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Best coapproximation and Schauder bases in Banach spaces

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1. Introduction

V. N. NIKOL'SKIĬ [8], [9] studied the problem of best approximation in Banach spaces with basis. The study was carried out further by J. R. RETHERFORD [13], [14]. He characterized (strictly) monotone bases and (strictly) comonotone bases by means of best approximation. He also characterized (strict) orthogonality and (strict) co-orthogonality in Banach spaces having unconditional bases by means of best approximation. Some further connections between best approximation and theory of bases can be found in the book of I. SINGER [17].

Another kind of approximation known as "Best coapproximation" was introduced by C. FRANCHETTI and M. FURI [2]. The work was continued on this topic by P. L. PAPINI and I. SINGER [10], [11], GEETHA S. RAO [3] and others. In this paper, some characterizations of bases in Banach spaces are obtained by means of best coapproximation. Certain kind of norms are introduced using best coapproximation in which the given bases are (strictly) monotone and (strictly) comonotone, respectively. Equivalent norms are provided in which the given bases possess the special properties. The analogous theory is detailed in Banach spaces having unconditional bases.

2. Notation and terminology

Let E be a Banach space. A sequence $\{x_n\}$ in E is a basis of E if for every $x\in E$ there exists a unique sequence of scalars $\{\alpha_n\}\subset \mathbf{K}$ such that

$$x=\sum_{i=1}^{\infty}\alpha_i x_i. \tag{1}$$

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A system (x_n, f_n), $\{x_n\}\subset E$, $\{f_n\}\subset E^*$ is biorthogonal if $f_i(x_j)=\delta_{ij}$. If (x_n, f_n) is a biorthogonal system with $\{x_n\}$ a basis in E, then (x_n, f_n) is a Schauder basis for E if for each $x\in E$,

(2) $$x = \sum_{i=1}^{\infty} f_i(x)\, x_i.$$

$\{f_n\}\subset E^*$ may some times be called an associated sequence of coefficient functionals (a.s.c.f.—here after). A sequence $\{x_n\}$ in E is a basic sequence if $\{x_n\}$ is a basis of the closed linear subspace $[x_n]$ of E where $[.]$ denotes the linear span of x_n's. A basis $\{x_n\}$ of E is unconditional if the convergence of (1) or (2) is unconditional, for each $x\in E$.

Let E be a Banach space, G be a linear subspace of E and $x\in E$. An element $g_0\in G$ is a best approximation of x from G if

(3) $$\|x-g_0\| \leqq \|x-g\| \quad (g\in G).$$

The set of best approximations of x from G is denoted by $P_G(x)$. An element $g_0\in G$ is a best coapproximation of x from G if

(4) $$\|g_0-g\| \leqq \|x-g\| \quad (g\in G).$$

The set of best coapproximations of x from G is denoted by $R_G(x)$. For a sequence $\{x_n\}$ of E, let

$$G_n = [x_1, x_2, \ldots, x_n] \quad \text{and} \quad G^n = [x_{n+1}, x_{n+2}, \ldots].$$

Let $\mathcal{D}=\{\{i_1, i_2, \ldots, i_n\}\subset\mathcal{N} \mid 1\leqq n<\infty\}$, where $\mathcal{N}$ denotes the set of all natural numbers. For a sequence $\{x_n\}$ of E, let

$$G_d = [x_i\colon i\in d] \quad \text{and} \quad G^d = [x_i\colon i\in \mathcal{N}\setminus d], \quad \text{for} \quad d\in\mathcal{D}.$$

A sequence $\{x_n\}$ is basic in E (an unconditional basic in E) if there is a K such that for every $m\leqq n$ $(d\subset d',\ d'\in\mathcal{D})$ and arbitrary scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ $(\{\alpha_i\}_{i\in d'})$

$$\Big\|\sum_{i=1}^{m} \alpha_i x_i\Big\| \leqq K\Big\|\sum_{i=1}^{n} \alpha_i x_i\Big\| \quad \Big(\Big\|\sum_{i\in d} \alpha_i x_i\Big\| \leqq K\Big\|\sum_{i\in d'} \alpha_i x_i\Big\|\Big).$$

Let (x_n, f_n) be a Schauder basis for E. Then $s_n(x)$ and $r_n(x)$ (respectively $s_d(x)$, $r_d(x)$) are defined as

$$s_n(x) = \sum_{i=1}^{n} f_i(x) x_i \quad \text{and} \quad r_n(x) = x - s_n(x)$$

$$\Big(s_d(x) = \sum_{i\in d} f_i(x) x_i \quad \text{and} \quad r_d(x) = x - s_d(x)\Big).$$

3. Characterization of monotone bases

Definition 3.1. Let E be a Banach space with a basis $\{x_n\}$. Then E is said to satisfy *Property* (A_1) if there exists no collection of scalars $\alpha_{n+1}, \alpha_{n+2}, \ldots, \alpha_{n+m}$, for all $n, m \in \mathcal{N}$, such that $\sum_{i=n+1}^{n+m} |\alpha_i| \neq 0$ and satisfying

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| = \left\|\sum_{i=1}^{n+m} \alpha_i x_i\right\| = \left\|\sum_{i=1}^{n} \alpha_i x_i + \frac{1}{2} \sum_{i=m+1}^{n+m} \alpha_i x_i\right\|.$$

Definition 3.2. Let E be a Banach space with a basis $\{x_n\}$. Then E is said to satisfy *Property* (A_2) if there exists no collection of scalars $\alpha_{m+1}, \alpha_{m+2}, \ldots, a_n$, for all $n, m \in \mathcal{N}$, such that $\sum_{i=m+1}^{n} |\alpha_i| \neq 0$ and satisfying

$$\left\|\sum_{i=n+1}^{\infty} \alpha_i x_i\right\| = \left\|\sum_{i=m+1}^{\infty} \alpha_i x_i\right\| = \left\|\sum_{i=n+1}^{\infty} \alpha_i x_i + \frac{1}{2} \sum_{i=m+1}^{n} \alpha_i x_i\right\|.$$

Remark 3.1. All Banach spaces having basis $\{x_n\}$ and are strictly convex satisfy Property (A_1) and Property (A_2). But there is no connection between Property (A_1) and Property (A_2) in general.

