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Hyponormal operators on uniformly convex spaces

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Dedicated to Professor Jun Tomiyama on his 60th birthdy

1. Introduction. Let X be a complex Banach space. We denote by X^* the dual space of X and by $B(X)$ the space of all bounded linear operators on X.

Let us set

$$\pi = \{(x,f)\in X\times X^*:\ \|f\| = f(x) = \|x\| = 1\}.$$

The *spatial numerical range* $V(T)$ and the *numerical range* $V(B(X), T)$ of $T\in B(X)$ are defined by

$$V(T) = \{f(Tx):\ (x,f)\in\pi\}$$

and

$$V(B(X),T) = \{F(T): F\in B(X)^* \text{ and } \|F\| = F(I) = 1\},$$

respectively.

Definition 1. If $V(T)\subset \mathbf{R}$, then T is called *hermitian*. An operator $T\in B(X)$ is called *hyponormal* if there are hermitian operators H and K such that $T=H+iK$ and the commutator $C=i(HK-KH)$ is non-negative, that is

$$V(C) \subset \mathbf{R}^+ = \{a\in\mathbf{R}:\ a \geqq 0\}.$$

An operator N is called *normal* if there are hermitian operators H and K such that $N=H+iK$ and $HK=KH$. A normal operator N on a Banach space X has the following properties:

(1) $\operatorname{co}\sigma(N)=\overline{V(N)}=V(B(X), N)$.

(2) If $Nx_n\to 0$ for a bounded sequence $\{x_n\}$ in X, then $Hx_n\to 0$ and $Kx_n\to 0$.

Definition 2. Let X be Banach space. X will be said to be *uniformly convex* if to each $\varepsilon>0$ there corresponds a $\delta>0$ such that the conditions $\|x\|=\|y\|=1$ and $\|x-y\|\geqq\varepsilon$ imply $\dfrac{\|x+y\|}{2}\leqq 1-\delta$.

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X will be said to be *uniformly c-convex* if for every $\varepsilon>0$ there is a $\delta>0$ such that $\|y\|<\varepsilon$ whenever $\|x\|=1$ and $\|x+\lambda y\|\leqq 1+\delta$ for all complex numbers λ with $|\lambda|\leqq 1$.

X will be said to be *strictly c-convex* if $y=0$ whenever $\|x\|=1$ and $\|x+\lambda y\|\leqq 1$ for all complex numbers λ with $|\lambda|\leqq 1$.

All uniformly convex spaces, for example $\mathscr{L}^p(S, \Sigma, \mu)$ and $\mathscr{C}_p(\mathscr{H})$ for $1<p<\infty$, are uniformly c-convex and all uniformly c-convex spaces are strictly c-convex.

$\mathscr{L}^1(S, \Sigma, \mu)$ and the trace class $\mathscr{C}_1(\mathscr{H})$ are the typical examples of uniformly c-convex spaces. See [7] and [9].

For an operator $T\in B(X)$, the spectrum, the approximate point spectrum, the point spectrum, the kernel, and the dual of T are denoted by $\sigma(T)$, $\sigma_\pi(T)$, $\sigma_p(T)$, Ker(T) and T^*, respectively.

For an operator $T=H+iK$ we denote the operator $H-iK$ by $\bar{T}$.

The following are well-known for $T\in B(X)$:

(1) $\overline{\text{co}}\, V(T)=V(B(X), T)$, where $\overline{\text{co}}\, E$ is the closed convex hull of E.

(2) $\text{co}\,\sigma(T)\subset\overline{V(T)}$, where $\text{co}\, E$ and $\bar{E}$ are the convex hull and the closure of E, respectively.

We now give a concrete example of a hyponormal operator on a uniformly c-convex space. Let $\mathscr{H}$ be a Hilbert space. Then the trace class $C_1(\mathscr{H})$ is a two sided ideal of $B(\mathscr{H})$.

