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Some aspects of nonstationarity. I

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I. Introduction

Recently, several contractive completion problems were considered in papers as [5], [6], [10], and in [5] an approach based on a lifting theorem for the representations of the algebra of upper triangular matrices was proposed.

The aim of the present paper is to point out another variant — a "nonstationary" one — for the lifting theorem of Sarason—Sz.-Nagy—Foiaş which can be also used for the above mentioned completion problems. Parametrizations with choice parameters and linear fractional maps are obtained also in this case.

The content of the paper is the following: in Section 2 we obtain a time-variant analog for some other basic results in Sz.-Nagy—Foiaş theory of contractions, as model for discrete time, time-variant linear systems. In Section 3 we describe the nonstationary variant of the lifting theorem and in the last section we show how some completion problems fit in our approach.

II. The marking model

In this section we are concerned with time-variant linear systems in the following state-space representation:

$$\begin{cases} x_{n+1} = T_n^* x_n + D_{T_n} u_n \\ y_n = D_{T_n^*} x_n - T_n u_n \end{cases} \quad n \in \mathbf{Z} \tag{1.1}$$

where $\{\mathscr{H}_n\}_{n\in\mathbf{Z}}$ is a given family of Hilbert spaces, $T_n \in \mathscr{L}(\mathscr{H}_{n+1}, \mathscr{H}_n)$ are contractions, $u_n \in \mathscr{D}_{T_n}$, $y_n \in \mathscr{D}_{T_n^*}$, $x_n \in \mathscr{H}_n$, and for a contraction $T \in \mathscr{L}(\mathscr{H}, \mathscr{H}')$ we use the standard notations $D_T = (I - T^*T)^{1/2}$ and $\mathscr{D}_T = \overline{D_T \mathscr{H}}$. Now, we consider the positive-definite kernel $\mathscr{S}$ associated by the algorithm in [7], Theorem 2.4, to the

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parameters $G_{i,i+1}=T_i$, $i\in\mathbf{Z}$, and zero in rest. Using Theorem 3.2 in [7], the Kolmogorov decomposition of $\mathscr{S}$ is simply given by the unitary operators

$$(1.2)\quad \begin{cases} W_n\colon \mathscr{K}_{n+1}\to\mathscr{K}_n \\ W_n(\dots,d_{*n-1},d_{*,n},\boxed{h_{n+1}},d_{n+1},d_{n+2},\dots)= \\ =(\dots d_{*,n-2},d_{*,n-1},\boxed{D_{T_n^*}d_{*,n}+T_nh_{n+1}},-T_n^*d_{*,n}+D_{T_n}h_{n+1},d_{n+1},d_{n+2},\dots)\end{cases}$$

where

$$\mathscr{K}_n=\dots\oplus\mathscr{D}_{T^*_{n-2}}\oplus\mathscr{D}_{T^*_{n-1}}\oplus\boxed{\mathscr{H}_n}\oplus\mathscr{D}_{T_n}\oplus\mathscr{D}_{T_{n+1}}\oplus\dots.$$

Let us pursue by introducing the main elements of the geometrical model of (1.1). Define the spaces

$$\mathscr{K}_n^+=\mathscr{H}_n\oplus\mathscr{D}_{T_n}\oplus\mathscr{D}_{T_{n+1}}\oplus\dots$$

and the isometries

$$(1.3)\qquad W_n^+\colon \mathscr{K}_{n+1}^+\to\mathscr{K}_n^+,\quad W_n^+=W_n|\mathscr{K}_{n+1}^+$$

and use the Wold decomposition for the family $\{W_k^+\}_{k\geq n}$. We denote $\mathscr{L}_n^+=\mathscr{K}_n^+\ominus\ominus W_n^+\mathscr{K}_{n+1}^+$ and according to the form (1.2) of the Kolmogorov decomposition, we identify $\mathscr{L}_n^+=W_n(\dots\oplus 0\oplus\mathscr{D}_{T_n^*}\oplus\boxed{0_{\mathscr{H}_{n+1}}}\oplus 0\oplus\dots)$.