Following [15], the next definition is introduced.

Definition 3.3. If $\{x_n\}$ is a basis for a Banach space E, then

(i) $\{x_n\}$ is *monotone* if $\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| \leq \left\|\sum_{i=1}^{n+m} \alpha_i x_i\right\|$ for all n and for all collections of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+m} \in \mathbf{K}$.

(ii) $\{x_n\}$ is *strictly monotone* if strict inequality holds in (i) whenever $\sum_{i=n+1}^{n+m} |\alpha_i| \neq 0$.

(iii) $\{x_n\}$ is *comonotone* if $\left\|\sum_{i=n}^{\infty} \alpha_i x_i\right\| \leq \left\|\sum_{i=m}^{\infty} \alpha_i x_i\right\|$ whenever $\sum_{i=n}^{\infty} \alpha_i x_i$ converges and for all collections of scalars $\alpha_m, \alpha_{m+1}, \ldots, \alpha_n, \alpha_{n+1} \ldots \in \mathbf{K}$.

(iv) $\{x_n\}$ is *strictly comonotone* if strict inequality holds in (iii) whenever $\sum_{i=m}^{n-1} |\alpha_i| \neq 0$.

Theorem 3.1. *Let E be a Banach space with a basis $\{x_n\}$. The following statements are true about $\{x_n\}$:*

(i) *$\{x_n\}$ is monotone if and only if $R_{G_n}(x) = \{s_n(x)\}$ $n = 1, 2, \ldots$.*

(ii) *$\{x_n\}$ is strictly monotone if and only if $R_{G_n}(x) = \{s_n(x)\}$ $n = 1, 2, \ldots$ and E satisfies Property (A_1).*

(iii) *$\{x_n\}$ is comonotone if and only if $R_{G^n}(x) = \{r_n(x)\}$ $n = 1, 2, \ldots$.*

(iv) *$\{x_n\}$ is strictly comonotone if and only if $R_{G^n}(x) = \{r_n(x)\}$ $n = 1, 2, \ldots$ and E satisfies Property (A_2).*

Proof. (i) $\{x_n\}$ is monotone, then $\|\sum_{i=1}^{n}\alpha_i x_i\| \leqq \|\sum_{i=1}^{n+1}\alpha_i x_i\|$ for all scalars $\alpha_{n+1}\in\mathbf{K}$. Then, if $x=\sum_{i=1}^{\infty}\alpha_i x_i\in E$, it follows that

$$\|\sum_{i=1}^{n}\alpha_i x_i-\sum_{i=1}^{n}\beta_i x_i\| = \|\sum_{i=1}^{n}(\alpha_i-\beta_i)x_i\| \leqq \|\sum_{i=1}^{n}(\alpha_i-\beta_i)x_i+\alpha_{n+1}x_{n+1}\| \leqq \dots$$

$$\dots \leqq \|\sum_{i=1}^{n}(\alpha_i-\beta_i)x_i+\sum_{i=n+1}^{\infty}\alpha_i x_i\| = \|x-\sum_{i=1}^{n}\beta_i x_i\|$$

for all $\sum_{i=1}^{n}\beta_i x_i=p(\neq s_n(x))\in G_n$. Thus $s_n(x)\in R_{G_n}(x)$. On the other hand, $E=G_n\oplus\oplus[x_{n+1},x_{n+2},\dots]$, $R_{G_n}^{-1}(0)\supset[x_{n+1},\dots]$ and $R_{G_n}^{-1}(0)\cap G_n=\{0\}$, where $R_{G_n}^{-1}(0)==\{y\in E\,|\,R_{G_n}(y)\ni 0\}$. Therefore $E=G_n\oplus R_G^{-1}(0)$ and $R_{G_n}(x)$ is unique for every $x\in E\setminus G_n$ (from [3]), i.e. $R_{G_n}(x)=\{s_n(x)\}$ as $s_n(x)\in R_{G_n}(x)$ $(x\in E)$.

If $R_{G_n}(x)=\{s_n(x)\}$, then for $x=\sum_{i=1}^{n+m}\alpha_i x_i$, it follows that

$$\|\sum_{i=1}^{n}\alpha_i x_i-p\| \leqq \|\sum_{i=1}^{n+m}\alpha_i x_i-p\| \quad (p\in G_n).$$

Since $0\in G_n$, it follows that

$$\|\sum_{i=1}^{n}\alpha_i x_i\| \leqq \|\sum_{i=1}^{n+m}\alpha_i x_i\|$$

for all collections of scalars $\alpha_1,\alpha_2,\dots,\alpha_n,\alpha_{n+1},\dots,\alpha_{n+m}\in\mathbf{K}$. Thus $\{x_n\}$ is monotone.

(ii) If $R_{G_n}(x)=\{s_n(x)\}$, then

$$\|\sum_{i=1}^{n}\alpha_i x_i\| \leqq \|\sum_{i=1}^{n+m}\alpha_i x_i\|$$

for all collections of scalars $\alpha_1,\alpha_2,\dots,\alpha_n,\alpha_{n+1},\dots,\alpha_{n+m}\in\mathbf{K}$ was proved.

If equality holds for some collection of scalars (say) $\alpha_1,\alpha_2,\dots,\alpha_n$, $\alpha_{n+1},\dots,\alpha_{n+m}\in\mathbf{K}$, with $\sum_{i=n+1}^{n+m}|\alpha_i|\neq 0$, then

$$\|\sum_{i=1}^{n}\alpha_i x_i\| = \|\sum_{i=1}^{n+m}\alpha_i x_i\|.$$

Consider $\left\|\sum_{i=1}^{n}\alpha_i x_i+\frac{1}{2}\sum_{i=1}^{n+m}\alpha_i x_i\right\|$. Since

$$R_{G_n}\left(\sum_{i=1}^{n}\alpha_i x_i+\frac{1}{2}\sum_{i=n+1}^{n+m}\alpha_i x_i\right) = \sum_{i=1}^{n}\alpha_i x_i, \tag{5}$$

it follows that

$$\left\|\sum_{i=1}^{n}\alpha_i x_i\right\| \leqq \left\|\sum_{i=1}^{n}\alpha_i x_i+\frac{1}{2}\sum_{i=n+1}^{n+m}\alpha_i x_i\right\|.$$

