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A characterization of o-distributive semilattices

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The notion of a distributive ordered set which generalizes the notion of a distributive lattice is introduced in [3], where there are shown some properties of such ordered sets. In [2] there are described ordered sets having a similar importance for distributive ordered sets as the pentagon and the diamond have for distributive lattices, i.e. on certain conditions they are not included in a distributive ordered set (e.g. as its strong subset) and each non-distributive ordered set contains at least one of those sets as an LU-subset. (For the definitions of an LU-subset and a strong subset see below.)

The aim of this paper is to describe the semilattices which are distributive ordered sets.

Let $A=(A, \leqq)$ be an ordered set. If $B \subseteq A$, then we denote

$$L_A(B) = \{x \in A;\ x \leqq b, \text{ for all } b \in B\},$$

$$U_A(B) = \{y \in A;\ y \geqq b, \text{ for all } b \in B\}.$$

If it is not a danger of misunderstanding, we write also $L(B)$ and $U(B)$ instead of $L_A(B)$ and $U_A(B)$. For $B=\{a_1, \ldots, a_n\}$ we use also the forms $L(B)=L(a_1, \ldots, a_n)$ and $U(B)=U(a_1, \ldots, a_n)$.

Definition 1. An ordered set A is called *distributive* if

$$L(U(L(a, c), L(b, c))) = L(U(a, b), c) \quad \text{for all } a, b, c \in A.$$

Remark 1. It is clear that in any ordered set A it holds $L(U(L(a, c), L(b, c))) \subseteq L(U(a, b), c)$ for all $a, b, c \in A$. Hence for the distributivity of an ordered set it suffices to verify only the identity with the opposite inclusion.

Remark 2. A lattice A is distributive if and only if it is a distributive ordered set. (See [3].)

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Recall that a semilattice $A=(A, \leqq, \vee)$ is called distributive (see [1, p. 135]) if for any $a, b, x\in A$ it holds the following condition:

If $x\leqq a\vee b$, then there exist $a_1, b_1\in A$, $a_1\leqq a$, $b_1\leqq b$ such that $x=a_1\vee b_1$.

To distinguish two notions of distributivity, a semilattice which is simultaneously a distributive ordered set will be called an o-distributive semilattice.

We will show a connection between these notions.

Proposition 1. *Every distributive semilattice is o-distributive.*

Proof. If $A=(A, \vee)$ is a semilattice, $a, b, c\in A$, then $L(U(a, b), c)=L(a\vee b, c)$. Let A be a distributive semilattice, $a, b, c, x\in A$, $x\leqq c$, $x\leqq a\vee b$. Then there exist $a_1, b_1\in A$, $a_1\leqq a$, $b_1\leqq b$ such that $x=a_1\vee b_1$. Let $y\in U(L(a, c), L(b, c))$. Then $a_1\leqq y$, $b_1\leqq y$, hence $x=a_1\vee b_1\leqq y$, and therefore $L(a\vee b, c)\subseteq L(U(L(a, c), L(b, c)))$.

Remark 3. The converse implication is not true. For example, the semilattice $A=\{a, b, c\}$, where $a<c$, $b<c$ (see Fig. 1), is o-distributive but it is not distributive.

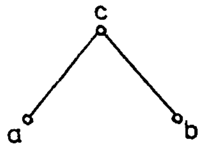

Fig. 1

Definition 2. a) A subset M of an ordered set A is said to be an *LU-subset* of A, if for each $a, b\in M$:

(i) $L_M(a, b)=\varnothing$ if and only if $L_A(a, b)=\varnothing$;

(ii) $U_M(a, b)=\varnothing$ if and only if $U_A(a, b)=\varnothing$.

b) A subsemilattice M of a semilattice $A=(A, \vee)$ which is an LU-subset of A (i.e. M satisfies the condition (i)) is called an LU-subsemilattice of A.

