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State extensions in transformation group C*-algebras

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Introduction. Let X be a compact (Hausdorff) space and G a discrete group acting on X as homeomorphisms: $x \to t(x)$, $(x \in X,\ t \in G)$. Throughout this paper we denote by A the C*-crossed product associated with the topological dynamics (G, X). Our purpose is to study state extensions in A. Since G is discrete, the algebra $C(X)$ of all continuous functions on X is regarded as a C*-subalgebra of A and the restriction of a state of A is a state of $C(X)$ again. We are interested in the correspondence of the family of states of $C(X)$ with that of A. So we study how to extend a state μ of $C(X)$ to a state or a tracial state of A. Of course μ is identified with a probability measure (throughout this paper, a measure means a Borel measure, which is always regular on the compact space X). Ultimately we get an equivalent condition for a probability measure and a G-invariant probability measure on X to be uniquely extended to a state and a tracial state of A respectively.

In Section 1, we prove that a probability measure μ on X has a unique state extension if and only if the measure $\mu(t(\cdot))$ is singular with repect to μ for all t in G except $t=e$. In Section 2, we prove that a G-invariant probability measure μ on X has a unique tracial state extension if and only if $\mu(X^t)=0$ for all t in G except $t=e$, where X^t is the set of fixed points of X for t. In the theory of C*-algebras, the unique tracial state plays an important rôle (cf. [6], [7]). Hence it seems to be interesting to consider the condition on $(G \cdot X)$ under which A has a unique tracial state. Those conditions are given as an application of our second result.

Notation. For a topological dynamics (G, X), we use s, t, u, m, n, e (=the identity) and x, y as elements of G and X respectively. We denote by G_x the isotropy group for x and X^t the set of fixed points for t, i.e., $G_x=\{t \in G\colon\ t(x)=x\}$ and $X^t = \{x \in X\colon\ t(x)=x\}$. The algebra $C(X)$ is the abelian C*-algebra with supremum norm and *-operation: $f^*(x)=\overline{f(x)}$, where the bar means complex conjugate. We denote by α_t the canonical *-automorphim of $C(X)$ induced by the action of t in G, i.e.,

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$\alpha_t(f)(x)=f(t^{-1}(x))$ for f in $C(X)$. Let $K(G, C(X))$ be the set of those functions of G into $C(X)$ which vanish outside finitely many elements. For t in G and f in $C(X)$, $f\delta_t$ means the function in $K(G, C(X))$ defined by $(f\delta_t)(t)=f$ and $(f\delta_t)(s)=0$ for $s\neq t$. Then every function Φ in $K(G, C(X))$ is of the form: $\Phi=\sum_{t\in F} f_t\delta_t$, where F is a finite subset of G. We consider $K(G, C(X))$ as a dense *-subalgebra of A by defining *-operation and multiplication as follows; $(\sum_{t\in F} f_t\delta_t)^*=\sum_{t\in F}\alpha_{t^{-1}}(\bar{f})\delta_{t^{-1}}$ and $(\sum_{t\in F_1} f_t\delta_t)(\sum_{s\in F_2} g_s\delta_s)=\sum_{t\in F_1}\sum_{s\in F_2} f_t\alpha_t(g_s)\delta_{ts}$.

For a measure μ on X the topological support $S(\mu)$ of μ means the smallest closed subset such that $\mu(f)=0$ for f in $C(X)$ with $\operatorname{supp}(f)\subset X-S(\mu)$. Given a family $\{\mu_t\}_{t\in G}$ of measures, we denote by $\psi=\bigoplus_{t\in G}\mu_t$ the linear functional on $K(G, C(X))$ defined by

$$\psi\left(\sum_{t\in F} f_t\delta_t\right)=\sum_{t\in F}\mu_t(f_t).$$

Since G is discrete, if ψ is positive definite then it is transform bounded on $K(G, C(X))$ in the sense of [3]. Hence ψ can be extended to a state of A, which is denoted by ψ again.

The action of G on X determines, addition to $\{\alpha_t\}$, a canonical transformation group on the state space of $C(X)$. Those are denoted by $\beta_t(\mu)$ for a state μ on $C(X)$, i.e., $\beta_t(\mu)(f)=\mu(\alpha_{t^{-1}}(f))$ for f in $C(X)$, which is regarded as a measure on X defined by $\beta_t(\mu)(E)=\mu(t^{-1}(E))$ for each Borel set E in X.