Given $A, B\in B(\mathscr{H})$ we define

$$\delta_{A,B}(T) = AT-TB \quad (T\in\mathscr{C}_1(\mathscr{H})).$$

Then $\delta_{A,B}$ is an operator on a uniformly c-convex space $\mathscr{C}_1(\mathscr{H})$. It is easy to see that if A and B^* are hyponormal then $\delta_{A,B}$ is a hyponormal operator on $\mathscr{C}_1(\mathscr{H})$ (see Theorem 4.3 in [9]).

The following theorem derives from Lemma 20.3 and Corollary 20.10 in [4].

Theorem A. *If H is hermitian and $Hx=0$ for $x\in X$ ($\|x\|=1$), then there exists $f\in X^*$ such that $(x,f)\in\pi$ and $H^*f=0$.*

2. Hyponormal operators on uniformly convex spaces. The following theorem was shown by K. Mattila [9].

Theorem B. *Let X be uniformly c-convex and let $T=H+iK$ be a hyponormal operator on X. If there exists a sequence $\{x_n\}$ of unit vectors in X such that*

$$(T-(a+ib))x_n \to 0,$$

then $(H-a)x_n\to 0$ and $(K-b)x_n\to 0$.

We shall show the following (converse to the theorem above):

Theorem 1. *Let X be uniformly convex and let $T=H+iK$ be a hyponormal operator on X.* (1) *If $a\in\sigma(H)$, then there exist some real number b and sequence $\{x_n\}$ of unit vectors for which $(H-a)x_n\to 0$ and $(K-b)x_n\to 0$, so that in particular, $a+ib\in\sigma(T)$.* (2) *Similarly, if $b'\in\sigma(K)$, then there exist some real number a' and sequence $\{y_n\}$ of unit vectors for which $(H-a')y_n\to 0$ and $(K-b')y_n\to 0$, so that in particular, $a'+ib'\in\sigma(T)$.*

We need the following

Theorem C ([9], Theorem 2.4). *Let X be strictly c-convex and let $C\geqq 0$ be hermitian. If $f(Cx)=0$ for some $(x,f)\in\pi$, then $Cx=0$.*

Proof of Theorem 1. (1) Since H is hermitian, so it follows that $a\in\sigma_\pi(H)$. Consider the extension space X^0 of X and the faithful representation $B(X)\to B(X^0)$: $T\to T^0$ in the sense of DE BARRA [1]. Then a is an eigenvalue of H^0. If x^0 is in $\operatorname{Ker}(H^0-a)$ such that $\|x^0\|=1$, then by Theorem A there exists $f^0\in X^{0*}$ such that $f^0(x^0)=\|f^0\|=1$ and $(H^0-a)^*f^0=0$.

Since T is hyponormal we can let that $C=i(HK-KH)\geqq 0$; then $C^0\geqq 0$ and

$$f^0(C^0x^0) = i\hat{x}\big(K^{0*}(H-a)^{0*}f^0\big)-if^0\big(K^0(H^0-a)x^0\big) = 0,$$

where $\hat{x}$ is the Gel'fand representation of x. Since the space X^0 is uniformly convex ([1], Theorem 4), by Theorem C, it follows that $C^0x^0=0$. Therefore, it is easy to see that $\operatorname{Ker}(H^0-a)$ is invariant for K^0. So there exist a sequence $\{x_n\}$ of unit vectors and a real number b such that $(H-a)x_n\to 0$ and $(K-b)x_n\to 0$.

(2) is the same. So the proof is complete.