Moreover, define $\mathscr{R}_n^+=\bigcap_{p=0}^{\infty}W_n\dots W_{n+p}\mathscr{K}_{n+p+1}^+$ and then

$$(1.4)\qquad \mathscr{K}_n^+=(\mathscr{L}_n^+\oplus\bigoplus_{p=1}^{\infty}W_n^+\dots W_{n+p-1}^+\mathscr{L}_{n+p}^+)\oplus\mathscr{R}_n^+.$$

Similar considerations take place for the spaces

$$\mathscr{K}_n^-=\dots\oplus\mathscr{D}_{T^*_{n-1}}\oplus\mathscr{H}_n$$

and the isometries

$$(1.5)\qquad W_n^-\colon \mathscr{K}_n^-\to\mathscr{K}_{n+1}^-,\quad W_n^-=W_n^*|\mathscr{K}_n^-.$$

We use the notation $\mathscr{L}_n^-=\mathscr{K}_n^-\ominus W_{n-1}^-\mathscr{K}_{n-1}^-$ and taking (1.2) into account, we identify $\mathscr{L}_n^-=W_n^*(\dots 0\oplus 0_{\mathscr{H}_n}\oplus\boxed{\mathscr{D}_{T_n}}\oplus 0\oplus\dots)$. It is also useful to denote the space $\dots\oplus 0\oplus\mathscr{D}_{T_n^*}\oplus\boxed{0}\oplus 0\oplus\dots\subset\mathscr{K}_{n+1}$ by $\mathscr{D}_{T_n^*}^{(-1)}$ and $\dots 0\oplus\boxed{0}\oplus\mathscr{D}_{T_n}\oplus 0\oplus\dots\subset\mathscr{K}_n$ by $\mathscr{D}_{T_n}^{(1)}$. Another application of the Wold decomposition for the family $\{W_k^-\}_{k\leq n-1}$ will produce a decomposition

$$\mathscr{K}_n^-=(\mathscr{L}_n^-\oplus\bigoplus_{p=1}^{\infty}W_{n-1}^-\dots W_{n-p}^-\mathscr{L}_{n-p}^-)\oplus\mathscr{R}_n^-.$$

Now define the spaces

$$\mathscr{K}_n^{\text{out}}=\bigoplus_{q=1}^{\infty}W_{n-1}^*\dots W_{n-q}^*\mathscr{D}^{(-1)}_{T^*_{n-q-1}}\oplus\mathscr{D}^{(-1)}_{T^*_{n-1}}\oplus\bigoplus_{p=0}^{\infty}W_n\dots W_{n+p}\mathscr{D}^{(-1)}_{T^*_{n+p}},$$

and

$$\mathcal{K}_n^{\text{inp}} = \bigoplus_{p=1}^{\infty} W_{n-1}^* \dots W_{n-p}^* \mathcal{D}_{T_{n-p}}^{(1)} \oplus \mathcal{D}_{T_n}^{(1)} \oplus \bigoplus_{q=0}^{\infty} W_n \dots W_{n+q} \mathcal{D}_{T_{n+q+1}}^{(1)}.$$

A usual condition in Sz.-Nagy—Foiaş theory (and also in system theory) is to ask $\mathcal{K}_n = \mathcal{K}_n^{\text{inp}} \vee \mathcal{K}_n^{\text{out}}$ for every $n \in \mathbf{Z}$. We have by direct computation using (1.2) that

$$\mathcal{K}_n \ominus (\mathcal{K}_n^{\text{inp}} \vee \mathcal{K}_n^{\text{out}}) =$$
$$= \{h \in \mathcal{H}_n \mid \dots \|T_{n-2} T_{n-1} h\| = \|T_{n-1} h\| = \|h\| = \|T_n^* h\| = \|T_{n+1}^* T_n^* h\| = \dots\}$$

which corresponds to Theorem I. 3.2 in [16].