1. State extensions. Let ψ be a state of A. For each element t in G, let μ_t denote the bounded linear functional of $C(X)$ defined by $\mu_t(f)=\psi(f\delta_t)$ for f in $C(X)$. Then it follows that $\psi=\bigoplus_{t\in G}\mu_t$.

Proposition 1.1. *Let $\psi=\bigoplus_{t\in G}\mu_t$ be a state of A. Then $\{\mu_t\}$ has the following properties:*

(1) *μ_e is a probability measure on X,*

(2) *μ_t is absolutely continuous with respect to μ_e and $\beta_t(\mu_e)$, and $S(\mu_t)\subset S(\mu_e)\cap t(S(\mu_e))$,*

(3) $\mu_{t^{-1}}(f)=\overline{\mu_t(\alpha_t(\bar{f}))}$.

Proof. (1) is trivial.

(2) By the Cauchy—Schwarz inequality, we have, for f in $C(X)$,

(a) $|\mu_t(f)|^2=|\psi(f\delta_t)|^2\leqq\psi(\delta_{t^{-1}}\delta_t)\psi(f\delta_e f\delta_e)=\mu_e(|f|^2)$

and

(b) $|\mu_t(f)|^2=|\psi(f\delta_t)|^2=\left|\psi\left(\delta_t\alpha_{t^{-1}}(f)\delta_e\right)\right|^2\leqq\psi\left(\alpha_{t^{-1}}(f)\overline{\alpha_{t^{-1}}(f)}\delta_e\right)\psi(\delta_e)=$
$=\mu_e\left(|\alpha_{t^{-1}}(f)|^2\right).$

By inequality (a) and the regularity of μ and μ_t, we have that μ_t is absolutely continuous with respect to μ_e and $S(\mu_t)\subset S(\mu_e)$. For f in $C(X)$ with $\mathrm{supp}(f)\subset X-t(S(\mu_e))$, it follows that $\mathrm{supp}(\alpha_{t^{-1}}(f))=t^{-1}(\mathrm{supp}(f))\subset t^{-1}(X-t(\mathrm{supp}(\mu_e))=X-\mathrm{supp}(\mu_e)$. By inequality (b), we have that μ_t is absolutely continuous with respect to $\beta_t(\mu_e)$ and $S(\mu_t)\subset t(S(\mu_e))$.

(3) Let $\Phi=\sum_{t\in F} f_t\delta_t$ be in $K(G, C(X))$, where $F=\{t_1, \ldots, t_n\}$. Then $\psi(\Phi^*\Phi)=\sum_{t,s\in F}\mu_{t^{-1}s}(\alpha_{t^{-1}}(\bar{f}_t f_s))\geqq 0$. Given each set of complex numbers $\{\lambda_i\}_{i=1}^n$, setting $\lambda_i f_{t_i}$ in place of f_{t_i}, we have

$$\sum_{i,j=1}^{n}\mu_{t_i^{-1}t_j}(\alpha_{t_i^{-1}}(\bar{f}_{t_i}f_{t_j}))\bar{\lambda}_i\lambda_j\geqq 0.$$

This means that the $n\times n$ matrix $(\mu_{t_i^{-1}t_j}(\alpha_{t_i^{-1}}(\bar{f}_{t_i}f_{t_j})))_{ij}$ is positive. Hence $\mu_{t^{-1}s}(\alpha_{t^{-1}}(\bar{f}_t f_s))=\overline{\mu_{s^{-1}t}(\alpha_{s^{-1}}(\bar{f}_s f_t))}$. Putting $s=e$ and $f_t=1$, $f_e=\alpha_t(f)$, we have $\mu_{t^{-1}}(f)=\overline{\mu_t(\alpha_t(\bar{f}))}$.

Let μ be a probability measure on X and ε the conditional expectation of A onto $C(X)$. Let $\tilde{\mu}(=\mu\circ\varepsilon)$ denote the canonical state extension of μ. In order to find a condition under which $\tilde{\mu}$ is the unique state extension, we consider the possibility of existence of another extension of μ.