Theorem 2. *Let X be uniformly convex and let $T=H+iK$ be a hyponormal operator on X. Then*

$$\operatorname{co}\sigma(T) = \overline{V(T)} = V(B(X),T).$$

Proof. It is well-known that $\operatorname{co}\sigma(T)\subset\overline{V(T)}\subset V\big(B(X),T\big)$. We assume that $\operatorname{Re}\sigma(T)\subset\{a\in\mathbf{R}: a\geqq 0\}$. Then, by Theorem 1, it follows that $\sigma(H)\subset\{a\in\mathbf{R}: a\geqq 0\}$. So it follows that $V\big(B(X),H\big)\subset\{a\in\mathbf{R}: a\geqq 0\}$ and so $\operatorname{Re}V\big(B(X),T\big)\subset\{a\in\mathbf{R}: a\geqq 0\}$. Since $\alpha T+\beta$ is hyponormal for every $\alpha,\beta\in\mathbf{C}$, it follows that $\operatorname{co}\sigma(T)= =V\big(B(X),T\big)$. So the proof is complete.

Theorem D ([9], Theorem 2.5). *Let X be uniformly c-convex and let $C\geqq 0$ be a hermitian operator on X. If there are sequences $\{x_n\}\subset X$ and $\{f_n\}\subset X^*$ such that $\|x_n\|=\|f_n\|=1$ for each n, $f_n(x_n)\to 1$ and $f_n(Cx_n)\to 0$, then $Cx_n\to 0$.*

Lemma 3. *Let $T=H+iK$ be a hyponormal operator. If $\overline{T}T$ and $T\overline{T}$ are not invertible, then $0\in\partial\sigma(\overline{T}T)$ and $0\in\partial\sigma(T\overline{T})$, respectively, where ∂ denotes 'the boundary of'.*

Proof. We may only prove that $\sigma(\overline{T}T)$ and $\sigma(T\overline{T})$ are included in the half-plane $\{\alpha\in\mathbf{C}\colon \operatorname{Re}\alpha\geqq 0\}$. Since $V(H^2)$ and $V(K^2)$ are included in $\{\alpha\in\mathbf{C}\colon \operatorname{Re}\alpha\geqq 0\}$, it follows that $V(\overline{T}T)=V(H^2+K^2+C)\subset V(H^2)+V(K^2)+V(C)\subset\{\alpha\in\mathbf{C}\colon \operatorname{Re}\alpha\geqq 0\}$, where $C=i(HK-KH)\geqq 0$. Therefore, $\sigma(\overline{T}T)$ is included in $\{\alpha\in\mathbf{C}\colon \operatorname{Re}\alpha\geqq 0\}$. Also, since $\sigma(\overline{T}T)-\{0\}=\sigma(T\overline{T})-\{0\}$, it follows that $\sigma(T\overline{T})\subset\{\alpha\in\mathbf{C}\colon \operatorname{Re}\alpha\geqq 0\}$.

So the proof is complete.

Lemma 4. *Let X be uniformly c-convex and let $T=H+iK$ be a hyponormal operator on X. If $\overline{T}T$ is not invertible, then $T\overline{T}$ is not invertible.*

Proof. By Lemma 3, there exists a sequence $\{x_n\}$ of unit vectors in X such that $\overline{T}Tx_n\to 0$. We let that $C=i(HK-KH)\geqq 0$. Then, for a sequence $\{f_n\}$ in X^* such that $(x_n, f_n)\in\pi$, we get that $f_n(Cx_n)\to 0$. So, by Theorem D, $Cx_n\to 0$. Therefore, $T\overline{T}x_n=(H^2+K^2-C)x_n\to 0$.

So the proof is complete.

Theorem 5. *Let X and X^* be uniformly c-convex and let $T=H+iK$ be a hyponormal operator on X. Then*

$$\sigma(T)=\{z\in\mathbf{C}\colon \bar{z}\in\sigma_\pi(\overline{T})\}.$$

Proof. Since $T-z$ is hyponormal for every $z\in\mathbf{C}$, it is sufficient to show that $0\in\sigma(T)$ if and only if $0\in\sigma_\pi(\overline{T})$. Assume that 0 belongs to $\sigma(T)$. By Lemma 4, we may assume that $T\overline{T}$ is not invertible.