Finally, we define the family of characteristic operators of the system (1.1) by the formula

$$(1.6) \qquad Q_n: \mathcal{K}_n^{\text{inp}} \to \mathcal{K}_n^{\text{out}}, \quad Q_n = P_{\mathcal{K}_n^{\text{out}}}^{\mathcal{K}_n} | \mathcal{K}_n^{\text{inp}}.$$

We obtain a first result concerning the geometry of the spaces $\mathcal{K}_n$.

2.1. Theorem. *For a system* (1.1) *satisfying the condition* $\mathcal{K}_n = \mathcal{K}_n^{\text{inp}} \vee \mathcal{K}_n^{\text{out}}$ *for every* $n \in \mathbf{Z}$, *the following relations hold:*

$$\mathcal{K}_n = \mathcal{K}_n^{\text{out}} \oplus \mathcal{R}_n^+,$$

$$\mathcal{H}_n = \mathcal{K}_n^+ \ominus \{Q_n u \oplus (I - Q_n) u \mid u \in \bigoplus_{p=1}^{\infty} W_n \dots W_{n+p-1} \mathcal{D}_{T_{n+p}}^{(1)}\}.$$

Proof. The first relation is obvious. For the second one, we remark that

$$\mathcal{K}_n^+ = \mathcal{H}_n \vee \bigvee_{p=1}^{\infty} W_n \dots W_{n+p-1} \mathcal{H}_{n+p}$$

and, as

$$W_n \mathcal{D}_{T_n^*}^{(-1)} \oplus W_n \mathcal{H}_{n+1} = \mathcal{H}_n \oplus \mathcal{D}_{T_n}^{(1)},$$

one obtains

$$\mathcal{K}_n^+ \subset \mathcal{H}_n \oplus \mathcal{D}_{T_n}^{(1)} \oplus W_n \mathcal{D}_{T_{n+1}}^{(1)} \oplus \dots.$$

The converse inclusion is clear, consequently

$$\mathcal{H}_n = \mathcal{K}_n^+ \ominus (\mathcal{D}_{T_n}^{(1)} \oplus W_n \mathcal{D}_{T_{n+1}}^{(1)} \oplus \dots),$$

which completes the proof.

Then we introduce the marking model. The marking operators appear as the main elements involved by the Kolmogorov decomposition of an arbitrary positive-definite kernel. In our case, define the spaces:

$$\mathcal{M}_+ = \bigoplus_{n \in \mathbf{Z}} \mathcal{D}_{T_n^*} \quad \text{and} \quad \mathcal{M}_- = \bigoplus_{n \in \mathbf{Z}} \mathcal{D}_{T_n}$$

and the operators

$$(1.7)\qquad \begin{cases} M_n^+: \bigoplus_{k\geq n+1} \mathscr{D}_{T_k^*} \to \bigoplus_{k\geq n} \mathscr{D}_{T_k^*}, \\ M_n^+(d_{*,n+1}, d_{*,n+2}, \ldots) = (0, d_{*,n+1}, d_{*,n+2}, \ldots) \end{cases}$$

$$(1.8)\qquad \begin{cases} M_n^-: \bigoplus_{k\geq n+1} \mathscr{D}_{T_k} \to \bigoplus_{k\geq n} \mathscr{D}_{T_k}, \\ M_n^-(d_{n+1}, d_{n+2}, \ldots) = (0, d_{n+1}, d_{n+2}, \ldots). \end{cases}$$

Our goal is to obtain identifications for $\mathscr{K}_n^{\mathrm{inp}}$, $\mathscr{K}_n^{\mathrm{out}}$, $\mathscr{H}_n$ and the characteristic operators in terms of the marking operators (1.7) and (1.8) and the marking spaces $\mathscr{M}_+$ and $\mathscr{M}_-$. For this aim, we introduce the following unitary operators

$$(1.9)\quad \begin{cases} \Phi_n^+: \mathscr{K}_n^{\mathrm{out}} \to \mathscr{M}_+, \\ \Phi_n^+(\ldots \oplus W_{n+1}^* d_{*,n-2}^{(-1)} \oplus d_{*,n-1}^{(-1)} \oplus W_n d_{*,n}^{(-1)} \oplus \ldots) = (\ldots, d_{*,-1}, \boxed{d_{*,0}}, d_{*,1}, \ldots), \end{cases}$$