Theorem 2. *Let a semilattice $A=(A, \vee)$ do not be o-distributive. Then it contains an LU-subsemilattice isomorphic to one of the ordered sets M_2, M_4, N_3, N_4.* (See Fig. 2.)

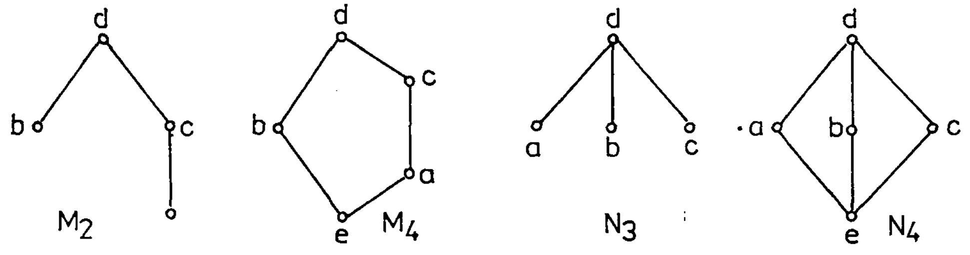

Fig. 2

Proof. If a semilattice A is not o-distributive, then there exist $a, b, c \in A$ such that

$$L(U(L(a, c), L(b, c))) \subset L(a \vee b, c).$$

I. Let $a < c$. Then $L(U(L(a, c), L(b, c))) = L(U(a, L(b, c)))$, and thus $L(U(a, L(b, c))) \subset L(a \vee b, c)$. Clearly $a \| b$, $b \| c$.

(a) Firstly let us suppose $L(b, c) = \varnothing$. Then there exists $x \in L(a \vee b, c)$ such that $x \not\leqq a$.

(α) Let $x > a$. Then $a \vee b = b \vee x$, $a \vee b > b$, $b \| x$. From that we also have $a \vee b > x$. Therefore the set $T_1 = \{a, b, x, a \vee b\}$ is a subsemilattice of A. Furthermore $L(a, b) \subseteq \subseteq L(b, x) \subseteq L(b, c) = \varnothing$, hence T_1 is an LU-subsemilattice of A isomorphic to M_2.

(β) Let $x \| a$. Let us denote $T_2 = \{a, b, a \vee x, a \vee b\}$. We have $a \vee x \leqq a \vee b$ and $a < a \vee x$. Furthermore $a \vee b \not\leqq c$. In the case $c < a \vee b$, we obtain $a \vee b \geqq \geqq a \vee x$, in the case $c \| a \vee b$, we have $a \vee x < c$, $a \vee x < a \vee b$. Therefore it always holds $a \vee x < a \vee b$. In addition, we have $b < a \vee b$. Let us show that $b \| a \vee x$. In fact, if $a \vee x \leqq b$, then $a < b$, a contradiction, and if $b < a \vee x$, then $a \vee b \leqq a \vee x$, a contradiction, too.

Therefore T_2 is a subsemilattice of A, and because $L(a, b) \subseteq L(b, a \vee x) \subseteq \subseteq L(b, c) = \varnothing$, T_2 is an LU-subsemilattice of A isomorphic to M_2.

(b) Let now $L(b, c) \neq \varnothing$ and let $v \in L(b, c)$. Since $L(U(a, L(b, c))) \subset L(a \vee b, c)$, there exist $x \in L(a \vee b, c)$, $y \in U(a, L(b, c))$ such that $x \not\leqq y$.

(α) Let $x > y$. Let us denote $T_3 = \{b, x, y, v, a \vee b\}$. Then from $a < x$ we obtain $a \vee b \leqq x \vee b$, and since evidently $x \vee b \leqq a \vee b$, we have $y \vee b = a \vee b$. Further it is clear that $v < b$ and $v < y$. Since $c \| b$, we have $x < a \vee b$. If $b \geqq x$, then $b > a$, and if $b \leqq x$, then $x = a \vee b$, hence it must hold $b \| x$. Analogously we can prove $b \| y$. But this means that T_3 is an LU-subsemilattice of A isomorphic to M_4.

