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Subvarieties of varieties generated by graph algebras

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1. Introduction. Graph algebras have been invented by C. SHALLON [14] to obtain examples of nonfinitely based finite algebras (see G. MCNULTY, C. SHALLON [5] for an account on these results, and K. BAKER, G. MCNULTY, H. WERNER [1] on the newer developments). To recall this concept, let $G=(V, E)$ be a (directed) graph with vertex set V and edges $E\subseteq V\times V$. Define the graph algebra $A(G)$ corresponding to G to have underlying set $V\cup\{\infty\}$, where ∞ is a symbol outside V, and two basic operations, a nullary operation pointing to ∞ and a binary one denoted by juxtaposition, given by $uv=u$ if $(u, v)\in E$ and $uv=\infty$ otherwise. One of the first examples of a nonfinitely based finite algebra has been a particular three element graph algebra, called Murskii's groupoid after its discoverer (see [6]). Here is its multiplication table and the picture of the corresponding graph G_0.

$A(G_0)$	∞	0	1
∞	∞	∞	∞
0	∞	∞	0
1	∞	1	1

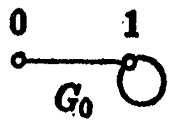

Figure 1

It has been observed by S. OATES-WILLIAMS and M. VAUGHAN-LEE [7] that the lattice of subvarieties of the variety generated by Murskii's groupoid is also very interesting. According to the main result of S. OATES-WILLIAMS [10], this lattice contains a chain isomorphic to that of the real numbers, hence it is uncountable, and satisfies neither the minimum nor the maximum condition.

Very little is known about lattices of subvarieties in general. An unexpected restriction has been revealed by W. A. LAMPE [3], and strengthened in the locally finite case by tame congruence theory (see R. MCKENZIE [4]). However, even the following two questions are open.

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Problem 1.1 (S. Oates-Williams and M. Vaughan-Lee [7]). Does there exist a finitely generated variety such that its lattice of subvarieties satisfies the descending chain condition but not the ascending chain condition?

Problem 1.2 (T. E. Hall). Is the intersection of two finitely generated varieties always finitely generated?

Notice that the second question also depends only on the lattice of subvarieties, since a locally finite variety is finitely generated iff it is not a union of a proper ascending chain of its subvarieties. By the results of S. Oates-Williams mentioned above, it seemed reasonable to look for examples answering these problems among graph algebras.

Definition 1.3. For a class $\mathcal{G}$ of graphs let $V(\mathcal{G})$ be the class of all graphs H for which $A(H)\in\mathbf{HSP}\,\{A(G)\colon G\in\mathcal{G}\}$. We call $\mathcal{G}$ a *graph variety* if $V(\mathcal{G})=\mathcal{G}$.

Obviously, V is a closure operator. In order to translate algebraic questions to the language of graphs, we need an internal characterization of graph varieties. For undirected graphs, this has been accomplished independently by E. W. Kiss [2] and in a preliminary version of R. Pöschel, W. Wessel [13] (see Section 2 for the exact definitions).

Theorem 1.4. *A class of undirected graphs is a graph variety if and only if it is closed under direct products, induced subgraphs, disjoint and directed unions.*

In E. W. Kiss [2] this theorem is applied to obtain an easy proof of the result of S. Oates-Williams [10] mentioned above about the variety generated by Murskii's groupoid. Theorem 1.4 has been generalized for the directed case by R. Pöschel [11].

The aim of this paper is twofold. In Section 2 we introduce a technique of investigating identities of graph algebras by using homomorphisms of graphs. This technique enables us to obtain in Section 3 the following result.

Theorem 1.5. *Let $\mathcal{G}$ be a class of graphs. Then the lattice of subvarieties of* $\mathbf{HSP}\,\{A(G)\colon G\in\mathcal{G}\}$ *is isomorphic to the lattice of subvarieties of $V(\mathcal{G})$.*

Let $\mathcal{G}$ be a fixed graph variety. For a subvariety $\mathcal{G}'$ of $\mathcal{G}$ define $\varphi(\mathcal{G}')=$ $=\mathbf{HSP}\,\{A(G)\colon G\in\mathcal{G}'\}$. Then φ is an order-preserving map from the lattice of subvarieties of $\mathcal{G}$ to the lattice of subvarieties of $\mathcal{V}=\varphi(\mathcal{G})$. By the definition of a graph variety, φ is one-to-one, and is clearly onto the poset of all subvarieties of $\mathcal{T}$ that are generated by their graph algebras. Since an order preserving bijection from a lattice to a poset is a lattice isomorphism, this argument shows that in order to prove Theorem 1.5 it is sufficient to establish the following assertion.

Theorem 1.6. *Every subvariety of a variety generated by graph algebras is also generated by its graph algebras.*

Theorem 1.5 enables us to investigate lattices of graph varieties instead of varieties of algebras. In Section 4 we provide machinery that makes it easier to determine these lattices, and give some examples. We shall obtain Theorem 1.4 as a corollary of these methods.

As an application of this machinery, in connection with Problem 1.1, we investigate the descending chain condition for lattices of subvarieties of varieties generated by undirected graph algebras. Recall that an algebra is called *critical,* if it is not contained in the variety generated by its proper subalgebras and homomorphic images. In Section 4, one of our basic tools will be the analogous concept of a strongly critical graph (see Definition 4.1). In the paper of S. OATES-WILLIAMS and M. VAUGHAN-LEE [7] it is shown that a locally finite variety has finitely many subvarieties iff it has finitely many critical algebras iff it satisfies the ascending *and* the descending chain condition for subvarieties.