$$(1.10)\quad \begin{cases} \Phi_n^-: \mathscr{K}_n^{\mathrm{inp}} \to \mathscr{M}_-, \\ \Phi_n^-(\ldots \oplus W_{n-1}^* d_{n-1}^{(1)} \oplus d_n^{(1)} \oplus W_n d_{n+1}^{(1)} \oplus \ldots) = (\ldots, d_{-1}, \boxed{d_0}, d_1, \ldots), \end{cases}$$

where

$$d_{*,n}^{(-1)} = (\ldots, 0, d_{*,n}, \boxed{0}, 0, \ldots) \in \mathscr{K}_{n+1}, \quad d_{*,n} \in \mathscr{D}_{T_n^*}$$

and

$$d_n^{(1)} = (\ldots, 0, \boxed{0}, d_n, 0, \ldots) \in \mathscr{K}_n, \quad d_n \in \mathscr{D}_{T_n}.$$

The first remark is that for every $n \in \mathbf{Z}$ we get

$$\Phi_n^+ Q_n (\Phi_n^-)^{-1} = \Theta$$

where Θ is the transfer operator of the system (1.1) — see [11] for definition. Θ is a lower triangular operator such that its matricial elements are $\Theta_{ij} = D_{T_j} T_{j+1} \ldots T_{i-1} D_{T_i^*}$ for $i \in \mathbf{Z}$, $j < i$ and $\Theta_{ii} = -T_i^*$, $i \in \mathbf{Z}$. Θ is a contraction and we obtain the following identification of $\mathscr{K}_n$ in the model given by the marking operators: first we define the unitary operator

$$(1.11)\qquad \Phi_{\mathscr{R}_n^+}: \mathscr{R}_n^+ \to \overline{D_\Theta \mathscr{M}_-}, \quad \Phi_{\mathscr{R}_n^+}(I - Q_n) k = D_\Theta \Phi_n^- k, \quad k \in \mathscr{K}_n^{\mathrm{inp}}$$

then

$$(1.12)\qquad \Psi_n: \mathscr{K}_n \to \mathscr{M}_+ \oplus \overline{D_\Theta \mathscr{M}_-}, \quad \Psi_n = \Phi_n^+ \oplus \Phi_{\mathscr{R}_n^+}$$

and Ψ_n is a unitary operator yielding a natural identification of $\mathscr{K}_n$ in the marking model. Moreover, we have the following result which constitutes the time-variant analogue of the Sz.-Nagy—Foiaş functional model of a contraction.

Theorem 2.2. *For a system* (1.1) *satisfying the condition* $\mathscr{K}_n = \mathscr{K}_n^{\mathrm{inp}} \vee \mathscr{K}_n^{\mathrm{out}}$ *for every* $n \in \mathbf{Z}$, *the following relations hold through the identifications* Ψ_n:

$$\mathscr{H}_n = \Big(\bigoplus_{k \geq n} \mathscr{D}_{T_k^*} \oplus \overline{D_\Theta \mathscr{M}_-}\Big) \ominus \Big\{\Theta v \oplus D_\Theta v \,|\, v \in \bigoplus_{k \geq n} \mathscr{D}_{T_k}\Big\},$$

$$T_n(u_+ \oplus v_-) = P_{\mathscr{H}_n}(M_n^+ u_+ \oplus M_n^- v_-), \quad u_+ \oplus v_- \in \mathscr{H}_n.$$

Proof. From (1.4), (1.12) and (1.11) it follows that

$$\Psi_n \mathscr{K}_n^+ = \Phi_n^+ \Big(\mathscr{L}_n^+ \oplus \bigoplus_{p=1}^{\infty} W_n^+ \dots W_{n+p-1}^+ \mathscr{L}_{n+p}^+\Big) \oplus \overline{D_\Theta \mathscr{M}_-}$$

and by (1.9) we have

$$\Phi_n^+ \Big(\mathscr{L}_n^+ \oplus \bigoplus_{p=1}^{\infty} W_n^+ \dots W_{n+p-1}^+ \mathscr{L}_{n+p}^+\Big) = \bigoplus_{k \geq n} \mathscr{D}_{T_n^*}.$$