(β) Let $x \| y$. Let us denote $T_4 = \{b, a \vee v, x \vee a \vee v, v, a \vee b\}$. Since $v < b$, $x \leqq a \vee b$ and $a < b$, we have $x \vee a \vee v \leqq a \vee b$. Let us suppose $x \vee a \vee v = a \vee b$. Then $x \vee a \vee v > b$, hence $c \vee x \vee a \vee v \geqq b \vee c$. But $c \vee x \vee a \vee v = c$, therefore $c \geqq b$, a contradiction. Thus it must be $x \vee a \vee v < a \vee b$.

Since $x \| y$, we obtain $x \not\leqq a \vee v$, hence $x \vee a \vee v \neq a \vee v$, and so $a \vee v < x \vee a \vee v$. Further it is evident that $v < a \vee v$, $v < b$, $b < a \vee b$. At the same time, if $b \geqq a \vee v$, then $b \geqq a$, and if $b \leqq a \vee v$, then $b \leqq c$, a contradiction. Thus $b \| a \vee v$. Similarly $x \vee a \vee v \| b$.

Therefore T_4 is an LU-subsemilattice of A isomorphic to M_4.

II. Now, we shall observe the case $a \| c$. It is evident that then $a \| b$ and $c \not\leqq b$. We can suppose $b \| c$, otherwise we would obtain the same results as for the case I.

(a) First let us suppose $a \vee b < a \vee b \vee c$, $a \vee c < a \vee b \vee c$, $b \vee c < a \vee b \vee c$.

(α) Let $L(a, b) = L(a, c) = L(b, c) = \varnothing$. Then $L(U(L(a, c), L(b, c))) = \varnothing$, but

$L(a\vee b, c)\neq\emptyset$. Let $x\in L(a\vee b, c)$. Then $R_1=\{x, a\vee b, a\vee c, b\vee c, a\vee b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_4.

(β) If e.g. $L(a, b)\neq\emptyset$, $d\in L(a, b)$, then $R_2=\{d, a\vee b, a\vee c, b\vee c, a\vee b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_4.

(b) Let $a\vee b=a\vee b\vee c$, $a\vee c<a\vee b$, $b\vee c<a\vee b$.

(α) Let $L(a, b)=L(a, c)=L(b, c)=\emptyset$. If $L(a\vee c, b)=\emptyset$, then $R_3=\{a, b, a\vee c, a\vee b\}$ is an LU-subsemilattice of A isomorphic to M_2.

If $L(a\vee c, b)\neq\emptyset$, $d\in L(a\vee c, b)$, then $R_4=\{d, b, a\vee c, b\vee c, a\vee b\}$ is an LU-subsemilattice of A isomorphic to M_4.

(β) If $L(a, b)\neq\emptyset$, $e\in L(a, b)$, then $R_5=\{e, b, a\vee c, b\vee c, a\vee b\}$ is an LU-subsemilattice of A isomorphic to M_4.

(γ) If e.g. $L(a, c)\neq\emptyset$, $f\in L(a, c)$, then $R_6=\{f, a, a\vee c, b\vee c, a\vee b\}$ is an LU-subsemilattice of A isomorphic to M_4.

(c) Let us suppose $a\vee b=a\vee c=a\vee b\vee c$, $b\vee c<a\vee b$.

(α) Let $L(a, b)=L(a, c)=L(b, c)=\emptyset$. If $L(a, b\vee c)=\emptyset$, then $R_7=\{a, b, b\vee c, a\vee b\}$ is an LU-subsemilattice of A isomorphic to M_2.