Given a measure μ on X and a characteristic function χ_E of a Borel set E in X, we define a measure $\chi_E\mu$ on X by

$$\chi_E\mu(f)=\int_E f\,d\mu \quad \text{for } f \text{ in } C(X) \quad (=\mu(\chi_E f)).$$

Then we have

$$\chi_E\beta_t(\mu)(f)=\int_{t^{-1}(E)} f(t(x))\,d\mu \quad (=\mu(\chi_{t^{-1}(E)}\alpha_{t^{-1}}(f))),$$

and it is easy to see that $S(\chi_E\beta_t(\mu))=\bar{E}\cap t(S(\mu))$, where $\bar{E}$ is the closure of E in X.

For t in G, let $\beta_t(\mu)=\beta_t(\mu)_a+\beta_t(\mu)_s$ be the Lebesgue decomposition of the positive measure $\beta_t(\mu)$ with respect to μ. Namely there exists a measurable subset C_t of X satisfying the condition: $\mu(X-C_t)=0$ and, for each Borel set E in X with $E\subset C_t$,

$$\beta_t(\mu)(E)=\beta_t(\mu)_a(E)=\int_E k_t(x)\,d\mu,$$

where k_t is the Radon—Nikodym derivative of $\beta_t(\mu)_a$ with respect to μ. Let $D_t=\{x\in C_t\colon k_t(x)>0\}$, $E_t=\{x\in C_t\colon k_t(x)\geqq 1\}$ and $F_t=\{x\in C_t\colon k_t(x)\leqq 1\}$. Since $k_{t^{-1}}(x)=1/k_t(t(x))$ for x in D_t, it follows that $t^{-1}(D_t)=D_{t^{-1}}$, $t^{-1}(E_t)=F_{t^{-1}}$ and $t^{-1}(F_t)=E_{t^{-1}}$. Using those facts we prove the following proposition, and applying it we show a characterization for μ to have a unique state extension.

Proposition 1.2. *Let μ be a positive measure on X. For a fixed t in G, let $\{\mu_s\}_{s\in G}$ be the family of measures on X defined as follows:*

(1) *In case $t\neq t^{-1}$, let $\mu_e=\mu$, $\mu_t=\chi_{E_t}\mu/2$, $\mu_{t^{-1}}=\chi_{F_{t^{-1}}}\beta_{t^{-1}}(\mu)/2$ and $\mu_s=0$ for $s\notin\{e, t, t^{-1}\}$,*

(2) *In case $t=t^{-1}$, let $\mu_e=\mu$, $\mu_t=(\chi_{E_t}\mu+\chi_{F_t}\beta_t(\mu))/2$ and $\mu_s=0$ for $s\notin\{e, t\}$.*

Then $\psi=\bigoplus_{s\in G}\mu_s$ is positive definite.

Proof. It is sufficient to prove the statement only in the case of (1). Let $\Phi=\sum_{m\in F} f_m\delta_m$ be in $K(G, C(X))$. Then we have

$$2\psi(\Phi^*\Phi) = 2\mu\Big(\sum_{m\in F}\alpha_{m^{-1}}(\bar f_m f_m)\Big)+\mu_t\Big(\sum_{m^{-1}n=t}\alpha_{m^{-1}}(\bar f_m f_n)\Big)+\mu_{t^{-1}}\Big(\sum_{m^{-1}n=t^{-1}}\alpha_{m^{-1}}(\bar f_m f_n)\Big).$$

The second term of the right hand side $=\displaystyle\int_{E_t}\sum_{m^{-1}n=t}\overline{f_m(m(x))}f_n(m(x))\,d\mu.$

The third term $=\displaystyle\sum_{m^{-1}n=t^{-1}}\chi_{F_{t^{-1}}}\beta_{t^{-1}}(\mu)\big(\alpha_{m^{-1}}(\bar f_m f_n)\big)=$

$$=\int_{t(F_{t^{-1}})}\sum_{m^{-1}n=t^{-1}}\alpha_t\alpha_{m^{-1}}(\bar f_m f_n)\,d\mu=\int_{E_t}\sum_{m^{-1}n=t^{-1}}\alpha_{n^{-1}}(\bar f_m f_n)\,d\mu=$$

$$=\int_{E_t}\sum_{m^{-1}n=t^{-1}}\overline{f_m(n(x))}f_n(n(x))\,d\mu=\int_{E_t}\sum_{m^{-1}n=t}\overline{f_n(m(x))}f_m(m(x))\,d\mu.$$

