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Embedding results pertaining to strong approximation

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1. The aim of the paper is to make a step toward answering an open problem of our previous paper [2] and to extend another result published in the same paper.

In order to quote the known results we have to recall some notions and notations.

Let $f(x)$ be a continuous and 2π-periodic function and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series. Let $s_n = s_n(x) = s_n(f; x)$ and $\tau_n = \tau_n(x) = \tau_n(f, x)$ denote the n-th partial sum and the classical de la Vallée Poussin mean of (1), i.e.

$$\tau_n(x) = \frac{1}{n} \sum_{k=n+1}^{2n} s_k(x), \quad n = 1, 2, \ldots .$$

We denote by $\|\cdot\|$ the usual supremum norm.

Let $\omega(\delta)$ be a nondecreasing continuous function on the interval $[0, 2\pi]$ having the properties: $\omega(0)=0$, $\omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_2)$ for any $0 \leqq \delta_1 \leqq \delta_2 \leqq \delta_1+\delta_2 \leqq 2\pi$. Such a function is called a modulus of continuity. The modulus of continuity of f will be denoted by $\omega(f; \delta)$.

We define the following classes of continuous functions:

$$H^\omega := \{f\colon \omega(f; \delta) = O(\omega(\delta))\},$$

$$S_p(\lambda) := \{f\colon \|\sum_{n=0}^{\infty} \lambda_n |s_n - f|^p\| < \infty\}$$

and

$$V_p(\lambda) := \{f\colon \|\sum_{n=1}^{\infty} \lambda_n |\tau_n - f|^p\| < \infty\},$$

where $\lambda = \{\lambda_n\}$ is a monotonic sequence of positive numbers and $0 < p < \infty$.

V. G. Krotov and the author ([1]) proved the following result.

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Theorem A. *If* $\{\lambda_n\}$ *is a positive monotonic sequence,* ω *is a modulus of continuity and* $0<p<\infty$, *then*

$$\sum_{k=1}^{n}(k\lambda_k)^{-1/p} = O\left(n\omega\left(\frac{1}{n}\right)\right) \tag{2}$$

implies

$$S_p(\lambda)\subset H^{\omega}. \tag{3}$$

Conversely, if there exists a number Q *such that* $0\leqq Q<1$ *and*

$$n^{Q}\lambda_n\uparrow, \tag{4}$$

then (3) *implies* (2).

Since the de la Vallée Poussin means τ_n usually approximate the function f, in the sup norm, better than the partial sums s_n do, so we may expect that under reasonable conditions the following embedding relations will hold

$$S_p(\lambda)\subset V_p(\lambda)\subset H^{\omega}. \tag{5}$$

In [2], A. Meir and me, verified some results pertaining to (5). More precisely the following theorems were proved:

Theorem B. *If* $p\geqq 1$ *and* $\{\lambda_n\}$ *is a monotonic (nondecreasing or nonincreasing) sequence of positive numbers satisfying the condition*

$$\lambda_n/\lambda_{2n}\leqq K^{*},\quad n=1,2,\ldots, \tag{6}$$

then

$$S_p(\lambda)\subset V_p(\lambda) \tag{7}$$

holds.

Theorem C. *Let* $\{\lambda_n\}$ *be a monotonic sequence of positive numbers, furthermore let* ω *be a modulus of continuity and* $0<p<\infty$. *Then condition* (2) *implies*

$$V_p(\lambda)\subset H^{\omega}. \tag{8}$$

If $p\geqq 1$ *and there exists a number* Q *such that* $0\leqq Q<1$ *and* (4) *holds, then, conversely,* (8) *implies* (2).

To decide whether $S_p(\lambda)\subset V_p(\lambda)$, i.e. (7), holds when $0<p<1$; it was left as an open problem.

Making many unsuccessful attempts to prove (7) or its converse, at the present time, I have the conjecture that neither $S_p(\lambda)\subset V_p(\lambda)$ nor $V_p(\lambda)\subset S_p(\lambda)$ hold generally, but I have not been able to verify this statement.

