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The distance to operators with a fixed index

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1. Introduction. Let H be a fixed complex separable Hilbert space. For any (bounded linear) operator T on H we define the nullity and deficiency, denoted nul T and def T, to be the dimensions of the kernels of T and T^*, respectively. Of course, the index of T, denoted ind T, is defined to be (nul $T-$def T), with $\infty-\infty$ understood to be 0. We denote the operator norm of T by $\|T\|$.

In [2] the basic properties of the minimum modulus and the essential minimum modulus were developed; the distances from an arbitrary operator to the invertible operators, denoted G, and to the Fredholm operators were determined using the essential minimum modulus. In [1] the methods of [2] were extended to compute the distance from T to the semi-Fredholm operators with index n. In [3] the conclusions and some methods from [2] were used to compute the distance from T to the Fredholm operators with index n, which we denote F_n. Unfortunately the false assertion that $(S^n S^{*(n)}G)^- = \bar{G}$ in the proof given in [3] leaves a gap in the argument. In this note we give a rather brief proof that establishes the results of [3] plus some new conclusions. Part of the method is a refinement of a device in [3]. Other papers that continue the research in [2] are [4] and [5].

2. Preliminaries. Let J_n denote the set of operators on H with index equal to the integer n. Let I_n denote all operators T in J_n with a finite value for either nul T or def T. Note that $J_n \supset I_n \supset F_n$ and $J_0 \supset I_0 \supset F_0 \supset G$. It is immediate from Theorem 3 of [2] that $J_0 \subset \bar{G}$ and, consequently,

$$\bar{J}_0 = \bar{I}_0 = \bar{F}_0 = \bar{G}.$$

We use notation like PG for $\{PB : B \in G\}$.

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Lemma 1. *Let S be a unilateral shift on H with multiplicity n (an integer) and let P denote the orthogonal projection SS^*. If $PB=B$ then*

(i) $\operatorname{dist}(B, G) = \operatorname{dist}(B, PG)$

(ii) $\operatorname{dist}(B, F_0) = \operatorname{dist}(B, F_0 \cap PF_0)$

(iii) $\operatorname{dist}(B, I_0) = \operatorname{dist}(B, I_0 \cap PI_0)$.

Proof. For $C \in G$ define C_λ to be $PC+\lambda QC$ where $Q=I-P$ and $\lambda \in (0, 1]$. For any vector $f \in H$ we have

$$\|(B-C_\lambda)f\|^2 = \|(B-PC)f\|^2 + \lambda^2 \|QCf\|^2.$$

It follows that

$$(\|B-PC\|^2 + \lambda^2 \|QC\|^2)^{1/2} \geqq \|B-C_\lambda\| \geqq \|B-PC\|.$$

Thus,

$$\inf \{\|B-C_\lambda\| : 0 < \lambda \leqq 1\} = \|B-PC\|.$$

It is routine to see that C_λ is one-to-one and onto; so $C_\lambda \in G$. This argument shows that

$$\operatorname{dist}(B, G) = \operatorname{dist}(B, PG).$$

Now we prove parts (ii) and (iii). It is readily verified that $PG \subset F_0$ and the containment $PG \subset PF_0$ is obvious. Thus, we have $PG \subset F_0 \cap PF_0$ and

$$\operatorname{dist}(B, PG) \geqq \operatorname{dist}(B, F_0 \cap PF_0) \geqq \operatorname{dist}(B, F_0).$$

Since $\bar{F}_0 = \bar{G}$ we know that

$$\operatorname{dist}(B, G) = \operatorname{dist}(B, F_0).$$

Now it follows that

$$\operatorname{dist}(B, F_0 \cap PF_0) = \operatorname{dist}(B, F_0).$$

The proof of part (iii) is identical to the proof of part (ii).

The next lemma will provide the remaining facts necessary to implement our method of proof for the main result.

Lemma 2. *Let S be a unilateral shift on H with multiplicity n (an integer) and let P denote the orthogonal projection SS^*. Then*

(i) $SI_n = I_0 \cap PI_0$, *and*

(ii) $SF_n = F_0 \cap PF_0$.