In a similar way,

$$\mathscr{D}_{T_n}^{(1)} \oplus W_n \mathscr{D}_{T_{n+1}}^{(1)} \oplus \dots = \bigoplus_{k \geq n} \mathscr{D}_{T_k}$$

and the first relation follows from Theorem 2.1. For the second relation we have to use the more remark that

$$\Phi_n^+ W_n (\Phi_{n+1}^+)^* \;|\; \bigoplus_{k \geq n+1} \mathscr{D}_{T_k^*} = M_n^+$$

$$\Phi_n^- W_n (\Phi_{n+1}^-)^* \;|\; \bigoplus_{k \geq n+1} \mathscr{D}_{T_k} = M_n^-.$$

Remark 2.3. The inverse problem of realization a given lower triangular contraction as a transfer operator of a certain system is treated for instance in [11] and [2].

III. Nonstationary lifting

In this section we describe a nonstationary variant for the lifting theorem of Sarason—Sz.-Nagy—Foiaş.

This variant is inspired by similar phenomena in the study of nonstationary processes (see [12], [13], [7]) and the difference from the "stationary" variant of Sarason—Sz.-Nagy—Foiaş is not structural, but only one of complexity. Consequently, we will have only to indicate the necessary changes, the proofs following the known ones. Fix two integers $-\infty \leq M < \infty$, $-\infty < N \leq \infty$, $M \leq N$ and two families $\{T_n\}_{M \leq n \leq N}$, $\{T_n'\}_{M \leq n \leq N}$ of contractions (the extremal indices are attained only for finite M and N), $T_n \in \mathscr{L}(\mathscr{H}_{n+1}, \mathscr{H}_n)$, $T_n' \in \mathscr{L}(\mathscr{H}_{n+1}', \mathscr{H}_n')$. Let $\{A_n\}_{M \leq n \leq N+1}$ be a family of contractions, $A_n \in \mathscr{L}(\mathscr{H}_n, \mathscr{H}_n')$ and suppose that it intertwines $\{T_n\}$ and $\{T_n'\}$, i.e.

$$T_n' A_{n+1} = A_n T_n$$

for $M \leqq n \leqq N$. For $\{T_n\}_{M \leqq n \leqq N}$ consider its associated kernel by the rule mentioned at the beginning of Section 2 and let $\{W_n\}_{M \leqq n \leqq N}$, $W_n \in \mathscr{L}(\mathscr{K}_{n+1}, \mathscr{K}_n)$ be its Kolmogorov decomposition, always written in the form (1.2). We have similar objects associated to $\{T'_n\}_{M \leqq n \leqq N}$. Now, the following result extends the lifting theorem of Sarason—Sz.-Nagy—Foiaş. Denote by P_n the orthogonal projection of $\mathscr{K}_n^+$ onto $\mathscr{H}_n$ and similarly, P'_n.

Theorem 3.1. *The set*

$$\mathrm{CID}\left(\{A_n\}_{M \leqq n \leqq N+1}\right) =$$
$$= \left\{\{B_n\}_{M \leqq n \leqq N+1} \mid B_n \text{ are contractions in } \mathscr{L}(\mathscr{K}_n^+, \mathscr{K}_n'^+),\ W_n'^+ B_{n+1} = B_n W_n^+,\ P'_n B_n = A_n P_n\right\}$$

is nonvoid.

