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A model for a general linear bounded operator between two Hilbert spaces

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The main result of this paper is a theorem asserting that every bounded linear operator between two Hilbert spaces is unitary equivalent with a certain particular operator, the "model", in a similar sense with that used for contractions in [5]. This is accomplished by proving a model theorem for a contraction between two Hilbert spaces inspired by the techniques used in Ch. I, Sec. 10 from [7] then by proving a model theorem for an invertible linear bounded operator between two Hilbert spaces whose inverse is a contraction and then by the use of the canonical decomposition of every linear bounded operator as a direct sum of a contraction, an operator whose inverse is a contraction and an isometry (see [4], [6]). The model for the contraction is used also to prove a result concerning dilation of the couple (T, T^*).

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1. A model for a contraction between two Hilbert spaces

Let $\mathcal{H}_1, \mathcal{H}_2$ be two separable Hilbert spaces and $T: \mathcal{H}_1 \to \mathcal{H}_2$ a contraction, that is a bounded linear operator with $\|T\| \leqq 1$. Then $T^*: \mathcal{H}_2 \to \mathcal{H}_1$ is also a contraction. Define

$$D = (I_{\mathcal{H}_1} - T^*T)^{1/2}, \quad D_* = (I_{\mathcal{H}_2} - TT^*)^{1/2}, \quad \mathcal{E}_1 = \overline{D\mathcal{H}_1}, \quad \mathcal{E}_2 = \overline{D_*\mathcal{H}_2}$$

where $I_{\mathcal{H}}$ denotes the identity operator in $\mathcal{H}$. The norms in the two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ will be denoted respectively by $\|\cdot\|_1, \|\cdot\|_2$.

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We observe that $((T^*T)^k)_{k=0}^{\infty}$ is a decreasing sequence of selfadjoint contractions, consequently $Q_1=\lim_k (T^*T)^k$ exists in the strong sense and $0\leqq Q_1 \leqq I_{\mathscr{H}_1}$. Since $Q_1(I_{\mathscr{H}_1}-T^*T)h=0$ for $h\in\mathscr{H}_1$, Q_1 is the orthogonal projection onto $\ker(I_{\mathscr{H}_1}-T^*T)$. Similarly $Q_2=\text{s}-\lim_k (TT^*)^k$ is the orthogonal projection onto $\ker(I_{\mathscr{H}_2}-TT^*)$. In particular $Q_1\mathscr{H}_1$ and $Q_2\mathscr{H}_2$ are closed subspaces of $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively.

The definitions of Q_1 and Q_2 show that

$$Q_1 = T^*Q_2T, \quad Q_2 = TQ_1T^*. \tag{1.1}$$

Let $W\colon Q_1\mathscr{H}_1\to Q_2\mathscr{H}_2$ be defined by

$$WQ_1h = Q_2Th, \quad h\in\mathscr{H}_1. \tag{1.2}$$

Then by (1.1) one can easily see that

$$\|WQ_1h\|_2 = \|Q_2Th\|_2 = \|Q_1h\|_1,$$

such that W is an isometry.

Since, by (1.1), $Q_2(\ker T^*)=\{0\}$, it results that $Q_2T\mathscr{H}_1$ is dense in $Q_2\mathscr{H}_2$, such that, by (1.2), W has dense range in $Q_2\mathscr{H}_2$. It results that W is a unitary operator. A computation shows (see [7] Ch. I, Sec. 10) that for every $h\in\mathscr{H}_1$

$$\sum_{k=0}^{n}\|D(T^*T)^kh\|_1^2+\sum_{k=1}^{n}\|D_*T(T^*T)^kh\|_2^2 =$$

$$= \sum_{k=0}^{n}((T^*T)^{2k}-(T^*T)^{2k+1}h, h)+\sum_{k=0}^{n}((T^*T)^{2k+1}-(T^*T)^{2k+2}h, h) =$$

$$= \|h\|_1^2-\|T^*T)^{n+1}h\|_1^2.$$

Taking limits we have

$$\|h\|_1^2 = \sum_{k=0}^{\infty}\|D(T^*T)^kh\|_1^2+\sum_{k=0}^{\infty}\|D_*T(T^*T)^kh\|_2^2+\|Q_1h\|_1^2, \quad h\in\mathscr{H}_1. \tag{1.3}$$

