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Further sharpening of inequalities of Hardy and Littlewood

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In a previous paper [2] we generalized some classical and very useful inequalities of HARDY and LITTLEWOOD [1]. One special case of our results states the following inequalities:

If $\lambda_n>0$ *and* $a_n\geqq 0$ $(n=1, 2, \ldots)$ *then we have*

$$\left.\begin{aligned} &(1)\qquad \sum_{n=1}^{\infty}\lambda_n\Big(\sum_{k=1}^{n}a_k\Big)^p \leqq p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\Big(\sum_{k=n}^{\infty}\lambda_k\Big)^p\\ &(2)\qquad \sum_{n=1}^{\infty}\lambda_n\Big(\sum_{k=n}^{\infty}a_k\Big)^p \leqq p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\Big(\sum_{k=1}^{n}\lambda_k\Big)^p \end{aligned}\right\}\quad \text{for}\quad p\geqq 1,$$

and

$$\left.\begin{aligned} &(3)\qquad \sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\Big(\sum_{k=n}^{\infty}\lambda_k\Big)^p \leqq 8\sum_{n=1}^{\infty}\lambda_n\Big(\sum_{k=1}^{n}a_k\Big)^p\\ &(4)\qquad \sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\Big(\sum_{k=1}^{n}\lambda_k\Big)^p \leqq 9\sum_{n=1}^{\infty}\lambda_n\Big(\sum_{k=n}^{\infty}a_k\Big)^p \end{aligned}\right\}\quad \text{for}\quad 0<p\leqq 1.$$

These inequalities reduce to those of Hardy and Littlewood if $\lambda_n=n^{-c}$ with $c>1$ in (1) and (3); and with $c\leqq 1$ in (2) and (4).

The factor p^p is best possible one, but 8 and 9 are not. In the present note we improve, among others, inequalities (3) and (4) proving that the constants 8 and 9 can be replaced by p^{-p} and this is the best possible one. It is easy to see that $p^{-p}\leqq \leqq e^{1/e}<1,45$ holds for any $0<p\leqq 1$. Having these improved inequalities (3) and (4), we can state that inequalities (1) and (2) hold for $p\geqq 1$ and their reversed ones hold for $0<p\leqq 1$.

In [2] it was assumed only that $\lambda_n\geqq 0$ and in this more general form we proved the following theorem.

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Theorem A. *Let* $a_n \geqq 0$ *and* $\lambda_n \geqq 0$ $(n=1, 2, \ldots)$ *be given. Let* $\nu_1 < \ldots < \nu_n < \ldots$ *denote the indices for which* $\lambda_{\nu_n} > 0$. *Let* N *denote the number of the positive terms of the sequence* λ_n, *provided this number is finite; in the contrary case set* $N = \infty$. *Set* $\nu_0 = 0$, *and if* $N < \infty$ *then* $\nu_{N+1} = \infty$. *Using the notations*

$$A_{m,n} := \sum_{i=m}^{n} a_i \quad \text{and} \quad \Lambda_{m,n} := \sum_{i=m}^{n} \lambda_i \quad (1 \leqq m \leqq n \leqq \infty),$$

we have the following inequalities:

$$\left.\begin{aligned} &(1') \quad \sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \leqq p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p \\ &(2') \quad \sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \leqq p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p \end{aligned}\right\} \text{for } p \geqq 1$$

(the constant p^p *being the best possible one) and*

$$\left.\begin{aligned} &(3') \quad \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p \leqq 8 \sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \\ &(4') \quad \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} \Lambda_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p \leqq 9 \sum_{n=1}^{\infty} \lambda_n A_{n,\infty}^p \end{aligned}\right\} \text{for } 0 < p \leqq 1.$$

The aim of this note is to prove

Theorem. *Under the assumptions of Theorem* A *the opposite inequalities of* (1') *and* (2') *hold for* $0 < p \leqq 1$, *and the constant* p^p *is best possible one, in this case, too.*

Theorem A and Theorem imply immediately

Corollary 1. *If* $\lambda_n > 0$ *and* $a_n \geqq 0$ *then* (1) *and* (2) *hold for* $p \geqq 1$, *and their opposite inequalities for* $0 < p \leqq 1$.