If $m^{-1}n=t$, since $k_{t^{-1}}(x)\leqq 1$ for x in $F_{t^{-1}}$, we have

$$\int_{F_{t^{-1}}}\alpha_{n^{-1}}(\bar f_n f_n)\,d\mu\geqq\int_{F_{t^{-1}}}\alpha_{n^{-1}}(\bar f_n f_n)k_{t^{-1}}\,d\mu=\int_{F_{t^{-1}}}\alpha_{n^{-1}}(\bar f_n f_n)\,d\beta_{t^{-1}}(\mu)_a=$$

$$=\int_{F_{t^{-1}}}\alpha_{n^{-1}}(\bar f_n f_n)\,d\beta_{t^{-1}}(\mu)=\int_{t(F_{t^{-1}})}\alpha_t\big(\alpha_{n^{-1}}(\bar f_n f_n)\big)\,d\mu=\int_{E_t}\alpha_{tn^{-1}}(\bar f_n f_n)\,d\mu=$$

$$=\int_{E_t}\alpha_{m^{-1}}(\bar f_n f_n)\,d\mu=\int_{E_t}\overline{f_n(m(x))}f_n(m(x))\,d\mu.$$

Hence, the first term $\geqq\displaystyle\sum_{m^{-1}n=t}\Big(\int_{E_t}\alpha_{m^{-1}}(\bar f_m f_m)\,d\mu+\int_{F_{t^{-1}}}\alpha_{n^{-1}}(\bar f_n f_n)\,d\mu\Big)\geqq$

$$\geqq\int_{E_t}\sum_{m^{-1}n=t}\big(\overline{f_m(m(x))}f_m(m(x))+\overline{f_n(m(x))}f_n(m(x))\big)\,d\mu.$$

Therefore it follows that

$$2\psi(\Phi^*\Phi) \geqq \int_{E_t} \sum_{m^{-1}n=t} \big(\overline{f_m(m(x))}f_m(m(x))+\overline{f_m(m(x))}f_n(m(x))+$$

$$+\overline{f_n(m(x))}f_m(m(x))+\overline{f_n(m(x))}f_n(m(x))\big)\,d\mu =$$

$$= \int_{E_t} \sum_{m^{-1}n=t} \overline{(f_m(m(x))+f_n(m(x)))}\,(f_m(m(x))+f_n(m(x)))\,d\mu \geqq 0.$$

Theorem 1.3. *Let μ be a probability measure on X. Then μ has a unique state extension if and only if $\beta_t(\mu)$ is singular with respect to μ for all t in G except $t=e$.*

Proof. Let $\psi = \bigoplus_{t\in G} \mu_t$ be a state extension of μ. By (2) of Proposition 1.1, each μ_t is absolutely continuous with respect to $\mu_e=\mu$ and $\beta_t(\mu)$. Hence the assumption on $\{\beta_t(\mu)\}_{t\in G}$ implies that $\mu_t=0$ for all $t\neq e$.

Next suppose that $\beta_t(\mu)$ is not singular with respect to μ for some $t\neq e$. Set $\psi=(\psi_t+\psi_{t^{-1}})/2$, where ψ_t and $\psi_{t^{-1}}$ are the states constructed in the above proposition corresponding to t and t^{-1} respectively. If $\mu(E_t)=0$, then $\mu(F_t)>0$, whereas we have

$$\mu(E_{t^{-1}}) = \mu(t^{-1}(F_t)) = \beta_t(\mu)_a(F_t) = \int_{F_t} k_t(x)\,d\mu > 0.$$

Hence ψ is a state extension of μ, which is different from $\tilde{\mu}=\mu\circ\varepsilon$.

In the following, we give an example of a state of $C(X)$ which has a unique state extension and whose topological support is the full space X.

Example 1.4. Let R_θ be an irrational rotation on the unit circle $[0, 1)$. Let $\{r_n\}_{n=1}^{\infty}$ be the set of all rational numbers in $[0, 1)$. We define a probability measure μ_Q on $[0, 1)$ by $\mu_Q(E)= \sum_{r_n\in E} 1/2^n$ for $E\subset[0, 1)$. Then $\{\beta_n(\mu_Q)\}_{n\in Z}$ are mutually singular and $S(\beta_n(\mu_Q))=[0, 1)$ for all n in Z. Namely μ_Q has a unique state extension but $S(\beta_n(\mu_Q))$ is the full space.

The theorem mentioned above gives a characterization for the pure state $\mu_{\{x\}}$ of $C(X)$ to have a unique pure state extension. Namely we have the following.