By similar computations

$$\|h'\|_2^2 = \sum_{k=0}^{\infty}\|D_*(TT^*)^kh'\|_2^2+\sum_{k=0}^{\infty}\|DT^*(TT^*)^kh'\|_1^2+\|Q_2h'\|_2^2, \quad h'\in\mathscr{H}_2. \tag{1.4}$$

For a Hilbert space $\mathscr{E}$, $H^2(\mathscr{E})$ denotes the vectorial Hardy space (see [7], Ch. V Sec. 1 or [5], Sec. 0). For

$$u(z) = \sum_{k=0}^{\infty} z^k a_k, \quad |z|<1$$

the norm is defined by

$$\|u\|^2_{H^2(\mathcal{E})} = \sum_{k=0}^{\infty} \|a_k\|^2_{\mathcal{E}}.$$

We denote by $S_{\mathcal{E}}$ the unilateral shift on $H^2(\mathcal{E})$, ([5] Sec. 0). Let

$$(1.5)\quad \begin{gathered} V_1\colon \mathcal{H}_1 \to H^2(\mathcal{E}_1)\oplus H^2(\mathcal{E}_2)\oplus Q_1\mathcal{H}_1, \\ V_1 h = [\sum_{k=0}^{\infty} z^k D(T^*T)^k h]\oplus[\sum_{k=0}^{\infty} z^k D_* T(T^*T)^k h]\oplus Q_1 h. \end{gathered}$$

From (1.3) we have $\|V_1 h\|^2 = \|h\|_1^2$, where the square of the norm in the direct sum is the sum of the squares of the norms of the components. Let

$$(1.6)\quad \begin{gathered} V_2\colon \mathcal{H}_2 \to H^2(\mathcal{E}_1)\oplus H^2(\mathcal{E}_2)\oplus Q_2\mathcal{H}_2 \\ V_2 h' = [\sum_{k=0}^{\infty} z^k DT^*(TT^*)^k h']\oplus[\sum_{k=0}^{\infty} z^k D_*(TT^*)^k h']\oplus Q_2 h', \quad h'\in\mathcal{H}_2. \end{gathered}$$

From (1.4) it follows that $\|V_2 h'\|^2 = \|h'\|_2^2$. From the previous definitions

$$(1.7)\quad V_2 Th = [\sum_{k=0}^{\infty} z^k DT^*(TT^*)^k Th]\oplus[\sum_{k=0}^{\infty} z^k D_*(TT^*)^k Th]\oplus Q_2 Th =$$

$$= [\sum_{k=0}^{\infty} z^k D(T^*T)^{k+1} h]\oplus[\sum_{k=0}^{\infty} z^k D_* T(T^*T)^k h]\oplus Q_2 Th = [S^*_{\mathcal{E}_1}\oplus I_{H^2(\mathcal{E}_2)}\oplus W]V_1 h$$

for ewery $h\in\mathcal{H}_1$, and

$$(1.8)\quad V_1 T^* h' = [\sum_{k=0}^{\infty} z^k D(T^*T)^k T^* h']\oplus[\sum_{k=0}^{\infty} z^k D_*(TT^*)^{k+1} h']\oplus Q_1 T^* h' =$$

$$= [I_{H^2(\mathcal{E}_1)}\oplus S^*_{\mathcal{E}_2}\oplus W^*]V_2 h'$$

for every $h'\in\mathcal{H}_2$. Therefore the following model theorem is proved.