On the other hand,

$$(6)\qquad \left\|\sum_{i=1}^{n}\alpha_i x_i+\frac{1}{2}\sum_{i=n+1}^{n+m}\alpha_i x_i\right\| \leqq \left\|\frac{1}{2}\sum_{i=1}^{n}\alpha_i x_i\right\|+\left\|\frac{1}{2}\sum_{i=1}^{n+m}\alpha_i x_i\right\| = \left\|\sum_{i=1}^{n}\alpha_i x_i\right\|.$$

From (5) and (6), it follows that

$$\left\|\sum_{i=1}^{n}\alpha_i x_i\right\| = \left\|\sum_{i=1}^{n}\alpha_i x_i+\frac{1}{2}\sum_{i=n+1}^{n+m}\alpha_i x_i\right\| = \left\|\sum_{i=1}^{n+m}\alpha_i x_i\right\|$$

for some collection of scalars $\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{n+m} \in \mathbf{K}$ with $\sum_{i=n+1}^{n+m}|\alpha_i| \neq 0$, in contradiction to the Property (Λ_1) satisfied by E. Thus $\{x_n\}$ is strictly monotone.

Proceeding to the other implication, if

$$\left\|\sum_{i=1}^{n}\alpha_i x_i\right\| < \left\|\sum_{i=1}^{n+m}\alpha_i x_i\right\|$$

for all collections of scalars $\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_{n+1}, \dots, \alpha_{n+m} \in \mathbf{K}$ with $\sum_{i=n+1}^{n+m}|\alpha_i| \neq 0$, then it is clear that it implies $R_{G_n}(x)=\{s_n(x)\}$ and Property (Λ_1) for E.

(iii) Consider $\{x_n\}$ is comonotone. Then

$$\left\|\sum_{i=n}^{\infty}\alpha_i x_i\right\| \leqq \left\|\sum_{i=m}^{\infty}\alpha_i x_i\right\|$$

for all collections of scalars $\alpha_m, \alpha_{m+1}, \dots, \alpha_n, \dots \in \mathbf{K}$ for which $\sum_{i=n}^{\infty}\alpha_i x_i$ is convergent. Then for $x=\sum_{i=1}^{\infty}\alpha_i x_i \in E$, it follows that,

$$\left\|r_n(x)-\sum_{i=n+1}^{\infty}\beta_i x_i\right\| = \left\|\sum_{i=n+1}^{\infty}\alpha_i x_i-\sum_{i=n+1}^{\infty}\beta_i x_i\right\| \leqq \left\|\sum_{i=n+1}^{\infty}(\alpha_i-\beta_i)x_i+\alpha_n x_n\right\| \leqq \dots$$

$$\dots \leqq \left\|\sum_{i=n+1}^{\infty}(\alpha_i-\beta_i)x_i+\sum_{i=1}^{n}\alpha_i x_i\right\| = \left\|x-\sum_{i=n+1}^{\infty}\beta_i x_i\right\|$$

for all $\sum_{i=n+1}^{\infty}\beta_i x_i = p(\neq r_n(x)) \in G^n$. Thus $r_n(x) \in R_{G^n}(x)$. But $E=G^n \oplus [x_1, x_2, \dots, x_n]$, $R_{G^n}^{-1}(0) \supset [x_1, x_2, \dots, x_n]$ and $R_{G^n}^{-1}(0) \cap G^n = \{0\}$. Therefore $E=G^n \oplus R_{G^n}^{-1}(0)$ and $R_{G^n}(x)$ is unique for all $x \in E \setminus G^n$. Thus $R_{G^n}(x)=\{r_n(x)\}$.

On the other hand if $R_{G^n}(x)=\{r_n(x)\}$, then for $x=\sum_{i=m+1}^{\infty}\alpha_i x_i$, it follows that

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i-p\Big\|\leqq\Big\|\sum_{i=m+1}^{\infty}\alpha_i x_i-p\Big\|$$

for all $p\in G^n$. Since $0\in G^n$, it follows that

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i=m+1}^{\infty}\alpha_i x_i\Big\|$$

for all collections of scalars $\alpha_{m+1},\dots,\alpha_{n+1},\dots\in\mathbf{K}$ for which $\sum_{i=n+1}^{\infty}\alpha_i x_i$ is convergent.

(iv) If $R_{G^n}(x)=\{r_n(x)\}$, then

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i=m+1}^{\infty}\alpha_i x_i\Big\|$$

for all scalars $\alpha_{m+1},\dots,\alpha_{n+1},\dots\in\mathbf{K}$ for which $\sum_{i=n+1}^{\infty}\alpha_i x_i$ is convergent. If equality holds for some $\alpha_{m+1},\dots,\alpha_{n+1},\dots\in\mathbf{K}$ (say) with $\sum_{i=m+1}^{n}|\alpha_i|\neq 0$, then

$$\Big\|\sum_{i=n+1}^{\infty}a_i x_i\Big\|=\Big\|\sum_{i=m+1}^{\infty}a_i x_i\Big\|.$$

Consider $\sum_{i=n+1}^{\infty}\alpha_i x_i+\frac{1}{2}\sum_{i=m+1}^{n}\alpha_i x_i$. Since

$$R_{G^n}\Big(\sum_{i=n+1}^{\infty}\alpha_i x_i+\frac{1}{2}\sum_{i=m+1}^{n}\alpha_i x_i\Big)=\sum_{i=n+1}^{\infty}\alpha_i x_i,$$

it is clear that

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i+\frac{1}{2}\sum_{i=m+1}^{n}\alpha_i x_i\Big\|. \tag{7}$$

On the other hand,

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i+\frac{1}{2}\sum_{i=m+1}^{n}\alpha_i x_i\Big\|\leqq\frac{1}{2}\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|+\frac{1}{2}\Big\|\sum_{i=m+1}^{n}\alpha_i x_i\Big\|=\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|. \tag{8}$$

From (7) and (8), it follows that

$$\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i\Big\|=\Big\|\sum_{i=n+1}^{\infty}\alpha_i x_i+\frac{1}{2}\sum_{i=m+1}^{n}\alpha_i x_i\Big\|=\Big\|\sum_{i=m+1}^{\infty}\alpha_i x_i\Big\|$$

for some collection of scalars $\alpha_{m+1},\dots,\alpha_{n+1},\dots\in\mathbf{K}$ with $\sum_{i=m+1}^{n}|\alpha_i|\neq 0$, contradicting Property (Λ_2) satisfied by E. Hence $\{x_n\}$ is strictly comonotone.