If $0 < p \leqq 1$ then we can reduce the restriction $\lambda_n > 0$ of Corollary 1 to $\lambda_n \geqq 0$, i.e. we can prove

Corollary 2. *For any* $a_n \geqq 0$ *and* $\lambda_n \geqq 0$, *if* $0 < p \leqq 1$, *we have*

$$(3'') \qquad \sum_{n=1}^{\infty} \lambda_n^{1-p} \Big(\sum_{k=n}^{\infty} \lambda_k\Big)^p a_n^p \leqq p^{-p} \sum_{n=1}^{\infty} \lambda_n \Big(\sum_{k=1}^{n} a_k\Big)^p$$

and

$$(4'') \qquad \sum_{n=1}^{\infty} \lambda_n^{1-p} \Big(\sum_{k=1}^{n} \lambda_k\Big)^p a_n^p \leqq p^{-p} \sum_{n=1}^{\infty} \lambda_n \Big(\sum_{k=n}^{\infty} a_k\Big)^p.$$

This is an immediate consequence of the facts that

$$\lambda_{\nu_n}^{1-p} A_{\nu_n,\infty}^p \Big(\sum_{i=\nu_{n-1}+1}^{\nu_n} a_i\Big)^p \geqq \sum_{i=\nu_{n-1}+1}^{\nu_n} \lambda_i^{1-p} A_{i,\infty}^p a_i^p \tag{5}$$

and

$$\lambda_{\nu_n}^{1-p} A_{1,\nu_n}^p \Big(\sum_{i=\nu_n}^{\nu_{n+1}-1} a_i\Big)^p \geqq \sum_{i=\nu_n}^{\nu_{n+1}-1} \lambda_i^{1-p} A_{1,i}^p a_i^p \tag{6}$$

obviously hold — since $\lambda_i=0$ if $i\neq\nu_n$ — so by the opposite inequalities of (1′) and (2′) to be proved in Theorem, regarding inequalities (5) and (6), we get (3″) and (4″).

Proofs

1. First we prove the opposite of (1′) for $0<p\leqq 1$, i.e.

$$\Sigma_1 := \sum_{n=1}^{\infty} \lambda_n A_{1,n}^p \geqq p^p \sum_{n=1}^{N} \lambda_{\nu_n}^{1-p} A_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p =: \Sigma_2. \tag{1.1}$$

We may assume that Σ_1 has a positive finite value. For $p=1$ (1.1) is obvious, we have only to interchange the order of the summations. If $0<p<1$ then we set the following notations:

$$\alpha_n := A_{\nu_{n-1}+1,\nu_n}, \quad \beta_0 = 0, \quad \beta_n := \sum_{k=1}^{n} \alpha_k, \quad \varrho_n := \lambda_{\nu_n},$$

and

$$R_n := A_{\nu_n,\infty}$$

for every $1\leqq n\leqq N$. If $N<\infty$ then let

$$R_{N+1} := \varrho_{N+1} := 0.$$

Then, for any positive integer $m(\leqq N)$, we have

$$\sum_{k=1}^{\nu_m} \lambda_k A_{1,k}^p = \sum_{n=1}^{m} \varrho_n \beta_n^p = \sum_{n=1}^{m} (R_n - R_{n+1})\beta_n^p = \sum_{n=1}^{m} R_n(\beta_n^p - \beta_{n-1}^p) - R_{m+1}\beta_m^p,$$

whence

$$\sum_{n=1}^{m} R_n(\beta_n^p - \beta_{n-1}^p) \leqq R_{m+1}\beta_m^p + \sum_{k=1}^{\nu_m} \lambda_k A_{1,k}^p \leqq \sum_{k=1}^{\infty} \lambda_k A_{1,k}^p = \Sigma_1 < \infty \tag{1.2}$$

follows for any $m\leqq N$.

Let μ be the smallest positive integer having the property $\beta_\mu>0$.

An easy calculation gives that

(1.3) $$\sum_{n=\mu}^{m} R_n(\beta_n^p-\beta_{n-1}^p) \geqq \sum_{n=\mu}^{m} R_n p\alpha_n\beta_n^{p-1} =$$

$$= p\sum_{n=\mu}^{m}(\varrho_n^{(1/p)-1}R_n\alpha_n)(\varrho_n^{1-(1/p)}\beta_n^{p-1}) =: p\Sigma_3.$$

Now using the classical inequality of Hölder with p and $\dfrac{p}{p-1}$ we get

(1.4) $$\Sigma_3 \geqq \Big(\sum_{n=\mu}^{m}\varrho_n^{1-p}R_n^p\alpha_n^p\Big)^{1/p}\Big(\sum_{n=\mu}^{m}\varrho_n\beta_n^p\Big)^{1-1/p}.$$