Let $L(a, b\vee c)\neq\emptyset$, $g\in L(a, b\vee c)$. Then $L(b, g)=L(c, g)=\emptyset$. If $b\vee g=c\vee g=b\vee c$, then $R_8=\{b, g, c, b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_3. If $b\vee g<b\vee c$, then $R_9=\{g, b\vee g, b\vee c, a\vee b, a\}$ is an LU-subsemilattice of A isomorphic to M_4.

(β) Let $L(a, b)\neq\emptyset$, $h\in L(a, b)$. Then $R_{10}=\{h, b, b\vee c, a\vee b, a\}$ is an LU-subsemilattice of A isomorphic to M_4. (Similarly for $L(a, c)\neq\emptyset$.)

(γ) Let $L(a, b)=L(a, c)=\emptyset$, $L(b, c)\neq\emptyset$. If $L(a, b\vee c)=\emptyset$, then R_7 is an LU-subsemilattice of A. Suppose $L(a, b\vee c)\neq\emptyset$, $g\in L(b, c)$, $h\in L(a, b\vee c)$. We have $h\vee g\not\equiv b$, $h\vee g\not\equiv c$, $h\vee g\leqq b\vee c$. Let $b<h\vee g$. If $h\vee g<b\vee c$, then $R_{11}=\{h, h\vee g, b\vee c, a\vee b, a\}$ is an LU subsemilattice of A isomorphic to M_4. If $h\vee g=b\vee c$, then $R_{12}=\{g, h, b, b\vee c\}$ is an LU-subsemilattice of A isomorphic to M_2. (For $c<h\vee g$, we can prove similarly.)

Let $b\| h\vee g$, $c\| h\vee g$. If $b\vee h\vee g=b\vee c$ and $c\vee h\vee g=b\vee c$, then $R_{13}=\{h, b, h\vee g, c, b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_4. If $b\vee h\vee g<b\vee c$ or $c\vee h\vee g<b\vee c$, respectively, then $R_{14}=\{h, c, b, b\vee h\vee g, b\vee c\}$ or $R_{15}=\{h, b, c, c\vee h\vee g, b\vee c\}$, respectively, is an LU-subsemilattice of A isomorphic to M_4.

(d) The case $a\vee c=b\vee c=a\vee b\vee c$, $a\vee b<a\vee c$ can be proved analogously as the case (c).

(e) Let us suppose $a\vee b=a\vee c=b\vee c=a\vee b\vee c$.

(α) If $L(a, b)=L(a, c)=L(b, c)=\emptyset$, then $R_{16}=\{a, b, c, a\vee b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_3.

(β) Let e.g. $L(a, b)\neq\emptyset$, $d\in L(a, b)$. If $d\vee c<a\vee b\vee c$, then $R_{17}=\{d, a, b, d\vee c, a\vee b\vee c\}$ is an LU-subsemilattice of A isomorphic to N_4.

Let $d \vee c = a \vee b \vee c$ and let $L(b, c) = \emptyset$ or $L(a, c) = \emptyset$, respectively. Then $R_{18} = \{d, b, c, a \vee b \vee c\}$ or $R'_{18} = \{d, a, c, a \vee b \vee c\}$, respectively, is an LU-subsemilattice of A isomorphic to M_2.

Finally, let us observe the case $L(a, b) \neq \emptyset$, $L(a, c) \neq \emptyset$, $L(b, c) \neq \emptyset$. Let $d \in L(a, b)$, $e \in L(a, c)$, $f \in L(b, c)$. If e.g. $L(e, f) \neq \emptyset$, $g \in L(e, f)$, then $R_{19} = \{g, a, b, c, a \vee b \vee c\}$ is an LU-subsemilattice of A isomorphic to N_4. Hence, let $L(d, e) = L(d, f) = L(e, f) = \emptyset$. Since $L(a \vee b, c) = L(c)$, it exists (by the assumption) an element $x \in U(L(a, c), L(b, c))$ such that $c \nleqq x$. For x we have $x \geqq e$, $x \geqq f$, thus it must be $c > e \vee f$. If now $a \vee f > c$, then $R_{20} = \{e, a, e \vee f, c, a \vee f\}$ is an LU-subsemilattice of A isomorphic to M_4.

