateral shift with nonzero weights. Then $\{T\}' = \mathscr{A}(T)$.*

Proof. Suppose e is a nonzero vector in X_0. Then $\{T^n e\}$ is a basis for X_n for $n = 0, 1, 2, \ldots$ Suppose $S \in \{T\}'$. Then $Se = \sum_{0 \leq k} a_k T^k e$ for some sequence $\{a_k\}$ of

scalars. Since $S \in \{T\}'$, it follows that

$$ST^n e = T^n Se = T^n \sum_{0 \leq k} a_k T^k e = \sum_{0 \leq k} a_k T^k (T^n e).$$

Hence the hypothesis of Theorem 3.1 is satisfied, which implies that $S \in \mathscr{A}(T)$.

Corollary 3.3. *If T is an operator-weighted shift on a normed linear space, then* $\{T\}' \cap \text{Alg Lat } T = \mathscr{A}(T)$.

Proof. It follows from Theorem 2.11 that if $S \in \{T\}' \cap \text{Alg Lat } T$, then S satisfies the hypothesis of Theorem 3.1.

Theorem 3.4. *Suppose T is an operator-weighted shift on a normed space X relative to the direct sum $X = \overline{\sum_n} X_n$. Let $M = \bigcup_n X_n$, and suppose for each x in M and each positive integer n with $T^n x \neq 0$, there is a y in M such that* sp $\{T^k x : k \geq 0\} \cap$ sp $\{T^k y : k \geq 0\} = \{0\}$ *and $T^n y \neq 0$. Then T is reflexive.*

Proof. Suppose $S \in \text{Alg Lat } T$. In view of Corollary 3.3, we need show only that $S \in \{T\}'$. We first show that if $x, y \in M$ and sp $\{T^k x : k \geq 0\} \cap$ sp $\{T^k y : k \geq 0\} = 0$, then $\{x, y\}$ is a shifted pair for T. Once this is done, it will follow from part 3(c) of Lemma 2.3 and the hypothesis of the theorem that $ST = TS$ on M, and, since $X = \overline{\text{sp}} \, M$, it will follow that $S \in \{T\}'$.

Suppose $x, y \in M$ and sp $\{T^k x : k \geq 0\} \cap$ sp $\{T^k y : k \geq 0\} = 0$. Suppose $x \in X_k$ and $y \in X_j$. By symmetry, it will suffice to show that if n is a positive integer and $T^n x \neq 0$, then $T^n x \notin \overline{\text{sp}} \, (\{T^i x : i \geq 0, \, i \neq n\} \cup \{T^i y : i \geq 0\})$. However, $T^n x \in X_{k+n}$, and if P is the projection onto X_{k+n}, then $T^n x \in \overline{\text{sp}} \, (\{T^i x : i \geq 0, \, i \neq n\} \cup \{T^i y : i \geq 0\})$ implies that $T^n x = PT^n x \in \overline{\text{sp}} \, (\{PT^i x : i \geq 0, \, i \neq n\} \cup \{PT^i y : i \geq 0\})$, and the last set is either 0 or sp $\{T^{n+k-j} y\}$. This violates the conditions that $T^n x \neq 0$ and sp $\{T^k x : k \geq 0\} \cap$ sp $\{T^k y : k \geq 0\} = 0$. It follows that $\{x, y\}$ is a shifted pair for T, and the proof is complete.

The following corollaries are immediate consequences of the theorem.

Corollary 3.5. *Operator-weighted shifts with injective operator weights of rank at least two are reflexive.*

Corollary 3.6. *Every direct sum of at least two scalar-weighted shifts with non-zero weights is reflexive.*

The following is a corollary of Theorem 2.10.

Corollary 3.7. *If T is a unilateral backwards operator-weighted shift with injective weights, and T has a shifted vector of infinite order, then T is reflexive.*

We will show that the unweighted (i.e., all weights 1) backwards unilateral shift

operator on the Hilbert space H^2 has no shifted vectors of infinite order. Our demonstration depends on the characterization of noncyclic vectors of the backwards shift in [12]. A vector f is noncyclic for the backward shift T in case there exists a meromorphic pseudocontinuation $f^\sim$ of f to the complement D_e of the closed unit disk in the Riemann sphere such that $f^\sim = G/H$, where G and H are bounded holomorphic functions on D_e. To say $f^\sim$ is a pseudocontinuation of f means that the radial limits of f and $f^\sim$ agree at almost every point of the unit circle. We need the following lemma.