Theorem 1.1. *Let $T\colon \mathcal{H}_1\to\mathcal{H}_2$ be a contraction. There exist the Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2$, the closed subspaces $\mathcal{K}_1\subset H^2(\mathcal{E}_1)\oplus H^2(\mathcal{E}_2)$, $\mathcal{K}_2\subset H^2(\mathcal{E}_1)\oplus H^2(\mathcal{E}_2)$ and the unitary operators*

$$V_1\colon \mathcal{H}_1 \to \mathcal{K}_1\oplus Q_1\mathcal{H}_1, \quad V_2\colon \mathcal{H}_2 \to \mathcal{K}_2\oplus Q_2\mathcal{H}_2, \quad W\colon Q_1\mathcal{H}_1 \to Q_2\mathcal{H}_2$$

such that

$$(1.9)\quad T = V_2^*(S^*_{\mathcal{E}_1}\oplus I_{H^2(\mathcal{E}_2)}\oplus W)V_1,$$

$$(1.10)\quad T^* = V_1^*(I_{H^2(\mathcal{E}_1)}\oplus S^*_{\mathcal{E}_2}\oplus W^*)V_2.$$

2. A model for the inverse of a contraction

Let $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ an invertible contraction. T^* is then invertible, too. We proceed to exhibit a model for T^{-1}.

Lemma 2.1.

$$\|Dh\|_1^2 = \sum_{n=1}^{\infty} \|D_*^n Th\|_2^2 \quad \textit{for every} \quad h \in \mathcal{H}_1. \tag{2.1}$$

Proof. First we observe that $\|D_*\| < 1$. Indeed,

$$\|D_*\|^2 = \sup_{\|h'\|_2=1} \|D_* h'\|_2^2 = \sup_{\|h'\|_2=1} (1 - \|T^* h'\|_1^2) = 1 - \inf_{\|h'\|_2=1} \|T^* h'\|_1^2 < 1.$$

Then $\|D_*^2\| < 1$, so $(I - D_*^2)^{-1} = \sum_{n=0}^{\infty} D_*^{2n}$. But $(I - D_*^2)^{-1} = (TT^*)^{-1}$ and so

$$\sum_{n=1}^{\infty} D_*^{2n} = D_*^2 (TT^*)^{-1}. \tag{2.2}$$

We observe that

$$T^* D_*^2 (TT^*)^{-1} T = T^*(I_{\mathcal{H}_2} - TT^*)(T^*)^{-1} = I_{\mathcal{H}_1} - T^*T = D^2.$$

Then

$$\|Dh\|_1^2 = (D^2 h, h)_1 = \big(T^* D_*^2 (TT^*)^{-1} Th, h\big)_1 = \big(D_*^2 (TT^*)^{-1} Th, Th\big)_2 =$$

$$= \Big(\sum_{n=1}^{\infty} D_*^{2n} Th, Th\Big)_2 = \sum_{n=1}^{\infty} (D_*^{2n} Th, Th)_2 = \sum_{n=1}^{\infty} \|D_*^n Th\|_2^2.$$

The lemma is proved.

From (1.3) and (2.1) it results

$$\|h\|_1^2 = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \|D_*^n T (T^*T)^k h\|_2^2 + \sum_{k=0}^{\infty} \|D_* T (T^*T)^k h\|_2^2 + \|Q_1 h\|_1^2 \tag{2.3}$$

for every $h \in \mathcal{H}_1$. From (1.4) and (2.1) it results

$$\|h'\|_2^2 = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \|D_*^n (TT^*)^{k+1} h'\|_2^2 + \sum_{k=0}^{\infty} \|D_* (TT^*)^k h'\|_2^2 + \|Q_2 h'\|_2^2 \tag{2.4}$$

for every $h' \in \mathcal{H}_2$.

Let $\mathcal{M} = \{u \in H^2(\mathcal{E}_2) \mid u(\lambda) = \sum_{n=1}^{\infty} \lambda^n D_*^n h',\ |\lambda| < 1,\ h' \in \mathcal{H}_2\}$. $\mathcal{M}$ is a closed subspace of $H^2(\mathcal{E}_2)$. Indeed, let $(u_j)_{j \ge 0}$ be a sequence in $\mathcal{M}$, $u_j \to u$, $u \in H^2(\mathcal{E}_2)$, $u_j(\lambda) = \sum_{n=1}^{\infty} \lambda^n D_*^n h_j'$, $|\lambda| < 1$; $u(\lambda) = \sum_{n=1}^{\infty} \lambda^n a_n$, $|\lambda| < 1$, then

$$\|u_j - u_k\|^2 = \sum_{n=1}^{\infty} \|D_*^n (h_j' - h_k')\|_2^2 = \|DT^{-1}(h_j' - h_k')\|_1^2 \to 0 \quad \text{as} \quad j, k \to \infty.$$