Proceeding to the other implication, if $\|\sum_{i=n+1}^{\infty}\alpha_i x_i\|<\|\sum_{i=m+1}^{\infty}\alpha_i x_i\|$ for all collections of scalars $\alpha_{m+1}, \ldots, \alpha_{n+1}, \ldots \in \mathbf{K}$ with $\sum_{i=m+1}^{n}|\alpha_i|\neq 0$ and $\sum_{i=n+1}^{\infty}\alpha_i x_i$ is convergent, then it is clear that $R_{G^n}(x)=\{r_n(x)\}$ for $x\in E$ and E satisfies Property (A_2).

Definition 3.4. The norm in a Banach space E with a basis $\{x_n\}$ is called a *CT-norm* (with respect to the basis $\{x_n\}$) if

(a) for every $x\in E$ and $n=1, 2, \ldots$, there exists a unique polynomial $R_{G_n}(x)= =\{s_n(x)\}$ of best coapproximation to x.

(b) E satisfies Property (A_1).

Observe that CT-norms will be denoted by $\|\cdot\|_{CT}$.

Definition 3.5. The norm in a Banach space E with a basis $\{x_n\}$ is called a *CK-norm* (with respect to the basis $\{x_n\}$) if

(a) for every $x\in E$, and $n=1, 2, \ldots$ there exists a unique polynomial complement $R_{G^n}(x)=\{r_n(x)\}$ of best coapproximation to x.

(b) E satisfies Property (A_2).

Note that CK-norms are denoted by $\|\cdot\|_{CK}$.

Definition 3.6. The norm in a Banach space E with a basis $\{x_n\}$ is called a *CTK-norm* (with respect to the basis $\{x_n\}$) if it is simultaneously a CT-norm and a CK-norm with respect to this basis.

CTK-norms are denoted by $\|\cdot\|_{CTK}$.

Lemma 3.1. *Let E be a Banach space with a basis $\{x_n\}$. The following statements are true:*

(i) *The norm in E is a CT-norm if and only if*

$$\|\sum_{i=1}^{n}\alpha_i x_i\|<\|\sum_{i=1}^{n+1}\alpha_i x_i\|$$

for all collections of scalars $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\in\mathbf{K}$ with $\alpha_{n+1}\neq 0$.

(ii) *The norm in E is a CK-norm if and only if*

$$\|\sum_{i=n+1}^{\infty}\alpha_i x_i\|<\|\sum_{i=n}^{\infty}\alpha_i x_i\|$$

for all sequence of scalars $\{\alpha_n\}_n^{\infty}$ with $\alpha_n\neq 0$ for which the series $\sum_{i=n}^{\infty}\alpha_i x_i$ is convergent.

(iii) *If the norm in E is CTK-norm, then*

$$\Big\|\sum_{i=l+1}^{n} \alpha_i x_i\Big\| < \Big\|\sum_{i=l}^{n+1} \alpha_i x_i\Big\|$$

for all collections of scalars $\alpha_l, \alpha_{l+1}, \ldots, \alpha_n, \alpha_{n+1} \in \mathbf{K}$ *with* $|\alpha_l| + |\alpha_{n+1}| \neq 0$.

Proof. The proof is clear from the proof of Theorem 3.1.

Example 3.1. A CK-norm which is not a CT-norm: The numbers

$$\|x\|_{\mathrm{CK}} = \max_{1 \leq n < \infty} \Big(\frac{1}{n}\sum_{i=1}^{n} |y_i| + \sup_{n+1 \leq j < \infty} |y_j|\Big) \quad (x = (y_i) \in c_0)$$

define a norm on c_0, equivalent to the initial norm of c_0. This norm $\|\cdot\|_{\mathrm{CK}}$ is a CK-norm but not a CT-norm with respect to the unit vector basis $\{x_n\}$ of c_0. On the other hand, it follows that

$$\|x_1 + x_2\|_{\mathrm{CK}} = \max\Big(1+1, \frac{1}{2}(1+1), \frac{1}{3}(1+1), \ldots\Big) = 2$$

$$\|x_1 + x_2 + x_3\|_{\mathrm{CK}} = \max\Big(1+1, \frac{1}{2}(1+1)+1, \frac{1}{3}(1+1+1), \frac{1}{4}(1+1+1), \ldots\Big) = 2.$$

Hence by Lemma 3.1, this is not a CT-norm.

Example 3.2. A CT-norm which is not a CK-norm: For every integer $n \geq 2$, let $\pi_{1,n}$ denote the collection of all permutations of the set

$$\{2, 3, \ldots, n-1, n+1, n+2, \ldots\}.$$

Then the numbers

$$\|x\|_{\mathrm{CT}} = \sup_{2 \leq n < \infty} \sup_{d \in \pi_{1,n}} \Big(\frac{|y_1|}{n2^n} + \sum_{i=2}^{\infty} \frac{|y_{d(i)}|}{2^i}\Big) \quad (x = (y_i) \in c_0)$$

define a norm on c_0, equivalent to the initial norm of c_0. This is a CT-norm but not a CK-norm with respect to the unit vector basis $\{x_n\}$ of c_0. The violation in the characterizing inequality of CK-norm in Lemma 3.1 was shown by I. SINGER [17].

Remark 3.2. The above examples show that there is no relation between CT-norms and CK-norms. That there can exist a basis and a norm which is a CT-norm but not a CK-norm and vice versa can be observed by the above examples.