Corollary 1.4. *Let $\mu_{\{x\}}$ be the Dirac measure on a point x of X. Then $\mu_{\{x\}}$ has a unique (pure) state extension if and only if $G_x=\{e\}$.*

Here we note that this result can be derived by Lemmas 4.19, 4.22 and 4.25 of [3]. (Though the second countability on G and X was assumed in [3], the proofs of these lemmas are still available here.)

Moreover ANDERSON [1] has given an equivalent condition for a pure state to have a unique pure state extension in a more general case. He proved that for any C*-

subalgebra D of a C*-algebra C, a pure state μ of D has a unique pure state extension to C if and only if C is D-compressible modulo μ, i.e.,

$$\inf\{\|dcd-e\| : d\in D, 0 \leqq d \leqq 1, \mu(d) = 1, e\in D\} = 0$$

for each c in C. Of course, in our case, this condition on $\mu_{\{x\}}$ is equivalent to $G_x=\{e\}$. In the case of state extension, Example 1.4 shows that the condition mentioned above is merely a sufficient condition for μ to have a unique state extension. In fact, since the identity is the only element in $C(X)$ with $\mu_Q(d)=1$, $0\leqq d\leqq 1$, we have $\|dcd-e\|=$ $=\|c-e\|\geqq \mathrm{dist}\,(c, C(X))>0$ for $c\notin C(X)$.

As a matter of course, it is interesting to study representations of A associated with states extended from $\mu_{\{x\}}$. Those are discussed in [4].

2. Tracial state extensions. Let μ be a G-invariant probability measure on X and $\psi=\bigoplus_{t\in G}\mu_t$ an extension of μ. We show a necessary and sufficient condition for μ to have a unique tracial state extension. First we consider the condition on $\psi=\bigoplus_{t\in G}\mu_t$ under which ψ is a tracial state.

Proposition 2.1. *Let $\psi=\bigoplus_{t\in G}\mu_t$ be a state extension of a probability measure μ on X. Then ψ is a tracial state of A if and only if $\{\mu_t\}_{t\in G}$ satisfies the following two conditions:*

(1) $S(\mu_t)\subset X^t$ *for all t in G,*

(2) $\beta_s(\mu_t)=\mu_{sts^{-1}}$ *for all s and t in G.*

Proof. Suppose that ψ is a tracial state. Then for f and g in $C(X)$, we have

$$(*)\quad \mu_{st}(f\alpha_s(g)) = \psi(f\alpha_s(g)\delta_{st}) = \psi(f\delta_s g\delta_t) = \psi(g\delta_t f\delta_s) = \psi(g\alpha_t(f)\delta_{ts}) = \mu_{ts}(g\alpha_t(f))$$

Putting $s=e$ in $(*)$, we have $\mu_t(fg)=\mu_t(g\alpha_t(f))$. If $x\notin X^t$, then there exists a neighbourhood U of x such that $U\cap t^{-1}(U)=\emptyset$. For any non-negative real-valued continuous function f on X with $\mathrm{supp}\,(f)\subset U$, we have $\mu_t(f)=\mu_t(\sqrt{f}\sqrt{f})=$ $=\mu_t(\sqrt{f}\,\alpha_t(\sqrt{f}))=0$. Thus $S(\mu_t)\subset X^t$. Next, put $t=us^{-1}$ and $g=1$ in $(*)$. Then we have

$$\mu_{sus^{-1}}(f) = \mu_u(\alpha_{us^{-1}}(f)) = \int_{X^u} f(su^{-1}(x))\,d\mu_u = \int_{X^u} f(s(x))\,d\mu_u =$$

$$= \mu_u(\alpha_{s^{-1}}(f)) = \beta_s(\mu_u)(f).$$

Conversely we suppose that $\{\mu_t\}_{t\in G}$ satisfies the conditions. Then we have

$$\mu_{st}(f\alpha_s(g)) = \beta_{t^{-1}}(\mu_{ts})(f\alpha_s(g)) = \mu_{ts}(\alpha_t(f)\alpha_{ts}(g)) =$$

$$= \int_{X^{ts}} f(t^{-1}(x))g((ts)^{-1}(x))\,d\mu_{ts} = \int_{X^{ts}} f(t^{-1}(x))g(x)\,d\mu_{ts} = \mu_{ts}(\alpha_t(f)g).$$

This implies that $\psi(f\delta_s g\delta_t)=\psi(g\delta_t f\delta_s)$. By the linearity and the continuity of ψ, $\psi(\Phi\cdot\Psi)=\psi(\Psi\cdot\Phi)$ for every Φ and Ψ in A.