We have $\|T^{-1}(h'_j-h'_k)\|_1^2=\|h'_j-h'_k\|_2^2+\|DT^{-1}(h'_j-h'_k)\|_1^2$. But, since $\|T^{-1}h'\|^2\geqq \geqq\|T\|^{-2}\|h'\|_2^2$ it results $(\|T\|^{-2}-1)\|h'_j-h'_k\|_2^2\to 0$ as $j,k\to\infty$, so there exists $h'=\lim_j h'_j$ and then $D_*^n h'_j\to D_*^n h'$ for every $n\geqq 1$ as $j\to\infty$. But $D_*^n h'_j\to a_n$ as $j\to\infty$, so $a_n=D_*^n h'$ and thus u is in $\mathcal{M}$.

Let $\tilde V_1\colon \mathcal{H}_1\to H^2(\mathcal{M})\oplus H^2(\mathcal{E}_2)\oplus Q_1\mathcal{H}_1$ be defined by

$$\tilde V_1 h=\Big[\sum_{k=0}^{\infty} z^k h_k\Big]\oplus\Big[\sum_{k=0}^{\infty} z^k D_* T(T^*T)^k h\Big]\oplus Q_1 h,\quad h\in\mathcal{H}_1 \tag{2.5}$$

where

$$h_k(\lambda)=\sum_{n=1}^{\infty}\lambda^n D_*^n T(T^*T)^k h,\quad \text{for}\quad |\lambda|<1. \tag{2.6}$$

(2.1) implies $\|h_k\|^2_{H^2(\mathcal{E}_2)}=\|D(T^*T)^k h\|_1^2$ and (2.3) implies $\|\tilde V_1 h\|^2=\|h\|_1^2$ for every $h\in\mathcal{H}_1$.

Let $\tilde V_2\colon \mathcal{H}_2\to H^2(\mathcal{M})\oplus H^2(\mathcal{E}_2)\oplus Q_2\mathcal{H}_2$ be defined by

$$\tilde V_2 f=\Big[\sum_{k=0}^{\infty} z^k f_k\Big]\oplus\Big[\sum_{k=0}^{\infty} z^k D_*(TT^*)^k f\Big]\oplus Q_2 f,\quad f\in\mathcal{H}_2 \tag{2.7}$$

where

$$f_k(\lambda)=\sum_{n=1}^{\infty}\lambda^n D_*^n (TT^*)^{k+1} f,\quad \text{for}\quad |\lambda|<1. \tag{2.8}$$

(2.1) implies $\|f_k\|^2_{H^2(\mathcal{E}_2)}=\|DT^*(TT^*)^k f\|_1^2$ and (2.4) implies $\|\tilde V_2 f\|^2=\|f\|_2^2$ for all $f\in\mathcal{H}_2$.

In order to find a model for T^{-1} we compute $\tilde V_1 T^{-1} f$ for $f\in\mathcal{H}_2$.

$$\tilde V_1 T^{-1} f=\Big[\sum_{k=0}^{\infty} z^k g_k\Big]\oplus\Big[\sum_{k=0}^{\infty} z^k D_*(TT^*)^k f\Big]\oplus Q_1 T^{-1} f \tag{2.9}$$

where

$$g_k(\lambda)=\sum_{n=1}^{\infty}\lambda^n D_*^n (TT^*)^k f,\quad \text{for}\quad |\lambda|<1. \tag{2.10}$$

Then

$$g_k(\lambda)-f_k(\lambda)=\sum_{n=1}^{\infty}\lambda^n D_*^{n+2} (TT^*)^k f\quad \text{for}\quad |\lambda|<1. \tag{2.11}$$