Lemma 3.8. *If f is a noncyclic vector for the backwards shift T, then $f \in M(Tf)^-$ if and only if $f^\sim(\infty) = 0$.*

Proof. The proof is virtually the same as that of Theorem 1 in [12]. Suppose

$$f(z) = \sum_{k \geq 0} a_k z^k, \quad G(z) = \sum_{k \geq 0} b_k z^{-k} \quad \text{and} \quad H(z) = \sum_{k \geq 0} c_k z^{-k}.$$

By multiplying both G and H by an appropriate power of z, we may assume that either $b_0 \neq 0$ or $c_0 \neq 0$. Then $f^\sim(\infty) = b_0/c_0$. (If $c_0 = 0$, then $f^\sim(\infty) = \infty$.) We have $f(e^{i\vartheta}) H(e^{i\vartheta}) = G(e^{i\vartheta})$ a.e., and hence as in [12],

$$a_0 c_0 + a_1 c_1 + \ldots = b_0,$$

$$a_1 c_0 + a_2 c_1 + \ldots = 0,$$

$$a_2 c_0 + a_3 c_1 + \ldots = 0,$$

etc. Thus if $f^\sim(\infty) \neq 0$, then $b_0 \neq 0$, and the preceding equations show that there exists a vector that is not orthogonal to f but is orthogonal to $T^k f$ for every $k \geq 1$. Thus f does not belong to $M(Tf)^-$.

Conversely suppose $f^\sim(\infty) = 0$. Let h be a vector that is orthogonal to $M(Tf)^-$. If c_k is the conjugate of the kth Fourier coefficient of h, then all but the first of the sequence of equations in the preceding paragraph hold. Thus if $H_0(z) = \sum_{k \geq 0} c_k z^{-k}$, then H_0 has radial limits at almost every point of the unit circle, and $H_0(e^{i\vartheta}) f(e^{i\vartheta}) = = G_0(e^{i\vartheta})$ defines a function G_0 in $L^1(d\vartheta)$ with $G_0(e^{i\vartheta}) = \sum_{k \geq 0} b_k e^{-ik\vartheta}$. It follows that if $G_0(z) = \sum_{k \geq 0} b_k z^{-k}$, then $g^\sim = G_0/H_0$ is a pseudocontinuation of f to D_e and G_0 and H_0 are quotients of bounded holomorphic functions since they are in H^2 and H^1 of D_e respectively. Since the pseudocontinuation of a function is unique, it follows that $g^\sim(\infty) = 0$, and thus $b_0 = 0$. Hence the first equation of the sequence in the preceding paragraph shows that f is also orthogonal to h, and it follows that $f \in M(Tf)^-$.

Proposition 3.9. *The only shifted vectors of the adjoint of the unweighted unilateral shift operator are polynomials.*

Proof. It will be shown that T has no noncyclic shifted vectors of infinite order. This will imply that it has no infinite order shifted vectors. For if f is cyclic, shifted and of infinite order, then Tf is shifted and of infinite order, but it is noncyclic.

Suppose f is any noncyclic vector of infinite order and $f^\sim$ is its meromorphic pseudocontinuation. Then since $Tf = (f - f(0))/z$, and since pseudocontinuations are determined by their radial limits (see [12]), it follows that $(Tf)^\sim = (f^\sim - f(0))/z$. Hence if $f^\sim$ has a pole at ∞ of order m, then $(Tf)^\sim$ has a pole at ∞ of order $m-1$, and consequently $(T^{m+1}f)^\sim(\infty) = 0$. By Lemma 3.8, $T^{m+1}f \in \overline{\mathrm{sp}} \{T^k f : k > m+1\}$. Since $T^{m+1}f \neq 0$, f is not a shifted vector.

4. Reflexive Graphs. In this section we study conditions that make graphs reflexive. We wish to consider a more general version of reflexivity than that of the preceding sections. A linear subspace $\mathscr{S}$ of $B(H)$ is reflexive if $T \in \mathscr{S}$ whenever $Tx \in [\mathscr{S}x]^-$ for every x in H. We say that a linear functional φ on $\mathscr{S}$ is elementary if there are vectors x, y in H such that $\varphi(S) = (Sx, y)$ for every S in $\mathscr{S}$. We say that $\mathscr{S}$ is weakly elementary if every weak-operator continuous linear functional is elementary on $\mathscr{S}$. (In [17] and [7], respectively the terms "property D" and "property A" used. Our notation agrees with that in [3].)