Proof. Because S maps H isometrically onto PH the deficiency of SA, for $A \in I_n$, is $(n + \operatorname{def} A)$ while $\operatorname{nul} SA = \operatorname{nul} A$. Thus, SA belongs to I_0 and since $PSA = SA$ we see that SA belongs to PI_0. Thus,

$$SI_n \subset I_0 \cap PI_0.$$

If the range of A is closed then the range of SA is closed and

$$SF_n \subset F_0 \cap PF_0.$$

Take $B \in I_0 \cap PI_0$ and let $A = S^*B$. Since $PB = B$, def B and nul B are not less than n. Because S^* maps PH isometrically onto H, it follows that

$$\operatorname{def} A = \operatorname{def}(S^*B) = \operatorname{def} B - n.$$

Since S is an isometry, we get

$$\operatorname{nul} A = \operatorname{nul}(SA) = \operatorname{nul}(SS^*B) = \operatorname{nul} B$$

and so

$$\operatorname{ind} A = n \quad \text{or} \quad A \in I_n.$$

Clearly $SA = B$ and we have proved that

$$SI_n = I_0 \cap PI_0.$$

The argument in the preceding paragraph shows that if $B \in F_0 \cap PF_0$ then $A \in F_n$ and consequently

$$SF_n = F_0 \cap PF_0.$$

3. Main results.

Theorem 3. *Let A be an operator on H and let n represent an integer. If $A \notin I_n$ then*

(i) $\operatorname{dist}(A, I_n) = \max\{m_e(A), m_e(A^*)\}$

(ii) $\operatorname{dist}(A, F_n) = \max\{m_e(A), m_e(A^*)\}$

(iii) $\operatorname{dist}(A, J_n) = \max\{m_e(A), m_e(A^*)\}$.

Proof of (i). Let n be a positive integer and let S be a unilateral shift on H with multiplicity of n. Let A be an operator belonging to I_m for $m \neq n$ and define B by $B = SA$. If π projects the ring of operators into the Calkin algebra then $\pi(S)$ is unitary. Regarding the Calkin algebra as an algebra of operators (as in Theorem 2 of [2]), we have

$$m_e(A) = m(\pi(A)) = m(\pi(B)) = m_e(B)$$
$$m_e(A^*) = m(\pi(A^*)) = m(\pi(B^*)) = m_e(B^*).$$

For $C \in I_n$ we have

$$\|B - SC\| = \|S(A-C)\| = \|A - C\|$$

and so by Lemma 2 we get

$$\operatorname{dist}(B, I_0 \cap PI_0) = \operatorname{dist}(A, I_n).$$

According to Lemma 1 it follows that

$$\operatorname{dist}(A, I_n) = \operatorname{dist}(B, I_0).$$

Since $\bar{I}_0 = \bar{G}$ we know that

$$\operatorname{dist}(A, I_n) = \operatorname{dist}(B, G) = \max\{m_e(B), m_e(B^*)\} = \max\{m_e(A), m_e(A^*)\}.$$

In the formula for dist (B, G) we used Theorem 3 of [2]. We should note that ind $B=0$ is not possible since $A \in I_m$ for $m \neq n$ and the multiplicity of S is n.

Another way that $A \notin I_n$ can occur is for precisely one of the quantities nul A or def A to be infinite. In that case precisely one of the quantities nul B or def B is infinite and, consequently, ind B is not zero. The only remaining possibility for the occurrence of $A \notin I_n$ is that both nul A and def A are infinite. In this case it follows from Theorem 2 of [2] that $m_e(A)=0=m_e(A^*)$. Since A belongs to J_0 and the closures of J_0 and I_0 coincide, we know that

$$\operatorname{dist}(A, I_0) = \operatorname{dist}(A, J_0) = 0 = \max\{m_e(A), m_e(A^*)\}.$$