Observe that $\mathcal{M}$ is invariant for $S^*_{\mathcal{E}_2}$ and let us denote

$$S_*=S^*_{\mathcal{E}_2}|_{\mathcal{M}}. \tag{2.12}$$

(2.11) becomes $g_k-f_k=S_*^2 g_k$, so

$$f_k=(I-S_*^2)g\ . \tag{2.13}$$

For a Hilbert space $\mathscr{E}$ and $A \in B(\mathscr{E})$ a linear bounded operator, we denote by $A_\times$ the operator of multiplication by A from $H^2(\mathscr{E})$ to $H^2(\mathscr{E})$:

$$(A_\times u)(z) = \sum_{k=0}^{\infty} z^k A u_k, \quad \text{for} \quad u(z) = \sum_{k=0}^{\infty} z^k u_k, \quad |z| < 1.$$

Lemma 2.2. *The operator* $(I_{\mathscr{M}} - S_*^2)_\times \colon H^2(\mathscr{M}) \to H^2(\mathscr{M})$ *is invertible.*

Proof. We will prove that $I_{\mathscr{M}} - S_*^2 \colon \mathscr{M} \to \mathscr{M}$ is invertible. Let $S_{\mathscr{E}_2} \colon H^2(\mathscr{E}_2) \to H^2(\mathscr{E}_2)$ be the unilateral shift

$$(S_{\mathscr{E}_2} u)(z) = \sum_{k=0}^{\infty} z^{k+1} u_k, \quad \text{for} \quad u(z) = \sum_{k=0}^{\infty} z^k u_k, \quad |z| < 1.$$

We observe first that $(S_*^2)^* = P_{\mathscr{M}} S_{\mathscr{E}_2}^2|_{\mathscr{M}}$.

Let $u \in \ker (I_{\mathscr{M}} - S_*^2)^* = \ker (I_{\mathscr{M}} - P_{\mathscr{M}} S_{\mathscr{E}_2}^2|_{\mathscr{M}})$. Then $u = P_{\mathscr{M}} u = P_{\mathscr{M}} S_{\mathscr{E}_2}^2 u \Leftrightarrow P_{\mathscr{M}}(u - S_{\mathscr{E}_2}^2 u) = 0$ or equivalently $(u - S_{\mathscr{E}_2}^2 u)$ is in $\mathscr{M}^\perp$ and this implies $(u - S_{\mathscr{E}_2}^2 u) \perp u$ from which it results

$$(u, u) = (S_{\mathscr{E}_2}^2 u, u). \tag{2.14}$$

Let $u(\lambda) = \sum_{n=1}^{\infty} \lambda^n D_*^n h'$, $|\lambda| < 1, h' \in \mathscr{H}_2$. (2.14) becomes

$$\sum_{n=1}^{\infty} \|D_*^n h'\|_2^2 = \sum_{n=3}^{\infty} (D_*^{n-2} h', D_*^n h') = \sum_{n=3}^{\infty} (D_*^{n-1} h', D_*^{n-1} h') = \sum_{n=2}^{\infty} \|D_*^n h'\|_2^2.$$

Then $\|D_* h'\|_2 = 0$ since the series are convergent by Lemma 2.1, so $D_* h' = 0$ and this implies $u = 0$, so

$$\ker (I_{\mathscr{M}} - S_*^2)^* = \{0\}. \tag{2.15}$$

Next we prove that $I_{\mathscr{M}} - S_*^2$ is bounded from below. Let $u(\lambda) = \sum_{n=1}^{\infty} \lambda^n D_*^n h'$, $h' \in \mathscr{H}_2$, $|\lambda| < 1$, then

$$\|(I_{\mathscr{M}} - S_*^2) u\|_{H^2(\mathscr{E}_2)}^2 = \sum_{n=1}^{\infty} \|(D_*^n - D_*^{n+2}) h'\|_2^2 = \sum_{n=1}^{\infty} \|D_*^n (TT^*) h'\|_2^2 =$$

$$= \sum_{n=1}^{\infty} \|(TT^*) D_*^n h'\|_2^2 \geqq c^2 \sum_{n=1}^{\infty} \|D_*^n h'\|_2^2 = c^2 \|u\|_{H^2(\mathscr{E}_2)}^2.$$

Here we used the fact that TT^*, being positive and invertible, is bounded from below, i.e.:

$$\|TT^* h'\|_2 \geqq c \|h'\|_2 \quad \text{for every } h' \in \mathscr{H}_2, \text{ with } c > 0.$$

So

$$\|(I_{\mathscr{M}} - S_*^2) u\|_{H^2(\mathscr{E}_2)} \geqq c \|u\|_{H^2(\mathscr{E}_2)}, \quad c > 0. \tag{2.16}$$

(2.15) and (2.16) prove that there exists $(I_{\mathcal{M}}-S_*^2)^{-1}\colon \mathcal{M}\to\mathcal{M}$ and then there exists $(I_{\mathcal{M}}-S_*^2)_\times^{-1}\colon H^2(\mathcal{M})\to H^2(\mathcal{M})$. So the lemma is proved.