Proof. Let $X_{ij}^{(n)}$ be the matrix of B_n, then writing the intertwining conditions, one gets:

$$X_{11}^{(n)} = A_n, \quad X_{1j}^{(n)} = 0, \quad j > 1$$
$$X_{ij}^{(n)} = 0, \quad j > i$$
$$X_{21}^{(n)} T_n + X_{22}^{(n)} D_{T_n} = D_{T'_n} A_{n+1}$$
$$X_{k1}^{(n)} T_n + X_{k2}^{(n)} D_{T_n} = X_{k-1,1}^{(n+1)}, \quad k \geqq 3$$

and

$$X_{ij}^{(n)} = X_{i-1,j-1}^{(n+1)}, \quad i, j \geqq 3.$$

Define the operators

$$S_{k-1,n}\colon \mathscr{D}_{T_n^*} \to \mathscr{D}_{T'_n} \quad S_{k-1,n} = X_{k1}^{(n)} D_{T_n^*} - X_{k2}^{(n)} T_n^* \tag{3.1}$$

such that the finite sections of B_n are contractions if and only if the operators

$$C_{kn} = \begin{bmatrix} A_n T_n \dots T_{n+k-1}, & \dots, & A_n T_n D_{T_{n+1}^*}, & A_n D_{T_n^*} \\ \dots, & & & S_{1n} \\ \dots, & & S_{1,n+1}, & S_{2,n+1} \\ & \ddots & & \\ D_{T'_{n+k-1}} A_{n+k}, & S_{1,n+k-1}, & \dots, & S_{k-1,n+k-1} \end{bmatrix} \tag{3.2}$$

are also contractions for $k \geqq 1$.

If we define $C_{0n} = A_n T_n = T'_n A_{n+1}$, then there exist contractions $Y_n\colon \mathscr{D}_{C_{0n}} \to \mathscr{D}_{T'_n}$ and $\tilde{Y}_n\colon \mathscr{D}_{T_n^*} \to \mathscr{D}_{C_{0n}^*}$ such that $A_n D_{T_n^*} = D_{C_{0n}^*} \tilde{Y}_n$ and $D_{T'_n} A_{n+1} = Y_n D_{C_{0n}}$ and using [4], [9] there exists an operator S_{1n} such that C_{1n} is a contraction. Now, the same approximation procedure as in [3] finishes the proof.

We can continue the analysis of the set $\mathrm{CID}\left(\{A_n\}_{M \leqq n \leqq N+1}\right)$ in order to derive results similar to those in [1], [3]. That is, a parametrization with a family of free parameters (generalizing the choice sequences in [3]) and a parametrization with lower triangular contractions — a Schur type formula — are obtained.

For a sequence of contractions $\{G_1, G_2, \dots\}$, $G_1 \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$, $G_k \in \mathcal{L}(\mathcal{D}_{G_{k-1}}, \mathcal{H}')$, $L(G_1, G_2, \dots)$ is the row contraction determined by these parameters,

$$L(G_1, G_2, \dots) = (G_1, D_{G_1^*} G_2, \dots, D_{G_1^*} \dots D_{G_{k-1}^*} G_k, \dots)$$

and similar considerations hold for column contractions, denoted by $C(G_1, G_2, \dots)$.

Theorem 3.2. *There exists a one-to-one correspondence between*

$$\mathrm{CID}\left(\{A_n\}_{M \leq n \leq N+1}\right)$$

and the families of contractions $\{G_{ij}\}$ such that $G_{1n} \in \mathcal{L}(\mathcal{D}_{G_{Y_n}}, \mathcal{D}_{G^*_{\tilde{Y}_n}})$, $M \leq n \leq N$ and $G_{ij} \in \mathcal{L}(\mathcal{D}_{G_{i-1,j}}, \mathcal{D}_{G^*_{i-1,j+1}})$ for $i \geq 2$, $M \leq j \leq N$. The correspondence is explicitely taken by the formula:

$$S_{kn} = L(Y_n, G_{1n}, G_{2n}, \dots, G_{k-1,n}) Q_{k-1,n} C(\tilde{Y}_{n-k+1}, G_{1,n-k+1}, G_{2,n-k+2}, \dots, G_{k-1,n-1}) +$$
$$+ D_{Y_n^*} D_{G_{1n}^*} \dots D_{G_{k-1,n}^*} G_{kn} D_{G_{k-1,n-1}} \dots D_{G_{1,n-k+1}} D_{\tilde{Y}_{n-k+1}},$$

where the operators Q_{kn} can be also described in terms of the parameters G_{ij}.

Proof. We only skech the beginning, the rest paralleling the proof in [8] of a slight modified variant of the main algorithm in [3].