Therefore, by Lemma 3, 0 belongs to $\partial\sigma(T\overline{T})$. It follows that there exists a sequence $\{x_n\}$ of unit vectors in X such that $T\overline{T}x_n\to 0$. Since T is hyponormal, by Theorem B it follows that $\overline{T}^2x_n\to 0$. By the spectral mapping theorem for approximate point spectrum, 0 belongs to $\sigma_\pi(\overline{T})$.

Conversely, assume that 0 belongs to $\sigma_\pi(\overline{T})$. Then it follows that $0\in\sigma(T\overline{T})=$ $=\sigma(\overline{T}^*T^*)$. Similarly, 0 belongs to $\sigma_\pi(\overline{T}^*T^*)$. Here, $\overline{T}^*$ is hyponormal on a uniformly c-convex space X^*. Therefore, 0 belongs to $\sigma(T^*)=\sigma(T)$.

So the proof is complete.

Theorem 6. *Let X be strictly c-convex and let $T=H+iK$ be a hyponormal operator on X. Suppose that λ is an extreme point of* co $\overline{V(T)}$ *such that $\lambda\in V(T)$. Let $f(Tx)=\lambda$ for some $(x,f)\in\pi$. Then $Tx=\lambda x$.*

Proof. Each linear mapping $u(z)=\alpha z+\beta$ $(z\in\mathbf{C})$, where $\alpha,\beta\in\mathbf{C}$, $\alpha\neq 0$, maps $V(T)$ onto $V(u(T))$ and $\overline{V(T)}$ onto $\overline{V(u(T))}$. In addition $u(T)$ is hyponormal. Hence, we can suppose that $\lambda\in\mathbf{R}$ and $\operatorname{Re}z\leqq\lambda$ $(z\in V(T))$. Since $f(Hx)=\lambda=$ $=\max\{\alpha\colon \alpha\in\overline{V(H)}\}$, it follows by Theorem C that $Hx=\lambda x$. If $x'\in\operatorname{Ker}(H-\lambda)$ such that $\|x'\|=1$, then there exists $f'\in X^*$ such that $(x',f')\in\pi$ and $(H-\lambda)^*f'=0$.

It follows that

$$f'(Cx') = i\hat{x}'\big(K^*(H-\lambda)^* f'\big) - i f'\big(K(H-\lambda)x'\big) = 0$$

where $C = i(HK-KH) \geqq 0$.

By Theorem C, $Cx'=0$. Hence, it follows that $(H-\lambda)Kx'=0$. Therefore, it is easy to see that $\operatorname{Ker}(H-\lambda)$ is invariant for K. Let K_1 be the restriction of K to $\operatorname{Ker}(H-\lambda I)$. Let $y \in \operatorname{Ker}(H-\lambda)$ with $\|y\|=1$ and $g \in (\operatorname{Ker}(H-\lambda))^*$ such that $\|g\| = g(y) = 1$. Then

$$Ty = \lambda y + iKy = \lambda y + iK_1 y \in \operatorname{Ker}(H-\lambda)$$

and

$$g(Ty) = \lambda + ig(K_1 y).$$

Here, $g(Ty) \in V(T)$. Since λ is an extreme point of $\operatorname{co}\overline{V(T)}$ and $\operatorname{Re} z \leqq \lambda$ $(z \in V(T))$, it follows that $V(K_1) \subset \mathbf{R}^+$ or $V(-K_1) \subset \mathbf{R}^+$. Let $f_1 = f|\operatorname{Ker}(H-\lambda)$. We have then $f_1(K_1 x) = f(Kx) = 0$ and $\|f_1\| = f_1(x) = 1$. Since $\operatorname{Ker}(H-\lambda)$ is strictly c-convex, it follows that $K_1 x = Kx = 0$, by Theorem C.

So the proof is complete.