Lemma 2.2, (1.2), (2.9) and (2.13) imply

$$\tilde{V}_1 T^{-1} f = [\sum_{k=0}^{\infty} z^k (I_{\mathcal{M}}-S_*^2)^{-1} f_k] \oplus [\sum_{k=0}^{\infty} z^k D_*(TT^*)^k f] \oplus W^{-1} Q_2 f =$$

$$= [(I_{\mathcal{M}}-S_*^2)_\times^{-1} \oplus I_{H^2(\mathcal{E}_2)} \oplus W^{-1}] \tilde{V}_2 f.$$

So we have proved

Theorem 2.3. *Let $T\colon \mathcal{H}_1\to\mathcal{H}_2$ be an invertible contraction. There exist the Hilbert spaces $\mathcal{E}_2$, $\mathcal{M}$, the subspaces (closed, linear) $\mathcal{K}_1$ and $\mathcal{K}_2$ of $H^1(\mathcal{M})\oplus H^2(\mathcal{E}_2)$ and the unitary operators $\tilde{V}_1\colon \mathcal{H}_1\to\mathcal{K}_1\oplus Q_1\mathcal{H}_1$, $\tilde{V}_2\colon \mathcal{H}_2\to\mathcal{K}_2\oplus Q_2\mathcal{H}_2$ such that*

$$T^{-1} = \tilde{V}_1^* [(I_{\mathcal{M}}-S_*^2)_\times^{-1} \oplus I_{H^2(\mathcal{E}_2)} \oplus W^{-1}] \tilde{V}_2 \tag{2.17}$$

where S_* is defined by (2.12) *and W by* (1.2).

3. A model for a general bounded linear operator

To apply the Theorems 1.1 and 2.3 to a general linear bounded operator $T\colon \mathcal{H}_1\to\mathcal{H}_2$, let us denote as in [4], [6]

$$D_T = [(I_{\mathcal{H}_1}-T^*T)^+]^{1/2}, \quad X_T = [(I_{\mathcal{H}_1}-T^*T)^-]^{1/2}$$

where, for $A=A^*$, $A^+=\dfrac{|A|+A}{2}$, $A^-=\dfrac{|A|-A}{2}$.

Let $\mathcal{D}_T=\overline{D_T\mathcal{H}_1}$ be the defect space of T, $\mathcal{D}_T^1=\ker(I-T^*T)$, $\mathcal{X}_T=\overline{X_T\mathcal{H}_1}$ the excess space of T, and consider the corresponding spaces $\mathcal{D}_{T*}$, $\mathcal{D}_{T*}^1$, $\mathcal{X}_{T*}$ for T^*.

Then $\mathcal{H}_1=\mathcal{D}_T\oplus\mathcal{X}_T\oplus\mathcal{D}_T^1$, $\mathcal{H}_2=\mathcal{D}_{T*}\oplus\mathcal{X}_{T*}\oplus\mathcal{D}_{T*}^1$ and from the relations $TD_T=D_{T*}T$, $TX_{T*}=X_{T_*}T$ (see the proof in [4]) it results $T\mathcal{D}_T\subset\mathcal{D}_{T*}$, $T\mathcal{X}_T\subset\mathcal{X}_{T*}$ and obviously $T\mathcal{D}_T^1\subset\mathcal{D}_{T*}^1$. Define the operators $T_1=T|_{\mathcal{D}_T}\colon \mathcal{D}_T\to\mathcal{D}_{T*}$, $T_2=T|_{\mathcal{X}_T}\colon \mathcal{X}_T\to\mathcal{X}_{T*}$ and $T_3=T|_{\mathcal{D}_T^1}\colon \mathcal{D}_T^1\to\mathcal{D}_{T*}^1$. T_1 is a strict contraction and $(|T_1|^n)_{m=1}^{\infty}$ converges strongly to 0 as $n\to\infty$ (see [4], [6]). T_2 is an invertible operator and T_2^{-1} is a contraction. T_3 is an isometry.