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*) $K, K_1, \ldots$ will denote positive constants, not necessarily the same at each occurence.

However it turned out that if one defined a new subclass of $V_p(\lambda)$, which one could call "strong" $V_p(\lambda)$-class, and denoted by $V_p^{(s)}(\lambda)$, i.e.

$$V_p^{(s)}(\lambda) := \left\{ f\colon \left\| \sum_{n=1}^{\infty} \lambda_n \left(\frac{1}{n} \sum_{k=n+1}^{2n} |s_k - f| \right)^p \right\| < \infty \right\},$$

then under restriction (6) $S_p(\lambda) \subset V_p^{(s)}(\lambda)$ also holds for $p \geqq 1$, and $S_p(\lambda) \supset V_p^{(s)}(\lambda)$ is already true for $0 < p \leqq 1$ if $\lambda_{2n} \leqq K\lambda_n$. First we prove these statements.

Compare the definitions of $V_p(\lambda)$ and $V_p^{(s)}(\lambda)$, it is obvious that for any positive p and for any λ

$$V_p^{(s)}(\lambda) \subset V_p(\lambda) \tag{9}$$

always holds.

It is clear that (8) and (9) imply

$$V_p^{(s)}(\lambda) \subset H^{\omega}. \tag{10}$$

Secondly we prove that (10) also implies relation (2) for any positive p if (4) holds.

This result is a mild sharpening of the second part of Theorem C for $p \geqq 1$; and by (9) it extends the previous statement for any positive p. The latter result is the more important one.

2. We prove the following results.

Theorem 1. *Let $\lambda = \{\lambda_n\}$ be a monotonic sequence of positive numbers. The following embedding relations hold:*

$$S_p(\lambda) \subset V_p^{(s)}(\lambda) \quad \textit{if} \quad p \geqq 1 \quad \textit{and} \quad \lambda_n = O(\lambda_{2n}); \tag{11}$$

and

$$S_p(\lambda) \supset V_p^{(s)}(\lambda) \quad \textit{if} \quad 0 < p \leqq 1 \quad \textit{and} \quad \lambda_{2n} = O(\lambda_n). \tag{12}$$

Theorem 2. *Let $\lambda = \{\lambda_n\}$ be a monotonic sequence of positive numbers, furthermore let ω be a modulus of continuity and $p > 0$. If there exists a number Q such that $0 \leqq Q < 1$ and* (4) *holds, then the embedding relation* (10) *implies relation* (2).

Theorem C and Theorem 2 convey as an immediate consequence the following result.

Corollary. *Under condition* (4) *the embedding relation* (8) *implies relation* (2) *for any positive p.*

3. To prove our theorems we require some lemmas.

Lemma 1 ([1]). *If $a_n \geqq 0$ and the function*

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin nx$$

belongs to the class H^ω, then

$$\sum_{k=1}^{n} ka_k = O\left(n\omega\left(\frac{1}{n}\right)\right).$$

Lemma 2. *If* $0<p<\infty$, $\lambda_n\uparrow$ *or* $\lambda_n\downarrow$ *and there exists a number* Q, $0\leqq Q<1$, *such that* (4) *holds, then the function*

$$f(x) := \sum_{n=1}^{\infty} \frac{1}{n}(n\lambda_n)^{-1/p} \sin nx$$

belongs to the class $V_p^{(s)}(\lambda)$.

Proof. It is easy to see that

$$\sum_{n=1}^{\infty} \frac{1}{n}(n\lambda_n)^{-1/p} = \sum_{n=1}^{\infty} n^{-(1+(1/p)(1-Q))}(n^Q\lambda_n)^{-1/p} < \infty,$$

so f is a continuous function, and $f(0)=f(\pi)=0$.