Given a state $\psi=\bigoplus_{t\in G}\mu_t$, in many cases it is easy to check whether the family $\{\mu_t\}_{t\in G}$ satisfies or not the conditions of the above proposition. However, given a family $\{\mu_t\}_{t\in G}$ of measures on X, it is not easy to see whether $\psi=\bigoplus_{t\in G}\mu_t$ is positive definite or not. Here we give a systematic construction of a (tracial) state extension. We denote by $H(G)$ the family of subgroups of G. Let J be a map of X into $H(G)$. We put $X_J^t=\{x\in X\colon J(x)\to t\}$ and denote by χ_t the characteristic function of X_J^t. When X_J^t is a measurable set, $\chi_t\mu$ is the measure on on X defined by $\chi_t\mu(f)=\int_{X_J^t} fd\mu=\int_X f\chi_t d\mu$ for f in $C(X)$.

Proposition 2.2. *Let J be a map of X into $H(G)$ with the properties:* (1) *X_J^t is a Borel set for all t in G, and* (2) *$J(x)\subset G_x$ for all x in X. Let μ be a probability measure on X and $\mu_t=\chi_t\mu$ for each t in G. Then $\psi=\bigoplus_{t\in G}\mu_t$ is positive definite. In addition, if μ is G-invariant and* (3) *$J(t(x))=tJ(x)t^{-1}$ for all t in G, then ψ is a tracial state.*

Proof. Let $\Phi=\sum_{t\in F}f_t\delta_t$ be in $K(G,C(X))$, where F is a finite subset of G. In case $t^{-1}s$ is in $J(x)\subset G_x$, $s(x)=t(x)$. Thus we have

$$\psi(\Phi^*\Phi)=\psi\Big(\sum_{t,s\in F}\delta_{t^{-1}}\bar{f}_t f_s\delta_s\Big)=\psi\Big(\sum_{t,s\in F}\alpha_{t^{-1}}(\bar{f}_t f_s)\delta_{t^{-1}s}\Big)=$$

$$=\sum_{t,s\in F}\int_X\overline{f_t(t(x))}f_s(t(x))\chi_{t^{-1}s}(x)\,d\mu=$$

$$=\int_X\sum_{t,s\in F}\overline{f_t(t(x))}f_s(s(x))\chi_{t^{-1}s}(x)\,d\mu.$$

For x in X, let $F=F_1\cup\ldots\cup F_n$ be the disjoint partition of F corresponding to the equivalent relation determined by the subgroup $J(x)$ of G, i.e., t and s belong to the same F_i if and only if $t^{-1}s\in J(x)$. Then we have

$$\sum_{s,t\in F}\overline{f_t(t(x))}f_s(s(x))\chi_{t^{-1}s}(x)=\sum_{i=1}^n\sum_{s,t\in F_i}\overline{f_t(t(x))}f_s(s(x))=$$

$$=\sum_{i=1}^n\Big|\sum_{t\in F_i}f_t(t(x))\Big|^2\geqq 0.$$

Hence $\psi(\Phi^*\Phi)$ is the integral of the non-negative function on X, so it follows that $\psi(\Phi^*\Phi)\geqq 0$.

Next we assume the additional condition. Then we have $s(X_J^t)=X_J^{sts^{-1}}$ for all s and t in G. By the G-invariance of μ, we get the following; for f in $C(X)$,

$$\beta_s(\mu_t)(f)=\mu_t(\alpha_{s^{-1}}(f))=\int_{X_J^t} f(s(x))\,d\mu=\int_{X_J^{sts^{-1}}} f(y)\,d\mu=\mu_{sts^{-1}}(f).$$

Since $S(\mu_t)\subset X^t$ by Condition (2), Proposition 2.1 implies that ψ is a tracial state.

In the following, we show several examples of J, μ and ψ treated in Proposition 2.2.

Example 2.3. Let $J(x)=\{e\}$ for all x in X. Then $X_J^e=X$ and $X_J^t=\emptyset$ for all $t\neq e$. Hence $\psi=\tilde{\mu}$ for each positive measure μ on X.