Theorem 3.2. *Let E be a Banach space with a basis* $\{x_n\}$ *and let* $\{f_n\} \subset E^*$ *be the a.s.c.f. Then the following statements hold.*

(i) *A CT-norm on E equivalent to the initial norm on E can be introduced by the formula*

$$\|x\|_{\mathrm{CT}} = \sum_{i=1}^{\infty} \frac{1}{2^i} \|f_i(x) x_i\| + \sup_{1 \leq n < \infty} \Big\|\sum_{i=1}^{n} f_i(x) x_i\Big\| \tag{9}$$

(ii) *A CK-norm on E equivalent to the initial norm on E can be introduced by the formula*

$$\|x\|_{\mathrm{CK}} = \max_{1 \leqq n < \infty} \left(\frac{1}{n} \sum_{i=1}^{n} \|f_i(x)x_i\| + \Big\| \sum_{i=n+1}^{\infty} f_i(x)x_i \Big\| \right) \tag{10}$$

and also another equivalent CK-norm on E, by the formula

$$\|x\|_{\mathrm{CK}} = \sum_{i=1}^{\infty} \frac{1}{2^i} \|f_i(x)x_i\| + \sup_{1 \leqq n < \infty} \Big\| \sum_{i=1}^{\infty} f_i(x)x_i \Big\| \tag{11}$$

(iii) *A CTK-norm on E equivalent to initial norm on E can be introduced by the formula*

$$\|x\|_{\mathrm{CTK}} = \sum_{i=1}^{\infty} \frac{1}{2^i} \|f_i(x)x_i\| + \sup_{1 \leqq n, m < \infty} \Big\| \sum_{i=1}^{n} f_i(x)x_i \Big\| \tag{12}$$

and another equivalent CTK-norm on E by the formula

$$\|x\|_{\mathrm{CTK}} = \sup_{1 \leqq n < \infty} \left\{ \Big\| \sum_{i=1}^{n} f_i(x)x_i \Big\|_{\mathrm{CK}} + \Big\| \sum_{i=n+1}^{\infty} f_i(x)x_i \Big\|_{\mathrm{CT}} \right\}. \tag{13}$$

Proof. The fact that all these numbers define a norm and all these norms are equivalent to the original norm on E was proved previously and can be found in [17]. Now that they actually have the property of CT, CK, CTK-norms is proved here.

(i)
$$\Big\| \sum_{i=1}^{n} \alpha_i x_i \Big\|_{\mathrm{CT}} = \sum_{i=1}^{n} \frac{1}{2^i} \|\alpha_i x_i\| + \max_{1 \leqq k \leqq n} \Big\| \sum_{i=1}^{k} \alpha_i x_i \Big\| <$$
$$< \sum_{i=1}^{n+1} \frac{1}{2^i} \|\alpha_i x_i\| + \max_{1 \leqq k \leqq n+1} \Big\| \sum_{i=1}^{k} \alpha_i x_i \Big\| = \Big\| \sum_{i=1}^{n+1} \alpha_i x_i \Big\|_{\mathrm{CT}}$$

for any scalars $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1} \in \mathbf{K}$ with $\alpha_{n+1} \neq 0$. Hence by Lemma 3.1, it follows that it is a CT-norm.

(ii). (ii)$_1$. Let $\{a_i\}_{i-1}^{\infty}$ be a sequence of scalars with $\alpha_{l-1} \neq 0$, such that $\sum_{i=l}^{\infty} \alpha_i x_i$ converges. Then it follows that for a suitable number n_0 with $l \leqq n_0 < \infty$,

$$\Big\| \sum_{i=l}^{\infty} \alpha_i x_i \Big\|_{\mathrm{CK}} = \frac{1}{n_0} \sum_{i=l}^{n_0} \|\alpha_i x_i\| + \Big\| \sum_{i=n_0+1}^{\infty} \alpha_i x_i \Big\| <$$
$$< \frac{1}{n_0} \sum_{i=l-1}^{n_0} \|\alpha_i x_i\| + \Big\| \sum_{i=n_0+1}^{\infty} \alpha_i x_i \Big\| \leqq \max_{l-1 \leqq n < \infty} \left(\frac{1}{n} \sum_{i=l-1}^{n} \|\alpha_i x_i\| + \Big\| \sum_{i=n+1}^{\infty} \alpha_i x_i \Big\| \right) =$$
$$= \Big\| \sum_{i=l-1}^{\infty} \alpha_i x_i \Big\|_{\mathrm{CK}}.$$

Hence from Lemma 3.1, it follows that this norm is a CK-norm.

(ii)$_2$.
$$\Big\|\sum_{i=l}^{\infty}\alpha_i x_i\Big\|_{CK} = \sum_{i=l}^{\infty}\frac{1}{2^i}\|\alpha_i x_i\| + \max_{l\leqq n<\infty}\Big\|\sum_{i=n}^{\infty}\alpha_i x_i\Big\| <$$
$$< \sum_{i=l-1}^{\infty}\frac{1}{2^i}\|\alpha_i x_i\| + \max_{l-1\leqq n<\infty}\Big\|\sum_{i=n}^{\infty}\alpha_i x_i\Big\| = \Big\|\sum_{i=l-1}^{\infty}\alpha_i x_i\Big\|_{CK}.$$

Hence from Lemma 3.1, it is clear that this norm is a CK-norm.

4. Characterization of bases

For a sequence $\{y_n\}$ in a Banach space E, let $P_n=[y_i\colon i\leqq n]$ and let $P=\bigcup_{n=1}^{\infty} P_n$. For $p\in P$, $p=\sum_{i=1}^{n}\alpha_i y_i$ for some n, let

$$s_m^n(p) = \begin{cases} \sum_{i=1}^{m}\alpha_i y_i & \text{if } m<n \\ p & \text{if } m\geqq n \end{cases}$$

and

$$r_m^n(p) = p - s_m^n(p).$$

Definition 4.1. The norm $\|\cdot\|$ of E is a

(i) *weak CT-norm* relative to $\{y_n\}$ if for each polynomial $p\in P$, $p=\sum_{i=1}^{n}\alpha_i y_i$ and each $m\leqq n$, the polynomial $\sum_{i=1}^{m}\alpha_i y_i$ is the unique best coapproximation to p from $[y_i\colon i\leqq m]$.