Recall that n is a positive integer. Let $\{f_1, f_2, \ldots\}$ be an orthonormal basis for $\ker A^* = (AH)^{\perp}$ and let the union of $\{g_1, \ldots, g_n\}$ and $\{e_1, e_2, \ldots\}$ be an orthonormal basis for ker A. Define C to coincide with A on $(\ker A)^{\perp}$, to be zero on $\{g_1, \ldots, g_n\}$, and to send e_j to εf_j for $j=1, 2, \ldots$. Clearly nul $C=n$, def $C=0$ and $C \in I_n$. Since $\|A-C\| = \varepsilon$ we see that

$$\operatorname{dist}(A, I_n) = 0 = \max\{m_e(A), m_e(A^*)\}.$$

We have now considered all instances of $A \notin I_n$ for n a positive integer.

If $n=0$ then the desired conclusion follows from the fact that $\bar{I}_0 = \bar{G}$ and the formula in Theorem 3 of [2] provided ind $A \neq 0$. Our hypothesis that $A \notin I_0$ implies that either ind $A \neq 0$ or both nul A and def A are infinite. In the latter case the preceding paragraph showed that

$$\operatorname{dist}(A, I_0) = \operatorname{dist}(A, J_0) = 0 = \max\{m_e(A), m_e(A^*)\}.$$

Thus, we have considered all instances of $A \notin I_0$.

For negative n we apply the preceding result to A^* and $I^*_{-n} = \{C^* : C \in I_{-n}\} = I_n$.

Proof of (ii). For $A \in I_m$ with $m \neq n$ and n a positive integer the differences in the proof are modest. We choose $C \in F_n$ rather than $C \in I_n$, we use part (ii) of Lemma 2 rather than part (i), we use part (ii) of Lemma 1 rather than part (iii), and we note that $\bar{F}_0 = \bar{G}$. Again if precisely one of the quantities nul A and def A are infinite then the same is true for B and ind $B \neq 0$.

The case of nul $A = \infty =$ def A is more complicated. In view of the construction given in the second paragraph of the proof of (i) the following will suffice. For any operator C such that def $C=0$ and nul $C=n$ where n is a positive integer, we have dist $(C, F_n)=0$. Let $C=UR$ be the usual polar factorization for C and let $E(\ )$ be the spectral measure for R. Define $R(\varepsilon)$ to coincide with R on $E([\varepsilon, \infty])H$ and let it agree with εI on $E([0, \varepsilon))H$. It is routine to see that $R(\varepsilon)$ is invertible and the kernel of $UR(\varepsilon)$ is $E(\{0\})H=\ker R=\ker C$. (Recall that U sends $(RH)^-$ isometrically onto $(CH)^-$ and $\ker U=(RH)^\perp=\ker R$.) Clearly

$$\|C-UR(\varepsilon)\| \leqq \|R-R(\varepsilon)\| \leqq 2\varepsilon$$

and $(UR(\varepsilon))\in F_n$. We conclude that dist $(C, F_n)=0$ and it follows that dist $(A, F_n)= =0=\max\{m_e(A), m_e(A^*)\}$.

If $n=0$ then the desired conclusion follows from the fact that $\bar{I}_0=\bar{F}_0$ and the formula has already been proved for dist (A, I_0).

Proof of (iii). Since $I_n=J_n$ for $n\neq 0$, this part follows from part (i) provided $n\neq 0$. Because the closures of J_0 and I_0 coincide we know that

$$\text{dist}\,(A, J_0) = \text{dist}\,(A, I_0) = \max\{m_e(A), m_e(A^*)\}.$$

The following corollary is immediate from Theorem 3.

Corollary. *For n an integer we have*

$$\bar{J}_n = \bar{I}_n = \bar{F}_n.$$

Unfortunately this method does not help in computing the distance to the semi-Fredholm operators with indices ∞ or $-\infty$. Indeed, for any isometry S we have

$$SI_\infty \cap I_0 = \emptyset,$$

in sharp contrast to part (i) of Lemma 2.

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