Example 2.4. Let x be a point of X and $\mu=\mu_{\{x\}}$. Let $J(x)=G_x$ and $J(y)=\{e\}$ for $y\neq x$. Then $X_J^t=\{x\}$ for $t(\neq e)\in G_x$ and $X_J^t=\emptyset$ for $t\notin G_x$. Then $\psi=\bigoplus_{t\in G_x}\mu_t$ is a pure state extension of $\mu_{\{x\}}$ (cf. Section 1).

Example 2.5. Let X consist of a single point $\{x\}$ and $J(x)=H$ for a (resp. normal) subgroup H of G. Then $X_J^t=X$ for $t\in H$ and $X_J^t=\emptyset$ for $t\notin H$. Since A is regarded as the group C*-algebra $C^*(G)$, ψ becomes a (resp. tracial) state of $C^*(G)$ with the property $\psi(\Phi)=\sum_{t\in H}\Phi(t)$ for $\Phi\in l^1(G)\subset C^*(G)$. In the case $H=\{e\}$, ψ is the conditional expectation ε. On the other hand, when $H=G$, ψ is a multiplicative linear functional of A, i.e., it is the trivial representation of $C^*(G)$.

Example 2.5 gives two typical tracial states of $C^*(G)$. However, in contrast to $C^*(G)$, the reduced C*-algebra $C_r^*(G)$ does not necessarily have two tracial states. In fact, Powers [10] has shown that the conditional expectation is the unique tracial state of $C_r^*(F_2)$ by using his result that $C_r^*(F_2)$ is simple.

Example 2.6. Let $J(x)=G_x$ for each x in X. Then we have $X_J^t=X^t$. Thus, if μ is a probability measure then $\psi=\bigoplus_{t\in G}\chi_{X^t}\mu$ is a state extension of μ. In addition, if μ is G-invariant then $\beta_s(\chi_{X^t}\mu)=\chi_{X^{sts^{-1}}}\mu$. Hence, by Proposition 2.1, ψ is a tracial state.

We get the following theorem by Proposition 2.1 and Example 2.6.

Theorem 2.7. *Let μ be a G-invariant probability measure on X. Then μ has a unique tracial state extension if and only if $\mu(X^t)=0$ for all t except $t=e$.*

Corollary 2.8. *The C*-crossed product A has a unique tracial state if and only if there exists exactly one G-invariant probability measure on X and $\mu(X^t)=0$ for all t except $t=e$.*

The standard theory of topological dynamics (cf. Chapter II of [2]) shows that the two conditions on (G, X) in Corollary 2.8 are independent. Now, in the theory of C*-algebras, faithful tracial states such as the unique tracial state of $C^*_r(F_2)$ have played an especially important rôle. Thus we consider faithfulness of tracial state extensions. In general, $\psi=\bigoplus_{t\in G}\chi_t\mu$ in Proposition 2.2 is not necessarily faithful. In fact the canonical homomorphism of $C^*(G)$ (cf. Example 2.5) is not faithful. Here let us assume that G is amenable. Let μ be a G-invariant faithful measure on X. Then the tracial state extension $\tilde{\mu}=\mu\circ\varepsilon$ is faithful because the GNS representation of A by $\tilde{\mu}$ is nothing but the C*-reduced crossed product on the Hilbert space $l^2(G)\otimes L^2(X,\mu)$, which is isomorphic to A (Theorem 7.7.7 of [5]). For $\psi=\bigoplus_{t\in G}\mu_t$ and $1>\omega>0$, let $\psi_\omega=\omega\tilde{\mu}+(1-\omega)\psi$. Then ψ_ω is a tracial state extension of μ and $\omega\tilde{\mu}\leqq\psi_\varepsilon$. Therefore ψ_ω is faithful on A. Then we get the following.

Corollary 2.9. *Suppose that G is amenable. Then A has a faithful unique tracial state if and only if there is exactly one G-invariant measure μ on X, which satisfies the properties:* (1) $S(\mu)=X$ *and* (2) $\mu(X^t)=0$ *for all t except $t=e$.*

If the support of the unique G-invariant measure is X, then (G, X) is minimal (cf. Chapter II (Exercise 7) of [2]). In addition, if G is abelian, $X^t=\emptyset$ since X^t is G-invariant. Then we have the following.

Corollary 2.10. *Suppose that G is abelian. Then A has a faithful unique tracial state if and only if there is exactly one G-invariant measure on X with $S(\mu)=X$.*

We note that the unique tracial state of the rotation C*-algebra is a prototype of Corollary 2.10 and a motivation of our discussion.

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