First of all, $Q_{0n} = -C_{0n}^*$ for $M \leq n \leq N$. Denote by F_{kn} the $k \times k$ principal submatrix of $C_{k+1,n}$ and by direct computations, we have

$$C_{2n} = \begin{bmatrix} F_{1n} & D_{F_{1n}^*} \tilde{\Omega}_{1n}^* C(Y_n, G_{1n}) \\ L(\tilde{Y}_{n+1}, G_{1,n+1}) \Omega_{1n} D_{F_{1n}} & S_{2,n+1} \end{bmatrix}$$

where Ω_{1n} and $\tilde{\Omega}_{1n}$ are obvious identifications of the defects of F_{1n}. Using once again [4], [9] we get the desired formula for S_{2n} with

$$Q_{1n} = -\Omega_{1n} F_{1n}^* \tilde{\Omega}_{1n}^*.$$

Then we compute

$$Q_{1n} \begin{bmatrix} D_{C_{0n}^*} & -C_{0n} \tilde{Y}_n^* \\ 0 & D_{\tilde{Y}_n^*} \end{bmatrix} = -\Omega_{1n} F_{1n}^* \tilde{\Omega}_{1n}^* \begin{bmatrix} D_{C_{0n}^*} & -C_{0n} \tilde{Y}_n^* \\ 0 & D_{\tilde{Y}_n^*} \end{bmatrix} =$$
$$= -\Omega_{1n} F_{1n}^* D_{F_{1n}^*} = -\Omega_{1n} D_{F_{1n}} F_{1n}^* = \begin{bmatrix} D_{C_{0,n+1}} & -C_{0,n+1}^* Y_{n+1} \\ 0 & D_{Y_{n+1}} \end{bmatrix} F_{1n}^*.$$

As in [8] we find

$$Q_{1n} = -\begin{bmatrix} C_{0n}^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_{Y_{n+1}^*} a_n & D_{Y_{n+1}^*} b_n \\ Y_{n+1}^* a_n & Y_{n+1}^* b_n \end{bmatrix}$$

with $a_n a_n^* + b_n b_n^* = I$ and we define the operator

$$V_{0n}: \mathscr{D}_{C_{0n}^*} \oplus \mathscr{D}_{Y_n^*} \to \mathscr{D}_{C_{0n}} \oplus \mathscr{D}_{Y_{n+1}}$$

$$V_{0n} = \begin{bmatrix} D_{Y_{n+1}^*} a_n & D_{Y_{n+1}^*} b_n \\ -Y_{n+1}^* a_n & -Y_{n+1}^* b_n \end{bmatrix}.$$

From now on we can continue as in [8].

Remark 3.3. Using Theorem 5.2 in [2] and Theorem 3.2 above, a parametrization of $\mathrm{CID}\left(\{A_n\}_{M \leqq n \leqq N+1}\right)$ with lower triangular contractions can be derived, together with corresponding Schur type formulae as in Corollary 6.1 in [3].

IV. Applications

In this section we show the way some completion problems can be solved using Theorem 3.1.

(A) For fixed operators $\left(C_{j+r,j} \in \mathscr{L}(\mathscr{H}_j, \mathscr{H}'_{j+r}) \mid j \geqq 0,\ 0 \leqq r \leqq N\right)$ we find conditions for the existence of lower triangular contractive extensions. This problem can be viewed as a "nonstationary" Carathéodory—Fejér problem and can be solved as in [15].

We define for $n \geqq 0$

$$T_n: \bigoplus_{k=1}^{N} \mathscr{H}'_{n+k} \to \bigoplus_{k=0}^{N-1} \mathscr{H}'_{n+k}$$

$$T_n = \begin{bmatrix} 0 & 0 \dots & 0 \\ I & 0 \dots & 0 \\ & \dots & \\ 0 & 0 \quad I & 0 \end{bmatrix}$$

and

$$A_n: \bigoplus_{k=0}^{N-1} \mathscr{H}_{n+k} \to \bigoplus_{k=0}^{N-1} \mathscr{H}'_{n+k}$$

$$A_n = \begin{bmatrix} C_{nn} & 0 & 0 \dots & 0 \\ C_{n+1,n} & C_{n+1,n+1} & 0 \dots & 0 \\ & \dots & & \\ C_{n+N-1,n} & \dots & & C_{n+N-1,n+N-1} \end{bmatrix}.$$