Let $a \vee f \parallel c$. If $a \vee f > a$, then $R_{21} = \{e, a, a \vee f, c, a \vee b \vee c\}$ is an LU-subsemilattice of A isomorphic to M_4. If $a \vee f = a$, then $R_{22} = \{f, a, b, c, a \vee b \vee c\}$ is an LU-subsemilattice of A isomorphic to N_4.

All remaining possibilities of the connections among a, b, c would lead to some variants of the preceding cases only.

Remark 4. In [2] it is proved for any ordered set A that if A is non-distributive, then it contains an LU-subset isomorphic to some of ordered sets M_1, M_2, M_3, M_4, M_5, M_6, N_1, N_2, N_3, N_4, N_5. (See Fig. 2 and 3.)

But for the case of semilattices, the constructions of respective LU-subsets from [2] do not lead to subsemilattices.

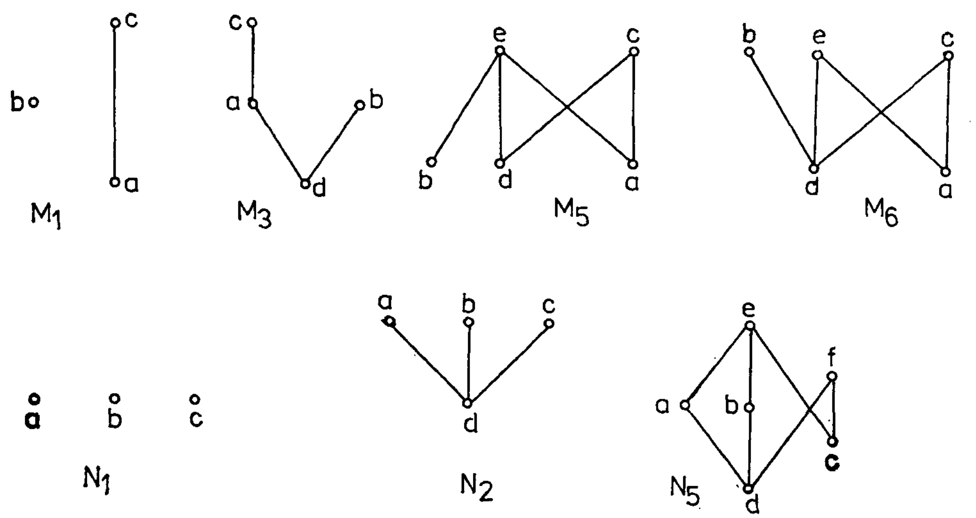

Fig. 3

Definition 3. A subset M of an ordered set A is called *strong* if for any $a, b \in M$ it holds:

(i) $L_A(U_M(a, b)) = L_A(U_A(a, b))$;

(ii) $U_A(L_M(a, b)) = U_A(L_A(a, b))$.

In [2] it is shown that if M is a strong subset of A such that $U_A(a, b) \neq \{1\}$ and $L_A(a, b) \neq \{0\}$ (where 1 or 0 denotes the greatest or the least element of A, respectively — if they exist), then M is an LU-subset of A. Furthermore, any strong subset of an ordered set A which is a semilattice with respect to the induced order, is a subsemilattice of A.

Therefore, the following theorem is similar to the converse of Theorem 2.

Theorem 3. *If a semilattice $A=(A, \vee)$ contains an LU-subsemilattice isomorphic to M_2 or to N_3, respectively, or if it contains a strong subsemilattice isomorphic to M_4 or to N_4, respectively, then A is non-o-distributive (and so non-distributive, too).*

Proof. The assertion follows from [2, Theorems 4 and 7]. It is clear that the non-distributivity of A for the cases of the strong subsemilattices M_2 and M_3 also directly follows from the fact that A is not (in those cases) lower directed.

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