We are going to prove that undirected graph algebras cannot provide an example requested in Problem 1.1, by determining all varieties of undirected graphs that satisfy the descending chain condition for subvarieties. There are seventeen of them, and all these indeed have only finitely many subvarieties. To formulate our result we have to define these graph varieties.

Let D_n be the undirected cycle (without loops) of length $n \geqq 3$, R_n the undirected (loopless) path of n edges $(n \geqq 0)$, and L_n the undirected path of $n+1$ vertices with a loop at every vertex $(n \geqq 0)$. Denote by $\mathcal{V}_0$ the graph variety generated by L_2 and R_3, and by $\mathcal{V}'_0$ the corresponding variety of graph algebras. We have computed the attice of subvarieties of $\mathcal{V}'_0$, it is shown on Figure 6 at the end of the paper.

Theorem 1.7. *Let $\mathcal{V}$ be a variety generated by a class of undirected graph algebras. The lattice of subvarieties of $\mathcal{V}$ satisfies the descending chain condition iff $\mathcal{V}$ is one of the seventeen subvarieties of $\mathcal{V}'_0$. Hence the descending chain condition implies the ascending chain condition for varieties generated by undirected graph algebras.*

There is no obvious relationship between the finite basis property and critical algebras. In the paper [7] an example is given of a finitely generated variety that is not finitely based, but which has only finitely many subvarieties. This variety is congruence permutable, but it is not inherently nonfinitely based.

In the paper K. BAKER, G. MCNULTY, H. WERNER [1] the finite basis property is investigated among varieties generated by undirected graph algebras. It turns out that there are only eleven such varieties that are finitely based and all the others are in fact inherently nonfinitely based. These eleven varieties are exactly the subvarieties of $\mathcal{V}'_1$, which is the variety corresponding to the graph variety $\mathcal{V}_1$ generated by

L_1 and R_2. Since $\mathscr{V}_1$ is a subvariety of $\mathscr{V}_0$, the lattice of subvarieties of $\mathscr{V}_1$ (calculated in [1]) is also shown on Figure 6 as an ideal in the lattice of subvarieties of $\mathscr{V}_0$. Thus the six remaining subvarieties of $\mathscr{V}_0$ have only finitely many subvarieties, but are inherently nonfinitely based.

Corollary 1.8. *Among inherently nonfinitely based varieties generated by undirected graph algebras, there are exactly six which have only finitely many subvarieties. Two of these six, the variety generated by the graph algebra of R_3 and the variety generated by the graph algebra of L_2, have the property that each of its subvarieties is finitely based.*

We conclude this section by two open questions.

Problem 1.9. Settle Problems 1.1 and 1.2 for varieties generated by graph algebras (or equivalently, for graph varieties).

Our proof of Theorem 1.6 is based on Lemma 3.1, which actually claims that every critical algebra in a variety generated by graph algebras is a graph algebra. The methods in Section 4 might therefore be sufficient to describe all critical algebras in these varieties.

Problem 1.10. Describe those graphs for which the corresponding graph algebra is a critical algebra.

2. Terms and identities. In this section we relate graphs to terms in the language of graph algebras and express the meaning of identities by graph theoretic properties. The proofs of the main results in later sections are all based on this translation. First we review our graph theoretic terminology.

By a graph we mean a pair $G=(V, E)$ with $E\subseteq V\times V$, where V is the set of vertices and E is the set of edges, we shall write $V(G)$ for V and $E(G)$ for E. Thus our graphs are directed without multiple edges, but may contain loops. A graph G is undirected if $(u, v)\in E$ implies that $(v, u)\in E$. If $v\in V(G)$, then we denote by $[v\rangle$ or by $[v\rangle_G$ the set of all vertices of G that are accessible from v via a directed path. A *root* of a graph G is a vertex v of G such that $[v\rangle=V(G)$. A graph G is *rootable* if it has a root. A *rooted graph* (G, v) is a pair, where G is a graph and v is a roof of G. The root of the rooted graph (G, v) is v.

Now let us review the basic graph theoretic constructions. Throughout the paper we shall assume that all classes of graphs are closed under isomorphisms.

Direct products. Let $G_i=(V_i, E_i)$ $(i\in I)$ be graphs. The product of these graphs, $G=(V, E)=\Pi\{G_i\colon i\in I\}$ is defined by $V=\Pi\{V_i\colon i\in I\}$ and $(\mathbf{u}, \mathbf{v})\in E$ iff $(\mathbf{u}_i, \mathbf{v}_i)\in E_i$ for all $i\in I$, where $\mathbf{u}_i$ is the i-th component of $\mathbf{u}$.

Induced subgraphs. Let $G=(V, E)$ be a graph. A *subgraph* of G is a graph H with $V(H)\subseteq V(G)$ and $E(H)\subseteq E(G)$. If $W\subseteq V$ is any subset, then the induced subgraph of G on W, denoted by $G\restriction W$, is $(W, E\cap(W\times W))$.