To prove that $f\in V_p^{(s)}(\lambda)$ we fix $0<x<\pi$ and choose N such that

$$\frac{1}{N+1} < x \leqq \frac{1}{N}.$$

We make the following estimates:

$$\sum_{n=1}^{\infty} \lambda_n \left\{\frac{1}{n}\sum_{k=n+1}^{2n} |s_k(x)-f(x)|\right\}^p \leqq$$

$$\leqq \sum_{n=1}^{N/2} \lambda_n \left\{\frac{1}{n}\sum_{k=n+1}^{2n}\left(\left|\sum_{m=k+1}^{N+1}\frac{1}{m}(m\lambda_m)^{-1/p}\sin mx\right| + \left|\sum_{m=N+2}^{\infty}\frac{1}{m}(m\lambda_n)^{-1/p}\sin mx\right|\right)\right\}^p +$$

$$+ \sum_{n=N/2}^{\infty} \lambda_n \left\{\frac{1}{n}\sum_{k=n+1}^{2n}\left|\sum_{m=k+1}^{\infty}\frac{1}{m}(m\lambda_m)^{-1/p}\sin mx\right|\right\}^p \equiv \sum\nolimits_1 + \sum\nolimits_2,$$

where

$$\sum\nolimits_1 \leqq K\sum_{n=1}^{N/2} \lambda_n \left\{\frac{1}{n}\sum_{k=n+1}^{2n}\left|\sum_{m=k+1}^{N+1}\frac{1}{m}(m\lambda_m)^{-1/p}\sin mx\right|\right\}^p +$$

$$+K\sum_{n=1}^{N/2} \lambda_n \left\{\frac{1}{n}\sum_{k=n+1}^{2n}\left|\sum_{m=N+2}^{\infty}\frac{1}{m}(m\lambda_m)^{-1/p}\sin mx\right|\right\}^p \equiv \sum\nolimits_{11} + \sum\nolimits_{12}.$$

First we assume that $\lambda_n\downarrow$. By our assumption, we can choose a positive Q such that $1>Q>1-p$ and $n^Q\lambda_n\uparrow$. Then $\dfrac{Q-1}{p}>-1$, so for any $n<k<N$ we have

that

$$\sum_{m=k+1}^{N+1} \frac{1}{m} (m\lambda_m)^{-1/p} \sin mx \leqq x \sum_{m=k+1}^{N+1} (m\lambda_m)^{-1/p} =$$

$$= x \sum_{m=k+1}^{N+1} (m^Q \lambda_m m^{1-Q})^{-1/p} \leqq x(n^Q \lambda_n)^{-1/p} \sum_{m=n+1}^{N+1} m^{(Q-1)/p} \leqq$$

$$\leqq Kx(n^Q \lambda_n)^{-1/p} N^{1+(Q-1)/p},$$

whence we get that

$$\sum\nolimits_{11} \leqq K_1 \sum_{n=1}^{N/2} \lambda_n x^p \lambda_n^{-1} n^{-Q} N^{p+Q-1} \leqq K_2 x^p N^p \leqq K_3.$$

Furthermore

$$\sum\nolimits_{12} \leqq \sum_{n=1}^{N/2} \lambda_n \left| \sum_{m=N+2}^{\infty} \frac{1}{m} (m\lambda_m)^{-1/p} \sin mx \right|^p \leqq$$

$$\leqq \sum_{n=1}^{N/2} \lambda_n \left\{ \sum_{m=N+2}^{\infty} \frac{1}{m} (m^Q \lambda_m m^{1-Q})^{-1/p} \right\}^p \leqq$$

$$\leqq \sum_{n=1}^{N/2} \lambda_n \left\{ (N^Q \lambda_N)^{-1/p} \sum_{m=N}^{\infty} m^{-1-(1-Q)/p} \right\}^p \leqq$$

$$\leqq \sum_{n=1}^{N} \lambda_n (N^Q \lambda_N)^{-1} N^{-(1-Q)} =$$

$$= N^{-1} \lambda_N^{-1} \sum_{n=1}^{N} n^Q \lambda_n n^{-Q} \leqq N^{Q-1} \sum_{n=1}^{N} n^{-Q} \leqq K.$$

