Acta Sci. Math., 54 (1990), 133—137

Contraction representations of semigroups in finite von Neumann algebras

ISTVÁN KOVÁCS and WILLIAM R. McMILLEN

Let $\mathfrak{H}$ be a complex Hilbert space, and let $\mathscr{B}(\mathfrak{H})$ be the algebra of all bounded linear operators of $\mathfrak{H}$. Furthermore, let $\mathscr{A}\subset\mathscr{B}(\mathfrak{H})$ be a von Neumann algebra. By a *contraction representation* of a semigroup S in $\mathscr{A}$ we mean a homomorphism $\pi\colon \mathrm{S}\to\mathscr{A}_1$, where $\mathscr{A}_1$ denotes the multiplicative semigroup of the unit ball of $\mathscr{A}$.

In connection with a previous result of the first author [4], Shigeru Itoh has recently suggested studying contraction representations of right reversible semigroups in finite von Neumann algebras [5]. A semigroup S is called *right reversible* if for any $s, t\in\mathrm{S}$, the set $\mathrm{S}s\cap\mathrm{S}t$ is not empty. If, in such a semigroup S, we define "$\geqq$" by $t\geqq s$ if and only if $t=s$ or $t\in\mathrm{S}s$, then S becomes a directed set which will be denoted by the same letter S.

Here we intend to study contraction representations of right reversible semigroups S under the condition that for each $t\in\mathrm{S}$, the orbit $\{t^n\}_{n\in\mathrm{N}}$ is cofinal in the directed set S [6]. Under this condition S will be called *archimedean.* The additive semigroup of the positive cone $\mathbf{R}^n_+$ of the n-dimensional euclidean space $\mathbf{R}^n$ is an example of a right reversible archimedean semigroup. The study of contraction representations of right reversible archimedean semigroups in finite von Neumann algebras (cf. [1]) will eventually lead us, as shown below, to generalize considerations carried out in [3] for a single contraction.

Before formulating our first result, let us agree to call an element T of a von Neumann algebra $\mathscr{A}$ *partially unitary* if there is an orthogonal projection E in $\mathscr{A}$ such that $T^*T=TT^*=E$ [3]. Furthermore, a contraction representation π of an archimedean semigroup S in $\mathscr{A}$ is called *partially unitary* (resp. *completely non-unitary*) if each element of $\pi(S)$ is a partially unitary (resp. completely non-unitary [2]) element of $\mathscr{A}$.

We now have

Received August 18, 1987.

Theorem 1 (Cf. [3], Th. 2 and Prop. 1). *Let* S *be a right reversible archimedean semigroup, and let π be a contraction representation of* S *in a finite von Neumann algebra $\mathcal{A}\subset\mathcal{B}(\mathfrak{H})$. Then, there is a unique orthogonal projection $P\in\mathcal{A}$ reducing π to a partially unitary representation π_u of* S *in $\mathcal{A}$. The orthognal projection $E=I-P$, in turn, reduces π to a completely non-unitary representation π_0 of* S *in $\mathcal{A}$ so that we have*

(1) $\pi=\pi_0+\pi_u$,

(2) $\pi_0(s)\pi_u(t)=\pi_u(t)\pi_0(s)=0 \quad (s, t\in \mathrm{S})$,

(3) $[\pi_0(s)]^n\to 0$ *strongly as* $n\to\infty$ *for every* $s\in\mathrm{S}$.

Proof. It is based upon methods used in [3] with the natural modifications.

Consider the nets $P_t=\pi^*(t)\pi(t)$ and $R_t=\pi(t)\pi^*(t)$ $(t\in\mathrm{S})$ [6]. Evidently, P_t and R_t are in the positive cone $\mathcal{A}^+$ of $\mathcal{A}$. Moreover, the nets P_t and R_t are downward directed. Let us prove this statement just for P_t since similar argument applies to R_t. Let $t, s\in\mathrm{S}$ be given arbitrarily. Then there is a $z\in\mathrm{S}$ such that $z\geqq t$ and $z\geqq s$ $(z\in \mathrm{S}s\cap\mathrm{S}t$ for instance). Thus, in particular, $z=s_1t$ with an appropriate s_1. Now, for every $x\in\mathfrak{H}$, we have

$$(P_z x|x)=(\pi^*(s_1t)\pi(s_1t)x|x)=\|\pi(s_1t)x\|^2=\|\pi(s_1)\pi(t)x\|^2\leqq$$
$$\leqq\|\pi(t)x\|^2=(\pi^*(t)\pi(t)x|x)=(P_tx|x);$$

hence $P_z\leqq P_t$. Observing that z can be also written as $z=s_2s$ with some $s_2\in\mathrm{S}$, a similar reasoning shows $P_z\leqq P_s$.