Disjoint unions. Let $G_i=(V_i, E_i)$ $(i\in I)$ be graphs. The disjoint union of these graphs, $G=(V, E)=\bigcup *\{G_i: i\in I\}$ is defined by $V=\{(g, i): i\in I, g\in V_i\}$ and $E=\{((u, i), (v, i)): i\in I, (\mathbf{u}, \mathbf{v})\in E_i\}$. Informally: consider isomorphic copies of our graphs on pairwise disjoint underlying sets and take the union of these graphs.

Directed unions. We do not wish to provide a complicated formal definition, since it will be sufficient to say that a class $\mathcal{G}$ of graphs is closed under directed unions if and only if it has the following property: an arbitrary graph G is in $\mathcal{G}$ if and only if all finite induced subgraphs of G are elements of $\mathcal{G}$.

Homomorphisms. A homomorphism $f: G=(V, E)\to H=(W, F)$ is a mapping $f: V\to W$ carrying edges to edges, that is, for which $(\mathbf{u}, \mathbf{v})\in E$ implies $(f(\mathbf{u}), f(\mathbf{v}))\in F$. Thus, an edge can be collapsed only to a vertex that has a loop.

Next we relate rooted graphs to terms. Let $T(X)$ be the set of all terms over a set X of variables in the type of graph algebras. A term is called trivial if the nullary operation ∞ occurs in it, these terms evaluate to ∞ in every graph algebra. For $t\in T(X)$, let $L(t)$ denote the leftmost variable of the term t.

Definition 2.1. For a nontrivial $t\in T(X)$ let $G(t)=(V(t), E(t))$ be the graph associated to t, defined as follows. The vertex set $V(t)$ is the set of variables occurring in t, and the set $E(t)$ of edges is defined inductively by $E(t)=\emptyset$ if t is a variable in $X\subseteq T(X)$, and $E(ts)=E(t)\cup E(s)\cup\{(L(t), L(s))\}$. The rooted graph corresponding to t is $(G(t), L(t))$.

$t = (x(uv))(((yu)y)x)$ $G(t)$:

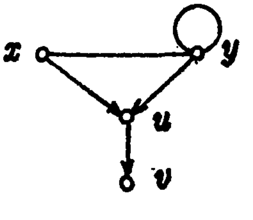

Figure 2

The following lemma has its precursor in the papers [5], [2], [13], [11].

Lemma 2.2. *Let $G=(V, E)$ be a graph, $t, s\in T(X)$ and $h: X\to A(G)$ an evaluation of the variables. Let the same h denote the unique extension of this evaluation to the algebra $T(X)$ of all terms.*

(1) *If* $t \in T(X)$ *is nontrivial, then* $(G(t), L(t))$ *is a finite rooted graph. Conversely, for every finite rooted graph* (G, v) *there exists* $t \in T(V(G))$ *with* $G(t) = G$ *and* $L(t) = v$.

(2) *If* t *is a trivial term, or if* h *takes the value* ∞ *on* X, *then* $h(t) = \infty$. *Otherwise, if* $h: G(t) \to G$ *is a homomorphism of graphs, then* $h(t) = h(L(t))$, *and if* h *is not a homomorphis of graphs, then* $h(t) = \infty$.

(3) *The identity* $s = t$ *is true in every graph algebra iff either both* s *and* t *are trivial terms, or none of them is trivial,* $G(s) = G(t)$, *and* $L(s) = L(t)$.

Proof. Let (G, v) be a finite rooted graph. We define the term $t(G, v)$ by induction on the number of edges of G. The set of variables $V(t)$ of $t(G, v)$ is $V(G)$. If $E(G) = \emptyset$, then set $t(G, v) = v$. If $E(G) \neq \emptyset$, then there is an edge $(v, u) \in E(G)$. Let H be the graph obtained by throwing this edge out and set $t(G, v) = t(H \restriction [v\rangle_H, v) \cdot t(H \restriction [u\rangle_H, u)$. An easy induction shows that $G(t(G, v)) = G$ and $L(t(G, v)) = v$, proving the second statement of (1). The rest of (1) and (2) can also be proved by a straightforward induction, it is left to the reader.

To show (3) assume first that $G(s) = G(t)$ and $L(s) = L(t)$ for a pair of nontrivial terms s and t. Then (2) shows that $s = t$ holds in every graph algebra. Conversely, assume that $s = t$ holds in every graph algebra, and that t is nontrivial. Let $G = G(t)$ and h the identity map of $V(t)$. Then (2) shows that $h(t) = h(L(t)) = L(t) \neq \infty$ in the graph algebra $A(G)$. Hence, $h(s) = h(t) \neq \infty$. Therefore s is nontrivial, $V(s) \subseteq V(t)$, and the identity map from $V(s)$ to $V(t)$ is a graph homomorphism from $G(s)$ to $G(t)$. By interchanging the role of t and s we see that $G(s) = G(t)$. Hence, $L(t) = h(L(t)) = h(t) = h(s) = h(L(s)) = L(s)$ and the proof of Lemma 2.2 is complete.

Our next aim is to give an equational base for the variety generated by all graph algebras. For two sets Σ and Σ' of identities we say that Σ graph implies Σ' if for every algebra A in the variety generated by all graph algebras $A \models \Sigma$ implies that $A \models \Sigma'$. We call Σ and Σ' graph equivalent if they graph imply each other.