(ii) *weak CK-norm* relative to $\{y_n\}$ if for each polynomial $p\in P$, $p=\sum_{i=1}^{n}\alpha_i y_i$ and each $m\leqq n$ the complementary polynomial $\sum_{i=m+1}^{n}\alpha_i y_1$ is the best coapproximation to p from $[y_i\colon m+1\leqq i\leqq n]$.

(iii) *weak CTK-norm* relative to $\{y_n\}$ if it is simultaneously a weak CK-norm and a weak CT-norm relative to $\{y_n\}$.

Remark 4.1. It is clear that if $\{x_n\}$ is a basis for E, then a CT-, CK-, CTK-norm with respect to $\{x_n\}$ is a weak CT-norm, weak CK-norm and weak CTK-norm relative to $\{x_n\}$. Example 3.1 shows that the converse is false.

Theorem 4.1. *Let $\{y_n\}$ be a non-zero sequence in a Banach space E with the norm $\|\cdot\|$. Then the following statements hold.*

(i) *The norm is a weak CT-norm relative to* $\{y_n\}$ *if and only if*

$$(*_1)\qquad \sup_n \sup\{\|s_n(p)\colon p\in P,\ \|p\| \leqq 1\} = 1.$$

(ii) *The norm is a weak CK-norm relative to* $\{y_n\}$ *if and only if*

$$(*_2)\qquad \sup_n \sup\{\|r_n(p)\|\colon p\in P,\ \|p\| \leqq 1\} = 1.$$

(iii) *The norm is a weak CTK-norm relative to* $\{y_n\}$ *if and only if*

$$\max[(*_1), (*_2)] = 1.$$

Remark 4.2. Here $s_n(p)$ and $r_n(p)$ will assume the roles of $s_n^k(p)$ and $r_n^k(p)$ whenever $p\in P$ is expressible in the form $p=\sum_{i=1}^{k} \alpha_i y_i$ for some $k\in\mathcal{N}$ and $(*_1)$ and $(*_2)$ denote the expressions on the left-hand side of the equations.

Proof of Theorem 4.1. (1) Suppose that $p=\sum_{i=1}^{n} \alpha_i y_i \in P$ and $(*_1)=1$. Let $y=\sum_{i=1}^{m} \beta_i y_i \in P_m$. If $\|p-y\|\neq 0$, let $p'=\|p-y\|^{-1}(p-y)$. Then it follows that $\|p'\|=1$. Therefore, $\|s_m^n(p')\|\leqq 1$ by property $(*_1)=1$. But $\left\|s_m^n(\|p-y\|^{-1}(p-y))\right\|\leqq 1$ implies

$$\|s_m^n(p-y)\| \leqq \|p-y\|,$$

i.e.

$$\|s_m^n(p)-y\| \leqq \|p-y\|,$$

i.e. $s_m^n(p)$ is a best coapproximation to p. But since $P_n=P_m\oplus[x_{m+1}, \dots, x_n]$, it follows that $R_{P_m}(p)$ is unique. Therefore $s_m^n(p)$ is the unique best coapproximation to p. On the other hand, if $\|p-y\|=0$, then $s_m^n(p)=p=y$ and the result is trivial.

Conversely if $s_m^n(p)$ is the unique best coapproximation to $p=\sum_{i=1}^{n} \alpha_i y_i$ for $m\leqq n$ and for $\|p\|\leqq 1$, it follows that $\|s_m^n(p)\|\leqq\|p\|\leqq 1$. Since only finite sums are dealt with, a $p\in P$ and n can be found such that $\|s_n(p)\|$ is nearly 1. Thus $(*_1)=1$.

(ii) and (iii). The proofs of (ii) and (iii) are similar and are omitted.

Theorem 4.2. *Let E be a Banach space with a basis $\{x_n\}$. A norm on E is a weak CK-norm if and only if*

$$\left\|\sum_{i=m}^{n} \alpha_i y_i\right\| \leqq \left\|\sum_{i=m-1}^{n} \alpha_i y_i\right\|$$

for arbitrary scalars $\alpha_{m-1}, \alpha_m, \dots, \alpha_n, \alpha_{m-1}\neq 0$ *and* $m, n=1, 2, \dots$.

Proof. The proof is similar to that of (ii) of Lemma 3.1.

Theorem 4.3. *The following statements about* $\{x_n\}$ *a sequence in a Banach space E with* $[x_i, i\in\mathcal{N}]=E$ *are equivalent:*

(i) $\{x_n\}$ *is a basis for E.*

(ii) *A weak CT-norm can be introduced relative to* $\{x_n\}$ *on E equivalent to the original norm on E.*

(iii) *A weak CK-norm can be introduced relative to* $\{x_n\}$ *on E equivalent to the original norm on E.*

(iv) *A weak CTK-norm can be introduced relative to* $\{x_n\}$ *on E equivalent to the original norm on E.*

Proof. (i) implies the other three was proved in the stronger form in Section 3 of this paper. If (ii) and (iii) implies (i), then (iv) also implies (i). So (ii) implies (i) is proved here as the other implication is similar.

Suppose $p\leqq q$, $\sum_{i=1}^{q}\alpha_i x_i\neq 0$, then by Theorem 4.1, it follows that

$$\Big\|s_p^q\Big(\Big\|\sum_{i=1}^{q}\alpha_i x_i\Big\|^{-1}\sum_{i=1}^{q}\alpha_i x_i\Big)\Big\|\leqq 1$$

(i.e.)

$$\Big\|s_p^q\Big(\sum_{i=1}^{q}\alpha_i x_i\Big)\Big\|\leqq\Big\|\sum_{i=1}^{q}\alpha_i x_i\Big\|$$

(i.e.)

$$\Big\|\sum_{i=1}^{p}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i=1}^{q}\alpha_i x_i\Big\|.$$

If $\sum_{i=1}^{q}\alpha_i x_i=0$, then, since the norm is a weak CT-norm,

$$\Big\|\sum_{i=1}^{p}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i=1}^{q}\alpha_i x_i\Big\|=0$$

implying

$$\sum_{i=1}^{p}\alpha_i x_i=0$$

for all $p\leqq q$. Thus Grinblyum's K-condition is satisfied with $K=1$.