Therefore, the nets P_t and R_t converge to elements P and R of $\mathcal{A}_1\cap\mathcal{A}^+$, respectively, in the strong topology [1]. In symbols:

(4) $$\lim_t P_t=\lim_t \pi^*(t)\pi(t)=P,\quad \lim_t R_t=\lim_t \pi(t)\pi^*(t)=R.$$

We claim that

(5) $$\pi^*(s)P\pi(s)=P \quad\text{and}\quad \pi(s)R\pi^*(s)=R \quad\text{for every } s\in\mathrm{S}.$$

We will prove the first statement. The second one can be proved similarly.

As $P=\lim_t \pi^*(t)\pi(t)$, it is natural to consider the net

$$\pi^*(s)\pi^*(t)\pi(t)\pi(s)=(\pi(t)\pi(s))^*\pi(t)\pi(s)=\pi^*(ts)\pi(ts),$$

where s is fixed and t runs over S. As long as we can prove that

$$Q(ts)=\pi^*(ts)\pi(ts)\quad(t\in\mathrm{S})$$

is a subnet of P_t, the first half of (5) will be proven. To do this, we prove

(i) $t_1 \leqq t_2 \Rightarrow t_1 s \leqq t_2 s$ (isotony);
(ii) $(ts)_{t \in S}$ is cofinal in S.

Ad (i). $t_1 \leqq t_2$ means that $t_2 \in St_1$, i.e. $t_2 = s't$, with some $s' \in S$, then, $t_2 s = s' t_1 s$, thus $t_2 s \in S t_1 s$ implying (i).

Ad (ii). Let $t_0 \in S$ be arbitrary, and consider an element from $St_0 \cap St_0 s$. This element can be written as $s' t_0 s = s'' t_0$. Let $s' t_0 = t_1$, then $t_1 s = s'' t_0 \in S t_0$, i.e., $t_0 \leqq t_1 s$, whence (ii).

Now, as in [3], by virtue of (4), one may prove that for every finite normal trace φ on $\mathscr{A}$, we have

$$\lim_t \varphi\big((\pi^*(t)\pi(t) - R)^*(\pi^*(t)\pi(t) - R)\big) = 0,$$

from which we conclude that

$$P = \lim_t \pi^*(t)\pi(t) = \lim \pi(t)\pi^*(t) = R \tag{6}$$

and that $P = R$ is an orthogonal projection of $\mathscr{A}$. The details are omitted. Also, it follows from (6) that

$$(I-P)\mathfrak{H} = \{x \in \mathfrak{H}: \lim_t \pi(t)x = 0\} = \{x \in \mathfrak{H}: \lim_t \pi^*(t)x = 0\}. \tag{7}$$

In fact, $x \in (I-P)\mathfrak{H}$ is equivalent to $Px = Rx = 0$. Then the conclusion is drawn from (4). In addition, for every $s \in S$, the operators $\pi(s)$ and $\pi^*(s)$ transform $(I-P)\mathfrak{H}$ into itself. To show this, we follow the arguments carried out to prove (5). Details are again omitted. Therefore, $I-P$, and thus P "reduces" each $\pi(s)$ $(s \in S)$, i.e., we have

$$P\pi(s) = \pi(s)P \quad (s \in S). \tag{8}$$

Using the techniques of [3], we may prove that

$$(\pi(s)P)^*(\pi(s)P) = (\pi(s)P)(\pi(s)P)^* = P, \tag{9}$$

i.e., each $\pi(s)P$ $(s \in S)$ is a partially unitary element of $\mathscr{A}$. For every $s \in S$, let