Theorem 4.1. *Suppose that $\mathscr{S}$ is a reflexive linear subspace of $B(H)$ and $\pi \colon \mathscr{S} \to B(M)$ is a linear mapping such that the set of elementary linear functionals φ on $B(M)$ for which $\varphi \circ \pi$ is elementary on $\mathscr{S}$ separates the points of $B(M)$. Then Graph $(\pi) = \{S \oplus \pi(S) : S \in \mathscr{S}\}$ is a reflexive linear subspace of $B(H \oplus M)$.*

Proof. Suppose that $A \in B(H \oplus M)$ and $Ae \in [\text{Graph } (\pi)e]^-$ for every vector e in $H \oplus M$. Clearly, we can write $A = B \oplus C$. Also, since $\mathscr{S}$ is reflexive, it is clear that $B \in \mathscr{S}$. Thus, by replacing A by $A - (B \oplus \pi(B))$, we can assume that $B = 0$. We need to show that $C = 0$. Suppose that $C \neq 0$. Then there is an elementary functional φ on $B(M)$ such that $\varphi(C) \neq 0$ and such that $\varphi \circ \pi$ is elementary on $\mathscr{S}$. Thus there are vectors x_1, x_2 in H and y_1, y_2 in M such $\varphi(T) = (Ty_1, y_2)$ for all T in $B(M)$ and such that $(Sx_1, x_2) = \varphi(\pi(S)) = (\pi(S)y_1, y_2)$ for all S in $\mathscr{S}$. Letting $e = x_1 \oplus y_1$, it follows that there is a sequence $\{S_n\}$ in $\mathscr{S}$ such that $(S_n \oplus \pi(S_n))e \to Ae$. Thus $S_n x_1 \to 0$ and $\pi(S_n)y_1 \to Cy_1$, Hence $0 \neq \varphi(C) = (Cy_1, y_2) = \lim (\pi(S_n)y_1, y_2) = \lim \varphi(\pi(S_n)) = \lim (S_n x_1, x_2) = 0$. This contradiction shows that $C = 0$.

Corollary 4.2. *If $\mathscr{S}$ is a weakly elementary reflexive linear subspace of $B(H)$ and $\pi \colon \mathscr{S} \to B(M)$ is a weakly continuous linear map, then Graph (π) is reflexive and weakly elementary.*

It was shown in [5] and [32] that if S is the unilateral shift operator and T is a contraction operator, then $S \oplus T$ is reflexive. An unsolved problem of P. R. HALMOS [18] asks whether every polynomially bounded operator is similar to a contraction,

and it was shown by W. MLAK [22] that every polynomially bounded operator is similar to the direct sum of a unitary operator and an operator with a weakly continuous H^∞ functional calculus.

Corollary 4.3. *If S is the unilateral shift operator and T is a polynomially bounded operator, then $S \oplus T$ is reflexive and weakly elementary.*

Proof. The result of the preceding corollary shows that $S \oplus T$ is reflexive and elementary when T has a weakly continuous H^∞ functional calculus. The direct sum of such an operator with a unitary operator must be reflexive and elementary by [17].

Suppose that S is a subnormal operator. A result of D. SARASON [27] says that there is a compactly supported Borel measure μ in the plane and an open subset Ω of the plane such that the weakly closed algebra generated by 1 and S is isomorphic to $L^\infty(\mu) \oplus H^\infty(\Omega)$. Call the set Ω the Sarason hull of S. It is shown in [8] that convergence in the weak operator topology in the $H^\infty(\Omega)$ summand implies uniform convergence on compact subsets of Ω. Thus if T is an operator whose spectrum is contained in Ω, then there is an appropriate $H^\infty(\Omega)$ functional calculus. The Sarason hull of the unilateral shift operator is the open unit disk. It was shown by R. OLIN and J. THOMSON [23] that the weakly closed algebra generated by a subnormal operator is weakly elementary. The proof of the preceding Corollary combined with the aforementioned facts yield the following.

Corollary 4.4. *If S is a subnormal operator and T is an operator whose spectrum is contained in the Sarason hull of S, then $S \oplus T$ is reflexive and weakly elementary.*

Note that the definitions of reflexivity and of being elementary for a linear subspace $\mathscr{S}$ of $B(H)$ makes sense when $\mathscr{S}$ is a subset of $B(H, K)$, the set of operators from the Hilbert space H to the Hilbert space K. In this way, it makes sense to talk of a subspace $\mathscr{S}$ of $B(H)$ having a reflexive (or elementary) restriction to a linear subspace M of H, i.e., $\mathscr{S}|M$ is reflexive.