Lemma 2.3. *Every identity in the language of graph algebras is graph equivalent to a finite set* Σ *of identities, such that each element of* Σ *belongs to one of the following types:*

(α) *Identities of the form* $s = t$ *with* $V(s) = V(t)$, $E(s) \subseteq E(t)$, *and* $L(s) = L(t)$;

(β) *Identities of the form* $s = t$, *with* $G(s) = G(t)$;

(γ) *Identities of the form* $s = \infty$ *with* s *nontrivial.*

Condition (γ) *can be replaced by*

(γ') *The identity* $xx = \infty$ *(where* x *is a variable).*

We shall prove this lemma after Lemma 2.6. It is very significant, because, combined with Lemma 2.2 (2), it explains very clearly and informally which properties of graphs can be expressed by identities in their graph algebras. Identities of type (α) say that if a certain configuration (that is, a homomorphic image of $G(s)$) exists in our graph, then certain other edges (those of $G(t)$) must also be in our graph. For example, the identity $xy=x(yx)$ expresses that our graph is undirected. Identities in (β) claim that a certain configuration (an image of $G(s)$) can exist only if two given vertices (the two leftmost variables) are collapsed. The identity $x(yx)=y(xy)$, expressing that our graph has no undirected edges other than loops, is of this kind. Finally the identities in (γ) say that a certain configuration cannot exist at all. In particular, $xx=\infty$ requires our graph to be loopless. These observations will be utilized in the proof of the forthcoming characterization of graph varieties.

To obtain graph implied identities we use substitutions. Let $t\in T(X)$ and $s_v\in T(X)$ for certain variables $v\in Y\subseteq X$. Then by $t[v\to s_v\colon v\in Y]$ we denote the term in $T(X)$ obtained from t by substituting s_v into the variable v, and leaving all other variables of t intact. The following lemma can be proved by an easy induction on the complexity of t.

Lemma 2.4. *With the notation above,*

$$V(t[v\to s_v\colon v\in V(t)]) = \bigcup\{V(s_v)\colon v\in V(t)\},$$

$$E(t[v\to s_v\colon v\in V(t)]) = \{(L(s_v),\ L(s_u))\colon (u,v)\in E(t)\}\cup(\bigcup\{E(s_v)\colon v\in V(t)\}),$$

$$L(t[v\to s_v\colon v\in V(t)]) = L(s_{L(t)}).$$

For every variable v and terms p, q, r the identity $p=r$ implies in every algebra, hence graph implies the identity $p[v\to q]=r[v\to q]$.

Definition 2.5. Let $s, t\in T(X)$ be nontrivial terms. For $v\in V(s)$, let s^v be any term (according to Lemma 2.2 (1)) with $G(s^v)=G(s)\restriction[v\rangle_{G(s)}$ and $L(s^v)=v$. For $v\notin V(s)$ set $s^v=v$. Define $t[s]=t[v\to s^v\colon v\in V(t)]$.

The following statement is an obvious consequence of Lemma 2.4.

Lemma 2.6. *Let $s, t\in T(X)$ be nontrivial terms. Then*

$$V(t[s]) = V(t)\cup(\bigcup\{V(s^v)\colon v\in V(t)\cap V(s)\}),$$

$$E(t[s]) = E(t)\cup(\bigcup\{E(s^v)\colon v\in V(y)\cap V(s)\}),$$

$$L(t[s]) = L(t).$$

In particular, if $V(s)\subseteq V(t)$, then $V(t[s])=V(t)$, $E(t[s])=E(t)\cup E(s)$, and if $V(t)\subseteq V(s)$ and $E(t)\subseteq E(s)$, then $G(t[s])=G(s^{L(t)})$.

The term $t[s]$ can therefore be considered as the "union" of the terms t and s. It is important to mention that if p, q, r are nontrivial terms, and $p=r$ holds in a graph algebra, then it does not follow in general that $q[p]=q[r]$ holds in the same graph algebra.

The proof of Lemma 2.3. Let $p=r$ be any identity of graph algebras. If both terms are trivial, or if $p=r$ holds in every graph algebra, then this identity is equivalent to the empty set of identities. So assume that, say, p is nontrivial. If r is trivial, then $p=r$ is equivalent to $p=\infty$, which is of type (γ). If not, then both p and r are nontrivial. If $V(p)\neq V(r)$, then without loss of generality we may assume that there is a variable $z\in V(r)$, $z\notin V(p)$. By making the substitution $z\to\infty$ in the identity $p=r$ we obtain $p=\infty$, which is therefore a consequence of $p=r$ in the variety generated by all graph algebras (in other words, $p=r$ graph implies $p=\infty$). Hence, by transitivity, $p=r$ is graph equivalent to $\{p=\infty,\ r=\infty\}$.

The other possibility is that $V(p)=V(r)$. Since $p[r]$ is defined by substitution, $p=r$ graph implies $p[r]=r[r]$ and $p[p]=r[p]$. On the other hand, $p[p]$ and p induce the same rooted graph by Lemma 2.6, and hence $p[p]=p$, as well as $r[r]=r$ holds in the variety generated by all graph algebras by Lemma 2.2 (3). Thus, $p=r$ graph implies $p[r]=r$, hence it graph implies $p=p[r]$ by transitivity. Therefore, again by transitivity, $p=r$ is graph equivalent to the set $\{p=p[r],\ p[r]=r[p],\ r=r[p]\}$. Of these identities, the first and third ones are of type (α) and the second one is of type (β) by Lemma 2.6.