$$\pi_u(s) = P\pi(s). \tag{10}$$

Now, if we let

$$\pi_0(s) = (I-P)\pi(s) \quad (s \in S), \tag{11}$$

then we evidently have (1). In fact, $\pi_0(s)+\pi_u(s)=(I-P)\pi(s)+P\pi(s)=\pi(s)$. By virtue of (8), it is evident that π_0 and π_u are representations of S in $\mathscr{A}$. Now, if we prove (3), we will surely know that each $\pi_0(s)$ $(s\in S)$ is completely non-unitary. To do this, let $I-P=E$ and fix a $\pi(s)$ arbitrarily. Then (3) can be equivalently formulated as follows: for every $x\in E\mathfrak{H}$ we have

$$[\pi(s)]^n x \to 0 \quad \text{as} \quad n \to \infty. \tag{12}$$

Now, since $[\pi(s)]^n=\pi(s^n)$, consider the function $g: \mathrm{N}\to \mathrm{S}$ defined as $g(n)=s^n$ $(n\in \mathrm{N})$. Then, by assumption, $g(\mathrm{N})$ is cofinal in S. Furthermore, g is also isotone; $n_1\leqq n_2$ $(n_1, n_2\in \mathrm{N})$ implies $g(n_1)\leqq g(n_2)$. In fact, it is enough to prove $g(n)\leqq g(n+1)$, i.e., $s^n\leqq s^{n+1}$. But this is evident since $s^{n+1}=ss^n$; thus $s^{n+1}\in Ss^n$, hence $s^n\leqq s^{n+1}$. So, $(\pi\circ g)(n)=\pi(s^n)$ is a subnet of $\pi(s)$, a fact from which (12) follows on account of (7). Moreover, the uniqueness of P follows from the observation that P (or $E=I-P$) simultaneously decomposes each contraction $\pi(s)$ $(s\in S)$ into a completely non-unitary and a unitary part, and the decomposition of this kind of contractions is canonical. Finally, (2) is an immediate consequence of the fact that the elements of $\pi_0(S)$ and $\pi_u(S)$ mutually operate on subspaces orthocomplement one to another. The proof is complete.

Theorem 2 (Cf. [3], Prop. 2). *Let* S *be a right reversible archimedean semigroup, and let* $s\to\pi^{(1)}(s)$ $\big($*resp.* $s\to\pi^{(2)}(s)\big)$ *be a contraction representation of* S *in a finite von Neumann algebra* $\mathscr{A}^{(1)}\subset\mathscr{B}(\mathfrak{H}^{(1)})$ $\big($*resp.* $\mathscr{A}^{(2)}\subset\mathscr{B}(\mathfrak{H}^{(2)})\big)$. *Let* X *be a bounded linear transformation of* $\mathfrak{H}^{(2)}$ *into* $\mathfrak{H}^{(1)}$ *such that* $\pi^{(1)}(s)X=X\pi^{(2)}(s)$ *for each* $s\in S$. *Now, if* $E^{(1)}$ *and* $E^{(2)}$ *are the orthogonal projections corresponding to* $E=I-P$ *in Theorem* 1, *then we also have* $E^{(1)}X=XE^{(2)}$.

Although the proof is similar to that of Prop. 2 in [3], we include it here for completeness.

Proof. First we prove that $E^{(1)}XE^{(2)}=XE^{(2)}$. Indeed, if $x_2\in E^{(2)}\mathfrak{H}_2$, i.e., $x_2=E^{(2)}x_2$ (on the orthocomplement of $E^{(2)}\mathfrak{H}$, the sides of the proposed equality are zero), then

$$[\pi^{(1)}(s)]^n Xx_2 = \pi^{(1)}(s^n)Xx_2 = X\pi^{(2)}(s^n)E^{(2)}x_2 = X[\pi_0^{(2)}(s)]^n x_2 \to 0 \quad (n\to\infty).$$

This means that $Xx_2=XE^{(2)}x_2\in E^{(1)}\mathfrak{H}_1$ implying $E^{(1)}XE^{(2)}x_2=XE^{(2)}x_2$, whence the assertion. A similar argument proves $E^{(2)}X^*E^{(1)}=X^*E_1$, from which

$$XE^{(2)} = E^{(1)}XE^{(2)} = (E^{(2)}X^*E^{(1)})^* = (X^*E^{(1)})^* = E^{(1)}X$$

follows. The proof is complete.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF SOUTH ALABAMA
MOBILE, AL 36688, USA