3. Doubly commuting n-tuples of hyponormal operators

Definition 3. For commuting operators T_1 and T_2 such that $T_j = H_j + iK_j$ (H_j and K_j hermitian, $j=1, 2$), T_1 and T_2 are called *doubly commuting* if $\overline{T}_1 T_2 = T_2 \overline{T}_1$. If T_1 and T_2 are doubly commuting, then H_j and K_j commute with H_l and K_l for $j \neq l$.

Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators on X. Let $\sigma(\mathbf{T})$ be the Taylor joint spectrum of $\mathbf{T}$. We refer the reader to Taylor [11].

The spatial joint numerical range $V(\mathbf{T})$ and the joint numerical range $V(B(X), \mathbf{T})$ of $\mathbf{T}$ are defined by

$$V(\mathbf{T}) = \{(f(T_1 x), \ldots, f(T_n x)) \in \mathbf{C}^n \colon (x, f) \in \pi\}$$

and

$$V(B(X), \mathbf{T}) = \{(F(T_1), \ldots, F(T_n)) \in \mathbf{C}^n \colon F \in B(X)^* \text{ and } \|F\| = F(I) = 1\}.$$

The *joint numerical radius* $v(\mathbf{T})$ and the *joint spectral radius* $r(\mathbf{T})$ of $\mathbf{T} = (T_1, \ldots, T_n)$ are defined by

$$v(\mathbf{T}) = \sup\{|z| \colon z \in V(\mathbf{T})\}$$

and

$$r(\mathbf{T}) = \sup\{|z| \colon z \in \sigma(\mathbf{T})\}.$$

Theorem E (V. Wrobel [14], Corollary 2.3). *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a commuting n-tuple of operators. Then*

$$\operatorname{co}\sigma(\mathbf{T}) \subset \overline{V(\mathbf{T})}.$$

Theorem 7. *Let X be uniformly convex, and let $\mathbf{T}=(T_1, \dots, T_n)$ be a doubly commuting n-tuple of hyponormal operators on X. Then*

$$\operatorname{co} \sigma(\mathbf{T}) = \overline{V(\mathbf{T})} = V(B(X), \mathbf{T}).$$

Proof. By Theorem E, it is clear that $\operatorname{co} \sigma(\mathbf{T}) \subset \overline{V(\mathbf{T})} \subset V(B(X), \mathbf{T})$. Assume that $\operatorname{co} \sigma(\mathbf{T}) \subsetneqq V(B(X), \mathbf{T})$. Suppose that $\alpha=(\alpha_1, \dots, \alpha_n) \in V(B(X), \mathbf{T}) - \operatorname{co} \sigma(\mathbf{T})$. Then there exists a linear functional Φ on $\mathbf{C}^n$ and a real number r such that

$$\operatorname{Re} \Phi(z) < r < \operatorname{Re} \Phi(\alpha) \quad (z \in \operatorname{co} \sigma(\mathbf{T})).$$

Let $\Phi(z)=t_{11}z_1+\dots+t_{1n}z_n$ $(z=(z_1, \dots, z_n) \in \mathbf{C}^n)$, and choose a non-singular $n \times n$ matrix M with $(t_{11}, \dots, t_{1n})$ as its first row. Then

$$\operatorname{Re} z_1 < r < \operatorname{Re} \beta_1 \quad (z = (z_1, \dots, z_n) \in \sigma(M\mathbf{T})),$$

where $(\beta_1, \dots, \beta_n)=M\alpha$. Therefore, $\operatorname{co} \sigma(\Sigma_j t_{1j}T_j) \subsetneqq V(B(X), \Sigma_j t_{1j}T_j)$. Since $\Sigma_j t_{1j}T_j$ is a hyponormal operator on a uniformly convex space, this yields a contradiction to Theorem 2.

So the proof is complete.

Corollary 8. *Let X be uniformly convex and let $\mathbf{T}=(T_1, \dots, T_n)$ be a doubly commuting n-tuple of hyponormal operators on X. Then $r(\mathbf{T})=v(\mathbf{T})$.*

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