5. Characterising orthogonal bases

Definition 5.1. Let E be a Banach space having a sequence $\{x_n\}$. Following [17],

(i) $\{x_n\}$ is *orthogonal* provided $\Big\|\sum_{i\in d_1}\alpha_i x_i\Big\|\leqq\Big\|\sum_{i\in d_2}\alpha_i x_i\Big\|$ for arbitrary $d_1, d_2\in\mathcal{D}$, with $d_1\subset d_2$ and arbitrary collection of scalars $\{\alpha_i\}_{i\in d_2}$.

(ii) $\{x_n\}$ is *strictly orthogonal* if the inequality is strict whenever $\sum_{i\in d_2\setminus d_1} |\alpha_i|\neq 0$.

(iii) $\{x_n\}$ is *coorthogonal* if $\left\|\sum_{i\in\mathcal{N}\setminus d_2}\alpha_i x_i\right\|\leqq\left\|\sum_{i\in\mathcal{N}\setminus d_1}\alpha_i x_i\right\|$ for arbitrary $d_1, d_2\in\mathcal{D}$ with $d_1\subset d_2$ and arbitrary collection of scalars $\{\alpha_i\}$ for which $\sum_{i\in\mathcal{N}}\alpha_i x_i$ is convergent.

(iv) $\{x_n\}$ is strictly coorthogonal if the inequality of (iii) is strict whenever $\sum_{i\in d_2\setminus d_1}|\alpha_i|\neq 0$.

Theorem 5.1. *Let E be a Banach space having an unconditional basis $\{x_n\}$. Then the following statements are true:*

(i) *$\{x_n\}$ is orthogonal if and only if $R_{G_d}(x)=\{s_d(x)\}$ for all $d\in\mathcal{D}$.*

(ii) *$\{x_n\}$ is strictly orthogonal if and only if $R_{G_d}(x)=\{s_d(x)\}$ for all $d\in\mathcal{D}$ and E has the property that there exist no scalars $\{\alpha_i\}_{i\in d_2\setminus d_1}$, for all $d_2, d_1\in\mathcal{D}$ with $d_2\supset d_1$ and $\sum_{i\in d_2\setminus d_1}|\alpha_i|\neq 0$ satisfying*

$$\left\|\sum_{i\in d_1}\alpha_i x_i\right\| = \left\|\sum_{i\in d_2}\alpha_i x_i\right\| = \left\|\sum_{i\in d_1}\alpha_i x_i+\frac{1}{2}\sum_{i\in d_2\setminus d_1}\alpha_i x_i\right\|.$$

(iii) *$\{x_n\}$ is strictly coorthogonal if and only if $R_{G^d}(x)=\{r_d(x)\}$ for all $d\in\mathcal{D}$ and E has the property that there exist no scalars $\{\alpha_i\}_{i\in d_2\setminus d_1}$, for all $d_2, d_1\in\mathcal{D}$ with $d_2\supset d_1$ and $\sum_{i\in d_2\setminus d_1}|\alpha_i|\neq 0$ satisfying*

$$\left\|\sum_{i\in\mathcal{N}\setminus d_2}\alpha_i x_i\right\| = \left\|\sum_{i\in\mathcal{N}\setminus d_1}\alpha_i x_i\right\| = \left\|\sum_{i\in\mathcal{N}\setminus d_2}\alpha_i x_i+\frac{1}{2}\sum_{i\in d_2\setminus d_1}\alpha_i x_i\right\|.$$

Proof. The proof is similar to Theorem 3.1 and is omitted.

Remark 5.1. Since the notions of orthogonal and coorthogonal bases are equivalent, the characterization of coorthogonal bases is omitted in Theorem 5.1. Analogous to Definitions 3.4, 3.5 and 3.6, one may call the norms satisfying the "if" parts of (ii), (iii) and (ii) and (iii) of Theorem 5.1 as CNT-, CNK-, CNTK-norms respectively. While every CNK-norm is a CNT-norm, the converse is not always true. Example 3.1 illustrates this.

Theorem 5.2. *Let E be a Banach space with an unconditional basis $\{x_n\}$. Then every norm in E in which $R_{G^d}(x)=\{r_d(x)\}$ and there exist no scalars $\{\alpha_i\}_{i\in d_2\setminus d_1}$, for all $d_2, d_1\in\mathcal{D}$ with $d_2\supset d_1$ and $\sum_{i\in d_2\setminus d_1}|\alpha_i|\neq 0$ satisfying*

$$\left\|\sum_{i\in\mathcal{N}\setminus d_2}\alpha_i x_i\right\| = \left\|\sum_{i\in\mathcal{N}\setminus d_1}\alpha_i x_i\right\| = \left\|\sum_{i\in\mathcal{N}\setminus d_2}\alpha_i x_i+\frac{1}{2}\sum_{i\in d_2\setminus d_1}\alpha_i x_i\right\|$$

also has $R_{G_d}(x)=\{s_d(x)\}$ and the property that there exist no scalars $\{\alpha_i\}_{i\in d_2\setminus d_1}$, *for all* $d_1, d_2 \in \mathscr{D}$ *with* $d_2 \supset d_1$ *and* $\sum_{i \in d_2 \setminus d_1} |\alpha_i| \neq 0$ *satisfying*

$$\Big\|\sum_{i \in d_1} \alpha_i x_i\Big\| = \Big\|\sum_{i \in d_2} \alpha_i x_i\Big\| = \Big\|\sum_{i \in d_1} \alpha_i x_i + \frac{1}{2} \sum_{i \in d_2 \setminus d_1} \alpha_i x_i\Big\|.$$

Proof. Every strictly coorthogonal basis is strictly orthogonal. This was proved by RETHERFORD [14]. Hence the theorem follows.