Suppose that $\mathscr{S}$ is a linear subspace of $B(H)$ and $x \in H$. We define $G_{\mathscr{S}}(x)$ to be the set of all vectors y in H such that

   (a) $[\mathscr{S}x]^- \cap [\mathscr{S}y]^- = \{0\}$,

   (b) $[\mathscr{S}x]^- + [\mathscr{S}y]^-$ is closed, and

   (c) if $\{S_n\}$ is a sequence in $\mathscr{S}$ such that $\|S_n x\| \to 0$ and $\{S_n y\}$ is norm convergent, then $S_n y \to 0$.

Note that (a) and (b) imply that the sum in (b) is direct sum of Banach spaces and that (c) implies that $\{Sx + Sy : S \in \mathscr{S}\}^-$ is a graph in this direct sum.

Theorem 4.5. *Suppose that M is a subspace of the Hilbert space H and $\mathscr{S}$ is a linear subspace of $B(H)$ such that*

(1) $\mathscr{S}|M$ is reflexive,

(2) $M+\text{span}\left(\cup\{G_{\mathscr{S}}(x): x\in M\}\right)$ is dense in H.

Then $\mathscr{S}$ is reflexive.

Proof. Suppose $T\in B(H)$ and $Te\in[\mathscr{S}e]^-$ for every e in H. It follows from (1) that $T|M\in\mathscr{S}|M$; hence we can assume that $T|M=0$. Suppose $x\in M$ and $y\in G_{\mathscr{S}}(x)$. Then there is a sequence $\{S_n\}$ in $\mathscr{S}$ such that $S_n(x+y)\to T(x+y)$. It follows from parts (a) and (b) of the definition of $G_{\mathscr{S}}(x)$ that $S_n x\to Tx=0$ and $S_n y\to Ty$. It follows from part (c) of the definition of $G_{\mathscr{S}}(x)$ that $Ty=0$. Thus, by (2), $T=0$, since $T=0$ on a dense subset of H.

Corollary 4.6. *Suppose $\mathscr{S}$ is a reflexive subspace of $B(H)$ and $\pi:\mathscr{S}\to B(M)$ is a linear map such that the set*

$$\{y\in M: \exists x\in H \text{ such that } \{Sx\oplus\pi(S)y: S\in\mathscr{S}\}^- \text{ is a graph}\}$$

is dense in M. Then Graph (π) *is reflexive and π is continuous with respect to the ultra-weak (and norm) topologies on $\mathscr{S}$ and $B(M)$.*

The preceding Corollary can also be used to recapture the result, that $S\oplus T$ is reflexive whenever S is the unilateral shift operator and T is a contraction [5], [32]. The basic idea is to let $\mathscr{S}$ be the unital weakly closed algebra generated by S and to define π by $\pi(\varphi(S))=\varphi(T)$ for each φ in H^∞. In the case $\|T\|<1$, it follows that if x is a unit vector in ker S^*, then $\{\varphi(S)x\oplus\varphi(T)y: \varphi\in H^\infty\}^-$ is a graph for every vector y. This follows from the fact that if $\{\varphi_n\}$ is a sequence in H^∞ and $\varphi_n(S)x\to\varphi(S)x$, then $\varphi_n\to\varphi$ in H^2 and thus uniformly on compact subsets of the open unit disk, which, by the Riesz functional calculus, implies that $\varphi_n(T)\to\varphi(T)$ in norm.

5. Questions and comments.

1. A Donoghue operator is a weighted shift on l^2 with square summable weights that tend monotonically to zero. It is easy to see that a backwards Donoghue operator has no shifted vectors of infinite order. For if f is a vector of infinite order, then D^*f is a cyclic vector. Does $(D\oplus D)^*$ have a shifted vector of infinite order, where D is a Donoghue operator?

2. Suppose S is the unilateral shift operator on l^2. What is the set of all Hilbert space operators T for which $S\oplus T$ is reflexive. This paper shows that the set contains all polynomially bounded operators and all operator-weighted shifts.

3. If T is a nonnilpotent Hilbert space operator that is a direct sum of nilpotent operators, must T be reflexive?

4. The results in this paper on reflexive graphs have been generalized in [15] and have been extended to prove that certain graphs are hyperreflexive. In particular, it is shown in [15] that if S is the unilateral shift operator on l^2 and T is polynomially

bounded, then $S \oplus T$ is hyperreflexive. In [9] K. DAVIDSON proved that the unilateral shift operator is hyperreflexive. What about the direct sum of two weighted shifts, or the operator-weighted shifts on Hilbert space considered in Theorem 3.4?

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