To conclude the proof of Lemma 2.3 we have to show that the set (γ) can be replaced by the set (γ'). Let $s=\infty$ be an identity with s nontrivial, G' the complete graph on $v(s)$ (that is, with $E(G')=V(s)\times V(s)$), and s' a term with $G(s')=G'$ and $L(s')=L(s)$. Then $s=\infty$ graph implies $s[s']=\infty$, but $s[s']=s'$ in all graph algebras. Hence $s=\infty$ is graph equivalent to $\{s=s',\ s'=\infty\}$. The first identity is clearly of type (α), and the second one is equivalent to $xx=\infty$. Indeed, by identifying all variables, $s'=\infty$ yields $xx=\infty$ by Lemma 2.4. Conversely, if $x\in V(s')$, then by Lemma 2.6 we have $s'[xx]=s'$ in the variety generated by all graph algebras, so $s'[xx]=s'[\infty]=\infty$.

3. The proof of Theorem 1.6. First we show that it is sufficient to establish the following lemma.

Lemma 3.1. *Let A be an algebra in a variety generated by graph algebras and let $p=r$ be an identity which fails in A. Then there exists a graph algebra $D\in\mathbf{HS}(A)$ such that $p=r$ fails in D.*

Indeed, assume that this lemma holds, let $\mathcal{V}$ be a subvariety of a variety generated by graph algebras, and let $\mathcal{W}$ be the variety generated by the graph algebras belonging to $\mathcal{V}$. To show $\mathcal{V}=\mathcal{W}$ let us assume that there is an algebra $A\in\mathcal{V}$, which is not in $\mathcal{W}$. Then there is an identity $p=r$ which holds in $\mathcal{W}$ but fails in A. By Lemma

3.1, there is a graph algebra D in $\mathbf{HS}(A)\subseteq\mathcal{V}$ such that $p=r$ fails in D. Therefore $D\in\mathcal{V}-\mathcal{W}$ contradicting to the definition of $\mathcal{W}$.

To prove Lemma 3.1 let A be an algebra in a variety generated by graph algebras and $p=r$ an identity that fails in A. Let $s=t$ be an identity satisfying the following conditions.

(a) The identity $s=t$ fails in A.

(b) The identity $p=r$ graph implies $s=t$.

(c) The term s is nontrivial, and either $t=\infty$ or $V(s)=V(t)$ and $E(s)\subseteq E(t)$.

(d) For any identity $s'=t'$ satisfying (a), (b), and (c) we have $|V(s')|\geqq|V(s)|$ and if $|V(s')|=|V(s)|$ then $|E(s')|\leqq|E(s)|$.

The existence of such an identity $s=t$ can be seen as follows. By Lemma 2.3 we obtain an equation $s=t$ satisfying (a), (b), and (c). Among these we select those with $|V(s)|$ being minimal, and among these one with $|E(s)|$ being maximal.

For the rest of this section we fix an identity $s=t$ satisfying (a), (b), (c), and (d). By (a) there exists an evaluation $h\colon V(s)\to A$ of the variables of s into A showing that $s=t$ fails in A. Denoting by the same h the unique extension of h to a homomorphism $T(V(s))\to A$ we therefore have $h(s)\neq h(t)$ in A. We fix this mapping h also.

Lemma 3.2. *For any nontrivial term q with $V(q)\subseteq V(s)$ and variable $z\in V(s)$, the equality $h(q[s])=h(s^z)$ implies that $E(q)\subseteq E(s)$.*

Proof. Consider the term $s'=s[z\to q[s],\ v\to s^v\colon v\in V(s),\ v\neq z]$, and let t' be obtained from t by the same substitution. Then $h(q[s])=h(s^z)$ implies that

$$h(s') = h\big(s[z \to s^z,\ v \to s^v\colon v\in V(s),\ v \neq z]\big) = h\big(s[s]\big) = h(s),$$

and similarly $h(t')=h(t[s])$. Condition (c) yields (using Lemma 2.2 (3) and Lemma 2.6) that $t[s]=t$ holds in every variety generated by graph algebras. Hence we have $h(t')=h(t)\neq h(s)=h(s')$, so the identity $s'=t'$ satisfies (a). As $s=t$ graph implies $s'=t'$, the identity $s'=t'$ satisfies (b). Let us calculate, using Lemmas 2.4 and 2.6, the graphs $G(s')$ and $G(t')$. First, it is clear from (c) that if t is trivial, then so is t', and if t is nontrivial, then $V(s')=V(t')$ and $E(s')\subseteq E(t')$, since we have made the same substitution on both sides. Therefore $s'=t'$ satisfies (c), too. It is also clear that $V(s')\subseteq V(s)$. So we must have equality here by (d), in particular, $z\in V(s')$. Hence $z\in V(q)$ or $z\in V(s^v)$ for some $v\neq z$. In either case, every edge $(z,u)\in E(s)$ is also in $E(s')$. On the other hand, if $(v,u)\in E(s)$ with $v\neq z$, then $(v,u)\in E(s^v)\subseteq E(s')$. Hence, $E(s)\subseteq E(s')$. So by (d), we have $E(s)=E(s')$. But clearly $E(q)\subseteq E(s')$, proving our lemma.