In order to obtain the model for T we apply Theorem 1.1 for T_1 with $\mathcal{H}_1$ replaced by $\mathcal{D}_T$ and $\mathcal{H}_2$ replaced by $\mathcal{D}_{T*}$ and Theorem 2.3 for T_2^{-1} with $\mathcal{H}_1$ replaced by $\mathcal{X}_{T*}$ and $\mathcal{H}_2$ replaced by $\mathcal{X}_T$.

4. Some results concerning the dilation of a contraction and its adjoint

Let $\mathscr{H}$ be a separable Hilbert space and $T: \mathscr{H} \to \mathscr{H}$ a contraction. For the sake of simplifying the presentation we suppose that

(4.1) $$(T^*T)^n \to 0 \quad \text{and} \quad (TT^*)^n \to 0 \quad \text{strongly, as} \quad n \to \infty.$$

The main results remain valid without this assumption. From (4.1), $\mathscr{E}_1 = \mathscr{E}_2 = \mathscr{H}$ and by Theorem 1.1 we have the subspaces $\mathscr{K}_1$ and $\mathscr{K}_2$ of $H^2(\mathscr{H}) \oplus H^2(\mathscr{H})$ and the unitary operators $V_1: \mathscr{H} \to \mathscr{K}_1$, $V_2: \mathscr{H} \to \mathscr{K}_2$ such that $V_2 T = (S^* \oplus I) V_1$ and $V_1 T^* = (I \oplus S^*) V_2$ (where we denoted $S^*_{\mathscr{H}}$ by S^* and $I_{H^2(\mathscr{H})}$ by I).

Define $J = V_1 V_2^*$. J is an unitary operator from $\mathscr{K}_2$ to $\mathscr{K}_1$. Using the (easy to prove) fact that $\dim \mathscr{K}_2 = \dim \mathscr{K}_1 = \infty$, the orthogonals being considered in $H^2(\mathscr{H}) \oplus \oplus H^2(\mathscr{H})$, we define $\tilde{J}: L^2(\mathscr{H}) \oplus L^2(\mathscr{H}) \to L^2(\mathscr{H}) \oplus L^2(\mathscr{H})$

(4.2) $$\tilde{J} = J \oplus (\text{unitary operator } \mathscr{K}_2^{\perp} \to \mathscr{K}_1^{\perp}) \oplus (\text{identity of } H^2_-(\mathscr{H}) \oplus H^2_-(\mathscr{H}))$$

(for the definition of $L^2(\mathscr{H})$ see [6], Ch. V); $H^2_-(\mathscr{H}) = L^2(\mathscr{H}) \ominus H^2(\mathscr{H})$).

Let Z^* be the backward shift on $L^2(\mathscr{H})$. if

$$u(z) = \sum_{n=-\infty}^{\infty} z^n u_n, \quad |z| = 1,$$

then

$$(Z^* u)(z) = \sum_{n=-\infty}^{\infty} z^n u_{n+1}, \quad |z| = 1.$$

Define

(4.3) $$U = \tilde{J}^* (I_{L^2(\mathscr{H})} \oplus Z^*), \quad V = (Z^* \oplus I_{L^2(\mathscr{H})}) \tilde{J}.$$

U and V are unitary operators on $L^2(\mathscr{H}) \oplus L^2(\mathscr{H})$. Let us identify $\mathscr{H}$ with $\mathscr{K}_2$ by the mean of V_2. Then we state

Theorem 4.1. *For every polynomial p in two variables,*

$$p(T, T^*) = P_{\mathscr{H}}\, p(V, U)|_{\mathscr{H}}$$

where by $P_{\mathscr{H}}$ we denote the projection onto $\mathscr{H}$.