To estimate $\sum_2$ we use the following inequality

$$\left| \sum_{m=k+1}^{\infty} \frac{1}{m} (m\lambda_m)^{-1/p} \sin mx \right| \leqq \frac{K}{kx} (k\lambda_k)^{-1/p}$$

for any $0<x<\pi$. Hence

$$\sum\nolimits_2 \leqq K_1 \sum_{n=N/2}^{\infty} \lambda_n n^{-p} x^{-p} n^{-1} \lambda_n^{-1} \leqq K_2 \sum_{n=N/2}^{\infty} n^{-p-1} x^{-p} \leqq K_3 (xN)^{-p} \leqq K_4.$$

Collecting these estimates we get that $f \in V_p^{(s)}(\lambda)$ in the case $\lambda_n \downarrow$.

The proof in the case $\lambda_n \uparrow$ is easier, then we can simply replace condition (4) by $\lambda_n \uparrow$ in some parts of the previous proof. Therefore we omit the details.

The proof is completed.

4. Now we can prove our theorems.

Proof of Theorem 1. For $p\geqq 1$, by Hölder's inequality, the inequality

$$\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|\leqq\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|^p\right\}^{1/p}$$

holds, whence

$$\sum_{n=1}^{\infty}\lambda_n\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|\right\}^p\leqq\sum_{n=1}^{\infty}\lambda_n\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|^p\right\}\leqq \tag{13}$$

$$\leqq\sum_{k=2}^{\infty}|s_k-f|^p\sum_{n=k/2}^{k}\lambda_n/n\equiv\sum\nolimits_3$$

follows. By $\lambda_n=O(\lambda_{2n})$ we have

$$\sum\nolimits_3\leqq K\sum_{k=2}^{\infty}\lambda_k|s_k-f|^p. \tag{14}$$

Inequalities (13) and (14) imply (11).

In the case $0<p\leqq 1$ we use the inequality

$$\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|\geqq\left\{\frac{1}{n}\sum_{k=n+1}^{2n}|s_k-f|^p\right\}^{1/p}, \tag{15}$$

which can also be proved by Hölder inequality, and the estimate

$$\lambda_k=O\Big(\sum_{n=k/2}^{k-1}\lambda_n/n\Big), \tag{16}$$

it follows from $\lambda_{2n}=O(\lambda_n)$. Then, by (15) and (16),

$$\sum_{n=2}^{\infty}\lambda_n|s_n-f|^p\leqq K\sum_{n=2}^{\infty}\Big(\sum_{k=n/2}^{n-1}\lambda_k/k\Big)|s_n-f|^p\leqq$$

$$\leqq K\sum_{k=1}^{\infty}\lambda_k/k\sum_{n=k+1}^{2k}|s_n-f|^p\leqq K\sum_{k=1}^{\infty}\lambda_k\left(\frac{1}{k}\sum_{n=k+1}^{2k}|s_n-f|\right)^p$$

holds, whence (12) clearly follows.

Proof of Theorem 2. Let us consider the function given in Lemma 2, i.e. let

$$f_0(x):=\sum_{n=1}^{\infty}\frac{1}{n}(n\lambda_n)^{-1/p}\sin nx.$$

Then, by Lemma 2, $f_0 \in V_p^{(s)}(\lambda)$. The assumption $V_p^{(s)}(\lambda) \subset H^\omega$ conveys that $f_0 \in H^\omega$ also holds. Hence, using Lemma 1, relation (2) follows, that is, (10) really implies (2).

The proof is completed.

References

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[2] L. LEINDLER and A. MEIR, Embedding theorems and strong approximation, *Acta Sci. Math.*, **47** (1984), 371—375.

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