Now set $B=\{h(s^v)\colon v\in V(s)\}$, let C be the subalgebra of A generated by B and $O=C-B$. Consequently, every element of B can be expressed as $h(q[s])$ for some term q with $V(q)\subseteq V(s)$. Let ϑ be the equivalence relation on C such that O

is a block of ϑ, and all other blocks (containing the elements of B) are singletons. To conclude the proof of Lemma 3.1 it is sufficient to establish the following claim.

Lemma 3.3. *The relation ϑ is a congruence of C and $D=C/\vartheta$ is a graph algebra in which the identity $s=t$ fails.*

Proof. First notice that $\infty\notin B$. Indeed, if $h(s^v)=\infty$ for a certain $v\in V(s)$, then $\infty=h(s[s])=h(s)$ and $\infty=h(t[s])=h(t)$ would yield $h(s)=h(t)$. Thus for every trivial term q we have $h(q[s])=\infty\in O$. Let q be nontrivial with $V(q)\subseteq V(s)$. Then by the last statement of Lemma 2.6 and by Lemma 3.2 we have

if $E(q)\subseteq E(s)$ then $h(q[s])=h(s^{L(q)})\in B$,

if $E(q)\nsubseteq E(s)$ then $h(q[s])\in O$.

Now let $c=h(q[s])$ and $c'=h(q'[s])$ be arbitrary elements of C. Then $cc'=h((qq')[s])$. If q or q' is trivial, then so is qq', hence $cc'\in O$. If both terms are nontrivial, then $E(q)\cup E(q')\subseteq E(qq')$. Therefore if c or $c'\in O$, then $E(qq')\nsubseteq E(s)$, thus $cc'\in O$. This proves that ϑ is a congruence, and that $D=C/\vartheta$ satisfies $x\infty=\infty x=\infty$ (where the value of the operation ∞ in D is the equivalence class $O=\infty/\vartheta$). If both c and c' are in B, then either $E(qq')\subseteq E(s)$, in which case $cc'=h((qq')[s])=h(s^{L(qq')})=h(s^{L(q)})=c$, or $E(qq')\nsubseteq E(s)$, and then $cc'\in O$. Therefore $D=C/\vartheta$ satisfies $xy\in\{x,\infty\}$, so it is a graph algebra.

Finally we show that $s=t$ fails in D. We have $h(s)=h(s[s])=h(s^{L(s)})\in B$, since $E(s)\subseteq E(s)$. Hence $h(s)$ sits in a singleton block of ϑ. So if $h(s)$ and $h(t)$ are congruent modulo ϑ, then $h(s)=h(t)$, and this is impossible since $h(s)\neq h(t)$ in A. Thus the proof is complete.

4. Graph varieties. In this section we investigate graph varieties. The concept of critical groups has proved useful in dealing with varieties of groups. We shall introduce an analogous concept in a slightly stronger form.

Definition 4.1. A finite graph G is called *strongly critical*, if it is not contained in the variety generated by all graphs $H\in \mathrm{V}(G)$ for which either $|V(H)|<|V(G)|$ or $|V(H)|=|V(G)|$ and $|E(H)|<|E(G)|$.

Lemma 4.2. *Every graph variety is generated by its strongly critical members.*

Proof. Suppose not, let $\mathcal{V}$ be a graph variety and $\mathcal{W}$ the subvariety of $\mathcal{V}$ generated by its strongly critical members. Since every graph variety is generated by its finite members (see Proposition 4.5 below), there exists a finite graph in $\mathcal{V}$, which is not in $\mathcal{W}$. Choose one with minimal $|V(H)|$, and among these one with minimal $|E(H)|$. Then H is strongly critical, so $H\in\mathcal{W}$, which is a contradiction.

Theorem 4.3. *Let $\mathcal{G}$ be a class of graphs and G a strongly critical member of $V(\mathcal{G})$. Then G is an induced subgraph of a direct product of members of $\mathcal{G}$.*

To prove this result we have to introduce concepts that make it easier to recognize graph varieties.

Definition 4.4. Let $G=(V, E)$ be a graph. Define the equivalence relation $\varrho(G)$ on V by calling two vertices equivalent if they have identical neighbourhoods. In notation: $(u, v)\in\varrho(G)$ iff for every vertex x we have $(u, x)\in E$ iff $(v, x)\in$ and $(x, u)\in E$ iff $(x, v)\in E$. For an equivalence relation $\vartheta\subseteq\varrho(G)$ define G/ϑ to be $(V/\vartheta, E')$, where $(u/\vartheta, v/\vartheta)\in E'$ iff $(u, v)\in E$.

Furthermore, define the equivalence relation $\tau(G)$ by $(u, v)\in\tau(G)$ iff $(u, v)\in\varrho(G)$ and either $u=v$ or there is no directed path from u to v. Finally, let $(u, v)\in\sigma(G)$ iff $(u, v)\in\varrho(G)$ and either $u=v$ or u is not a root of G. We say that G is *reduced* if $\tau(G)=0_V$ and *strongly reduced* if $\sigma(G)=0_V$.

Clearly, $\tau(G)\subseteq\sigma(G)$. To see that these are indeed equivalence relations notice that for any block B of $\varrho(G)$, if there is a directed path in G from a vertex of B to another vertex v of G, then there is a directed path from any vertex of B to v by the definition of $\varrho(G)$. On Figure 3 a graph G is shown with vertices $(u, v)\in\sigma(G)-\tau(G)$ and $(u', v')\in\tau(G)$.