The proof relies on direct computation and is omitted. Next we show that in the case of a normal contraction T satisfying the hypothesis (4.1), the operator $\tilde{J}$ of (4.2) can be choosed such that the operators U and V defined in (4.3) commute.

Theorem 4.2. *Let $T: \mathscr{H} \to \mathscr{H}$ be a normal contraction satisfying $(TT^*)^n \to 0$ strongly as $n \to \infty$. Then the operator $\tilde{J}$ in* (4.2) *can be constructed such that U and V defined in* (4.3) *satisfy $UV = VU$.*

Proof. The proof that follows was suggested by the referee, replacing the more complicated original one. T normal implies $D_*=D$ and $\mathscr{E}_1=\mathscr{E}_2=\mathscr{H}$ by hypotheses $(TT^*)^n\to 0$.

Let $T=\hat{W}R$ be the polar decomposition of T. Then $\hat{W}$ can be a unitary operator, $\hat{W}R=R\hat{W}$ and $\hat{W}D=D\hat{W}$. Define the operator $\hat{U}$ on $H^2(\mathscr{H})$ by

$$\hat{U}\Big(\sum_{k=0}^{\infty} z^k h_k\Big) = \sum_{k=0}^{\infty} z^k \hat{W}^2 h_k.$$

$\hat{U}$ is a unitary operator that commutes with S^*, the backward shift on $H^2(\mathscr{H})$. The operator $\tilde{U}$ defined by

$$\tilde{U} = \begin{pmatrix} 0 & I_{H^2(\mathscr{H})} \\ \hat{U} & 0 \end{pmatrix}$$

with respect to $H^2(\mathscr{H})\oplus H^2(\mathscr{H})$ is a unitary operator that satisfies

$$(S^*\oplus S^*)\tilde{U} = \tilde{U}(S^*\oplus S^*). \tag{4.4}$$

Then

$$\tilde{U}\Big(\Big(\sum_{k=0}^{\infty} z^k DT^*(TT^*)^k h\Big)\oplus\Big(\sum_{k=0}^{\infty} z^k D(TT^*)^k h\Big)\Big) =$$

$$= \tilde{U}\Big(\Big(\sum_{k=0}^{\infty} z^k D\hat{W}^* R^{2k+1} h\Big)\oplus\Big(\sum_{k=0}^{\infty} z^k D_* R^{2k} h\Big)\Big) = \Big(\sum_{k=0}^{\infty} z^k D_* R^{2k} h\Big)\oplus\Big(\sum_{k=0}^{\infty} z^k D\hat{W}RR^{2k} h\Big) =$$

$$= \Big(\sum_{k=0}^{\infty} z^k D(T^*T)^k h\Big)\oplus\Big(\sum_{k=0}^{\infty} z^k D_* T(T^*T)^k h\Big) =$$

$$= V_1 V_2^* \Big(\Big(\sum_{k=0}^{\infty} z^k DT^*(TT^*)^k h\Big)\oplus\Big(\sum_{k=0}^{\infty} z^k D_*(TT^*)^k h\Big)$$

for every $h\in\mathscr{H}$. This shows that $\tilde{U}\mathscr{K}_2=\mathscr{K}_1$ and $\tilde{U}|_{\mathscr{K}_2}=V_1V_2^*$. Since $\tilde{U}$ is a unitary operator it results $\tilde{U}\mathscr{K}_2^{\perp}=\mathscr{K}_1^{\perp}$ and so we can choose $\tilde{J}$ such that

$$\tilde{J}|_{H^2(\mathscr{H})\oplus H^2(\mathscr{H})} = \tilde{U}.$$

For this $\tilde{J}$ we have, due also to (4.4),

$$UV|_{H^2(\mathscr{H})\oplus H^2(\mathscr{H})} = \tilde{U}^*(S^*\oplus S^*)\tilde{U} = S^*\oplus S^* = VU|_{H^2(\mathscr{H})\oplus H^2(\mathscr{H})}.$$

Since by (4.2), (4.3) the same is true for $H^2_-(\mathscr{H})\oplus H^2_-(\mathscr{H})$ it results $UV=VU$ and the theorem is proved.

We remark at the end that we can drop the assumption (4.1) from Theorems 4.1 and 4.2 without altering the results.

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