We have that $T_n A_{n+1} = A_n T_n$ and if we suppose that A_n are contractions, we can use Theorem 3.1 in order to show that there exists a family of contractions

$\{B_n\}_{n\geqq 0}$ such that

$$M_n B_{n+1} = B_n M_n, \quad P_n B_n = A_n P_n$$

where M_n are marking operators as those given by (1.7) or (1.8).

Using an adaptation of Lemma V. 3.2 in [16], $\{B_n\}_{n\geqq 0}$ gives rise to a contractive lower triangular extension of the given family of operators.

Proposition 4.1. *In order that the family $(C_{j+r,j} \mid j\geqq 0,\ 0\leqq r\leqq N)$ has a contractive lower triangular extension it is necessary and sufficient that A_n are contractions for $n\geqq 0$.*

Moreover, Theorem 3.1 and Remark 3.3 give parametrizations for all the solutions.

(B) Theorem 3.1 can be used to solve completion problems with a finite number of data, those named as Nehari completions in [5]. We indicate here (for simplicity) only the very particular case of completing

$$\begin{bmatrix} C_{00} & C_{01} \\ C_{10} & \end{bmatrix}$$

to a contraction. Take

$$A_0 = (C_{00}, C_{01}), \quad A_1 = \begin{bmatrix} C_{00} \\ C_{10} \end{bmatrix}, \quad T_0' = (I, 0), \quad T_0 = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

then $T_0' A_1 = A_0 T_0$. Moreover,

$$W_0'^{+} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \quad W_0^{+} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

and if A_0 and A_1 are supposed to be contractions, then Theorem 3.1 asserts the existence of a contraction $\begin{bmatrix} C_{00} & C_{01} \\ C_{21} & C_{22} \end{bmatrix}$ such that

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} C_{00} \\ C_{10} \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Consequently, $C_{21} = C_{10}$ and a contractive completion of the given (C_{00}, C_{01}, C_{10}) is

$$\begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{22} \end{bmatrix}.$$

This shows that Theorem 3.1 (together with the parametrization in Theorem 3.2) for $M=N=0$ is equivalent with [4] and [9].

(C) Another application here is an extension of Theorem 5 in [5] and of a similar result in [14].

Proposition 4.2. *Let A and B be two lower triangular operators, $A \in \mathscr{L}(\bigoplus_{n\in\mathbf{Z}} \mathscr{H}_n, \bigoplus_{n\in\mathbf{Z}} \mathscr{H}_n'')$, $B \in \mathscr{L}(\bigoplus_{n\in\mathbf{Z}} \mathscr{H}_n, \bigoplus_{n\in\mathbf{Z}} \mathscr{H}_n')$. Then a necessary and sufficient condition for the existence of a lower triangular contraction $C \in \mathscr{L}(\bigoplus_{n\in\mathbf{Z}} \mathscr{H}_n', \bigoplus_{n\in\mathbf{Z}} \mathscr{H}_n'')$ such that $A=CB$ is that $A^*A \leqq B^*B$.*

Proof. Take $A_n = A \mid \bigoplus_{k\geqq n} \mathscr{H}_k$, $B_n = B \mid \bigoplus_{k\geqq n} \mathscr{H}_k$ and M_n, M_n' and M_n'' be marking operators as (1.7) and (1.8) such that

$$M_n' A_{n+1} = A_n M_n, \quad M_n'' B_{n+1} = B_n M_n.$$

Since A and B are lower triangular, then $A_n^* A_n \leqq B_n^* B_n$ for $n \in \mathbf{Z}$ and there exist uniquely determined contractions X_n: $\overline{\operatorname{Ran} B_n} \to \overline{\operatorname{Ran} A_n}$ such that $A_n = X_n B_n$. From now on we can follow [14].

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