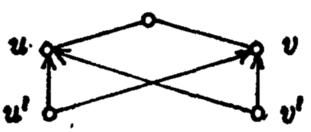

Figure 3

Recall that a graph G is called rootable if it has a vertex from which all other vertices can be accessed by a directed path. The following result not only implies Theorem 4.3 directly, but also enables us to decide in many cases whether a given graph belongs to a certain variety of graphs.

Proposition 4.5. *Let $\mathcal{G}$ be a class of graphs and $\mathcal{V}=\mathbf{V}(\mathcal{G})$.*

(1) *$G\in\mathcal{V}$ iff every finite induced subgraph of G belongs to $\mathcal{V}$.*

(2) *$G\in\mathcal{V}$ iff every rootable induced subgraph of G belongs to $\mathcal{V}$.*

(3) *$G\in\mathcal{V}$ iff the reduced graph $G/\tau(G)$ belongs to $\mathcal{V}$.*

(4) *If G is finite, rootable, and strongly reduced, then $G\in\mathcal{V}$ iff G is an induced subgraph of a direct product of graphs from $\mathcal{G}$.*

(5) *If G is strongly critical, then it is finite, rootable, and strongly reduced.*

Before proving this proposition, we collect some well-known or elementary facts about homomorphisms of graphs. The proofs are left to the reader.

Lemma 4.6. *Let G be a graph and $\mathcal{G}$ a family of graphs.*

(1) *The graph G is an induced subgraph of a direct product of members of $\mathcal{G}$ if and only if there exists a nonempty set Φ of graph homomorphisms from G to members of $\mathcal{G}$ satisfying the following two conditions.*

(a) *For every two different vertices u and v of G there is $f \in \Phi$ such that $f(u)$ and $f(v)$ are different.*

(b) *For every non-edge (u, v) of G there is $f \in \Phi$ such that $(f(u), f(v))$ is also a non-edge.*

(2) *Let $\vartheta \subseteq \varrho(G)$ be an equivalence relation and $f: G \to G/\vartheta$ the natural homomorphism. Then for any graph H and mapping $h: V(H) \to V(G)$, h is a graph homomorphism from H to G if and only if $f \circ h$ is a graph homomorphism from H to G/ϑ. Consequently, if an identity $s=t$ of type (α), (β), or (γ) holds in $A(G/\vartheta)$ but fails in $A(G)$ under an evaluation map $h: V(s) \to G$, then $G(s)=G(t)$, h is a homomorphism, $u=h(L(s)) \neq h(L(t))=v$, and $(u, v) \in \vartheta$.*

Now we prove Proposition 4.5. By Lemma 2.3, (1) and (2) are clear, since all graphs corresponding to terms are finite and rootable. To see (3) notice first that $A(G/\vartheta)$ is a homomorphic image of $A(G)$ for every $\vartheta \subseteq \varrho(G)$. Conversely, assume that an identity $s=t$ of type (α), (β), or (γ) holds in $A(G/\tau(G))$ but fails in $A(G)$. Apply Lemma 4.6 (2) with $\vartheta=\tau(G)$, Since there is a directed path in $G(s)$ from $L(s)$ to $L(t)$, there is one in G from u to v, since h is a graph homomorphism, and this contradicts to $(u, v) \in \tau(G)$. Thus (3) is proved.

One direction of the statement in (4) is well-known. Induced subgraphs correspond to subalgebras, and the graph algebra corresponding to a direct product of graphs can be obtained as a homomorphic image of the direct product of the corresponding graph algebras, this construction is described first in the paper of S. Oates-Williams and M. Vaughan-Lee [7]. For the other direction suppose that G is a finite, rootable, strongly reduced graph belonging to $\mathcal{V}$. In order to embed G as an induced subgraph of a product of members of $\mathcal{G}$, we apply Lemma 4.6 (1). First let (u, v) be a non-edge of G and G' the graph obtained by adding (u, v) to G. Since G is finite and rootable, there exists terms t, t' with the same leftmost variable such that $G=G(t)$ and $G'=G(t')$. The identity map of $V(G)$ shows by Lemma 2.2 (2) that $t=t'$ fails in $A(G)$. Hence it fails in a member of $\mathcal{G}$, too, yielding a homomorphism f of $G=G(t)$ into that member, which must carry the non-edge (u, v) into a non-edge. Hence condition (b) of Lemma 4.6 (1) is satisfied.

Now let u and v be two vertices of G that are collapsed by every homomorphism occurring in (b). If (u, x) is a non-edge, then there is a homomorphism f such that $(f(u), f(x))$ is also a non-edge. But $f(u)=f(v)$, hence (v, x) must also be a non-edge since f is a homomorphism. Similarly, if (x, u) is a non-edge, then (x, v) is also a non-edge. Hence, $(u, v) \in \varrho(G)$. But G is strongly reduced, hence u, as well as v, are roots of G. Let t_u, t_v be terms (according to Lemma 2.2 (1)) with $G(t_u)=G(t_v)=G$,

$L(t_u)=u$, $L(t_v)=v$. Then $t_u=t_v$ fails in $A(G)$, hence it fails in a member of $\mathscr{G}$. Therefore there exists a homomorphism f of G into that member such that $f(u)\neq f(v)$.