Theorem 5.3. *Let E be a Banach space with an unconditional basis* $\{x_n\}$ *and let* $\{f_n\} \subset E^*$ *be the a.s.c.f. Then a norm* $\|\cdot\|_*$ *on E can be introduced, equivalent to the initial norm on E, in which* $R_{G^d}(x) = \{r_d(x)\}$ *and there exist no scalars* $\{\alpha_i\}_{i \in d_2 \setminus d_1}$, *for all* $d_2, d_1 \in \mathscr{D}$ *with* $d_2 \supset d_1$ *and* $\sum_{i \in d_2 \setminus d_1} |\alpha_i| \neq 0$ *satisfying*

$$\Big\|\sum_{i \in \mathscr{N} \setminus d_2} \alpha_i x_i\Big\| = \Big\|\sum_{i \in \mathscr{N} \setminus d_1} \alpha_i x_i\Big\| = \Big\|\sum_{i \in \mathscr{N} \setminus d_2} \alpha_i x_i + \frac{1}{2} \sum_{i \in d_2 \setminus d_1} \alpha_i x_i\Big\|,$$

by the formula

$$\|x\|_* = \sum_{i=1}^{\infty} \frac{1}{2^i} \|f_i(x) x_i\| + \sup_{\{i_1, i_2, \dots, i_n\} \in \mathscr{D}} \Big\|\sum_{j=1}^{n} f_{i_j}(x) x_{i_j}\Big\|.$$

Proof. The equivalence of norms follows from I. SINGER [17, p. 554]. To prove that $\|\cdot\|_*$ has the required properties, it will be sufficient by Theorem 5.1 to prove that $\{x_n\}$ is strictly coorthogonal in this norm. Let $d_1, d_2 \in \mathscr{D}$ with $d_1 \subset d_2$ and $\sum_{i \in d_2 \setminus d_1} |\alpha_i| \neq 0$ be such that $\sum_{i \in \mathscr{N} \setminus d_2} \alpha_i x_i$ converges. Then it follows that

$$\mathscr{N} \setminus d_1 = (\mathscr{N} \setminus d_2) \cup (d_2 \setminus d_1).$$

Hence

$$\Big\|\sum_{i \in \mathscr{N} \setminus d_2} \alpha_i x_i\Big\|_* = \sum_{i \in \mathscr{N} \setminus d_2} \frac{1}{2^i} \|\alpha_i x_i\| + \sup_{\{i_1, i_2, \dots, i_n\} \in \mathscr{D} \cap \mathscr{N} \setminus d_2} \Big\|\sum_{j=1}^{n} \alpha_{i_j} x_{i_j}\Big\| <$$

$$< \sum_{i \in \mathscr{N} \setminus d_1} \frac{1}{2^i} \|\alpha_i x_i\| + \sup_{\{i_1, i_2, \dots, i_n\} \in \mathscr{D} \cap \mathscr{N} \setminus d_1} \Big\|\sum_{j=1}^{n} \alpha_{i_j} x_{i_j}\Big\| = \Big\|\sum_{i \in \mathscr{N} \setminus d_1} \alpha_i x_i\Big\|_*.$$

Thus $\{x_n\}$ is strictly coorthogonal in $\|\cdot\|_*$ and the proof is complete.

6. Characterization of unconditional bases

Let $\{y_n\}$ be a sequence in a Banach space E. Let $P_d = [y_i]_{i \in d}$ where $d = \{i_1, i_2, \dots, i_n\} \subset \mathscr{N}$, i.e., $d \in \mathscr{D}$ and $P = \bigcup_{d \in \mathscr{D}} P_d$. For $p \in S$, let $p = \sum_{i \in d} \alpha_i y_i$, then

$$s_{d'}^{d}(p) = \begin{cases} \sum_{i \in d'} \alpha_i x_i & \text{if } d' \subset d \\ p & \text{if } d \subset d' \end{cases}$$

$$r_{d'}^{d}(p) = p - s_{d'}^{d}(p).$$

Remark 6.1. $s_{d'}^{d}(p)$ is not defined whenever $d'\cap d\neq\emptyset$ and neither $d\subset d'$ or $d'\subset d$ hold.

Definition 6.1. A norm on E is a

(i) *weak CNTK-norm* relative to $\{y_n\}$ if for each polynomial $p\in P$, $p=\sum_{i\in d}\alpha_i y_i$ and for each $d'\subset d$, the polynomial $\sum_{i\in d'}\alpha_i y_i$ is the unique best coapproximation to p from $[y_i]_{i\in d'}$.

Remark 6.1. It should be noted here that analogous definitions of weak CNT-, weak CNK-norms coincide with that of weak CNTK-norm.

Theorem 6.1. *Let $\{y_n\}$ be a non-zero sequence in a Banach space E with norm $\|\cdot\|$. The norm is a weak CNTK-norm relative to $\{y_n\}$ if and only if*

$$\sup_d \sup \{\|s_d(p)\|:\ p\in P,\ \|p\|\leqq 1\}=1.$$

Proof. Similar to the proof of Theorem 4.1.

Remark 6.2. $s_d(p)$ will assume the role of $s_d^{d'}(p)$ whenever $p=\sum_{i\in d'}\alpha_i y_i\in P$.

Theorem 6.2. *The following statements about a sequence $\{y_n\}$ in a Banach space E with $[y_i]_{i\in\mathcal{N}}=E$ are equivalent:*

(i) *$\{y_n\}$ is an unconditional basis of E.*

(ii) *A weak CNTK-norm relative to $\{y_n\}$ can be introduced on E equivalent to the original norm on E.*

Proof. The proof is similar to that of Theorem 4.3.

7. Remarks

Let E be a Banach space with norm $\|\cdot\|$. A sequence $\{M_i\}$ of non-trivial subspaces of E is called a decomposition of E provided for each $x\in E$, there exists a unique sequence $\{x_i\}$ such that $x_i\in M_i$ and $\sum_{i=1}^{\infty}x_i=x$, the convergence being in the norm topology. It is also possible to define for each i, a projection $P_i\colon E\to M_i$ by $P_i(x)=x_i$. If each projection is continuous, then the pair $\{M_i, P_i\}$ is called a Schauder decomposition. The notions of a Schauder basis and a Schauder decomposition are almost similar in the view point of approximation theory. Best approximation and Schauder decompositions were studied by P. K. Jain and K. Ahmad [4], [5], [6]. Hence the analogous results of best coapproximation and bases in Banach spaces can be carried over to Schauder decompositions. Though the results look different, the idea is the same. Therefore the analogous results, even though known to the authors, are not elaborated.

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