By (1.2), (1.3) and (1.4) we have

$$\sum_{k=1}^{\infty}\lambda_k A_{1,k}^p \geqq p\Big(\sum_{n=1}^{\infty}\varrho_n^{1-p}R_n^p\alpha_n^p\Big)^{1/p}\Big(\sum_{n=1}^{\infty}\varrho_n\beta_n^p\Big)^{1-1/p} =$$

$$= p\Big(\sum_{n=1}^{\infty}\lambda_{\nu_n}^{1-p}A_{\nu_n,\infty}^p A_{\nu_{n-1}+1,\nu_n}^p\Big)^{1/p}\Big(\sum_{n=1}^{\infty}\lambda_{\nu_n}\Big(\sum_{k=1}^{\nu_n}a_k\Big)^p\Big)^{1-1/p},$$

whence (1.1) obviously follows.

2. Secondly we prove the opposite of (2') for $0<p\leqq 1$, i.e.

(1.5) $$\Sigma_4 := \sum_{n=1}^{\infty}\lambda_n A_{n,\infty}^p \geqq p^p\sum_{n=1}^{N}\lambda_{\nu_n}^{1-p}A_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p =: \Sigma_5.$$

As before the case $p=1$ is trivial and we may assume that Σ_4 has a positive finite value. Using the previous notations and let $\alpha_n^* := A_{\nu_n,\nu_{n+1}-1}$, $\gamma_n := \sum_{k=n}^{N}\alpha_k^*$ and $B_n := A_{1,\nu_n} = \sum_{k=1}^{n}\varrho_k$; furthermore let ω denote the greatest natural number, if there exists, for which $A_{\nu_\omega,\infty}>0$, otherwise let $\omega:=\infty$. If $\Omega:=\min(N,\omega)$, then for any $1\leqq n\leqq k\leqq\Omega$ $\gamma_n=A_{\nu_n,\infty}>0$ and we have the estimation

(1.6) $$\sum_{n=1}^{k+1}\varrho_n\gamma_n^p = \sum_{n=1}^{k+1}(B_n-B_{n-1})\gamma_n^p =$$

$$= \sum_{n=1}^{k}B_n(\gamma_n^p-\gamma_{n+1}^p)+B_{k+1}\gamma_{k+1} \geqq \sum_{n=1}^{k}B_n(\gamma_n^p-\gamma_{n+1}^p).$$

By (1.6) it is clear that for any $k\leqq\Omega$

(1.7) $$\Sigma_4 \geqq \sum_{n=1}^{k}B_n(\gamma_n^p-\gamma_{n+1}^p) \geqq \sum_{n=1}^{k}B_n p\alpha_n^*\gamma_n^{p-1} =$$

$$= p\sum_{n=1}^{k}(\varrho_n^{1/p-1}B_n\alpha_n^*)(\varrho_n^{1-1/p}\gamma_n^{p-1})$$

holds. Using again the classical inequality of Hölder with p and $\frac{p}{p-1}$ we obtain, by (1.7), that

$$\Sigma_4 \geqq p\Big(\sum_{n=1}^{k} \varrho_n^{1-p} B_n^p (\alpha_n^*)^p\Big)^{1/p} \Big(\sum_{n=1}^{k} \varrho_n \gamma_n^p\Big)^{1-1/p} =$$

$$= p\Big(\sum_{n=1}^{k} \lambda_{\nu_n}^{1-p} A_{1,\nu_n}^p A_{\nu_n,\nu_{n+1}-1}^p\Big)^{1/p} \Big(\sum_{n=1}^{k} \lambda_{\nu_n} A_{\nu_n,\infty}^p\Big)^{1-1/p},$$

whence (1.5) follows immediately.

3. To verify that the constant p^p in the opposite inequalities of (1′) and (2′) for $0<p\leqq 1$ is best possible it is enough to consider e.g. the following special case:

$$a_n := \big(\log(n+1)\big)^{-2/p} \quad \text{and} \quad \lambda_n = n^{-1-p} \quad \text{for} \quad n \geqq N \to \infty$$

and everything is zero for $n<N$.

The proof of Theorem is completed.

References

[1] G. H. Hardy—J. E. Littlewood, Elementary theorems concerning power series with positive coefficients and moment constants of positive functions, *J. reine angew. Math.*, **157** (1927), 141—158.

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