To conclude the proof of (4) we have to ensure that the set Φ of homomorphisms is not empty. If it is empty, then G has no non-edges, hence it is a complete graph, and has no two different vertices, hence G is the one element complete graph. In that case, the identity $xx=\infty$ fails in G, hence it fails in a member of $\mathscr{G}$. Therefore that member has at least one loop, so G is an induced subgraph of it.

Finally we have to prove (5), so let G be strongly critical. By (1), (2), and (3) we see that G is finite, rootable, and reduced. If G is not strongly reduced, then there exist different vertices u and v of G such that $(u, v)\in\varrho(G)$, but u is not a root of G. Let $S=G\restriction[u\rangle_G$, ϑ the equivalence relation with single nontrivial block $\{u, v\}$, and $F=G/\vartheta$. It is sufficient to prove that G is in the variety generated by S and F (which is a contradiction, since G is strongly critical).

Suppose otherwise. Then by Lemma 4.6 (2) there is identity $s=t$ that holds in F and S but fails in G, moreover, $G(s)=G(t)$, and $u=h(L(s))\neq h(L(t))=v$ holds for a graph homomorphism $h\colon G(s)\to G$. Now h maps $G(s)$ into $[u\rangle=V(S)$, and this yields an evaluation showing that $s=t$ fails in S. This contradiction finishes the proof of Proposition 4.5.

Let G be undirected, finite, and connected. Then every vertex is a root, hence G is strongly reduced. Thus Theorem 1.4 is an easy corollary of Proposition 4.5.

We introduce some notation for graphs that provide important examples. First, we deal with the directed case. Let P_n be the directed (loopless) path of $n+1$ vertices, that is, of length n, and let C_n be the directed cycle of n certices. Furthermore, let $G(n, k, m)$ denote the graph obtained by connecting a vertex of C_n to a vertex of C_m by a directed path of length k (see Figure 4 below). Thus, $G(n, k, m)$ has $n+k+m-1$ vertices and $n+k+m$ edges.

Example 4.7. The only subvarieties of the variety $\mathscr{V}$ generated by all P_n ($n\in\omega$) are the varieties $\mathbf{V}(P_m)$ ($m\in\omega$). The only subvarieties of $\mathbf{V}(C_n)$ are the varieties $\mathbf{V}(P_m)$ for $m<n-1$ and $\mathbf{V}(C_n)$ itself. Hence the lattices of subvarieties of $\mathscr{V}$, $\mathbf{V}(P_n)$, and $\mathbf{V}(C_n)$ ($n\in\omega$) are all chains.

Proof. In any direct product of cpoies of P_m and C_n the out-degree of every vertex is at most one. Hence if u is an element of such a product, then $[u\rangle$ is isomorphic to some P_m or to C_n. Hence, every strongly critical member is a path or a cycle.

Example 4.8. We have $G(n, k+1, m)\in\mathbf{V}(G(n, k, m))$ but $G(n, k, m)\notin \mathbf{V}(G(n, k+1, m))$ for $n, k, m\geqq 1$. Hence $\mathbf{V}(G(n, k, m))$ does not satisfy the descending chain condition for subvarieties.

Proof. It is straightforward to check that every graph $G(n, k, m)$ is finite rootable, and strongly reduced. So by Proposition 4.5 (4) we may apply Lemma 4.6 (1).

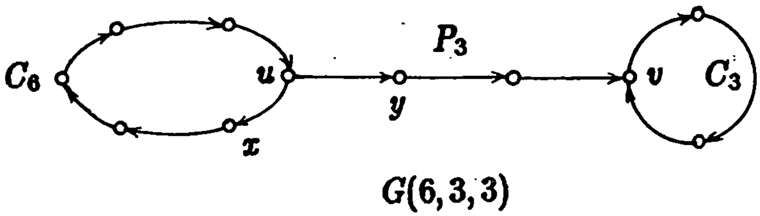

Figure 4

We show $G(n, k, m) \notin \mathbf{V}(G(n, k+1, m))$ first. Let u be the only vertex of $G(n, k, m)$ with out-degree two, v the only one with in-degree two, and let u' and v' be the vertices of $G(n, k+1, m)$ defined analogously. Denote by x and y the endpoints of the edges starting from u and let $f\colon G(n, k, m) \to G(n, k+1, m)$ be a homomorphism with $f(x) \neq f(y)$. Then $f(u)$ has out-degree two, hence $f(u)=u'$. If x is the vertex on the first cycle of $G(n, k, m)$ and y is the one on the intermediate path (cf. Figure 4), then there is a directed path from x to u, but there is no such path from y to u. Therefore $f(x)$ must be on the first cycle of $G(n, k+1, m)$, hence $f(y)$ is not on this cycle. Since all vertices except u and u' have out-degree one, starting from y we see that all the elements of the intermediate path of $G(n, k, m)$ are mapped to the intermediate path of $G(n, k+1; m)$. Thus $f(v)$ is still on the intermediate path of $G(n, k+1, m)$ and not on its second cycle. This is impossible however, since in $G(n, k, m)$ there is a directed path from v to v. This contradiction shows that there is no homomorphism from $G(n, k, m)$ to $G(n, k+1, m)$ that separates x and y, hence $G(n\ k, m) \notin \mathbf{V}(G(n, k+1, m))$ indeed.

To show that $G(n, k+1, m) \in \mathbf{V}(G(n, k, m))$ consider two homomorphisms from $G(n, k+1, m)$ to $G(n, k, m)$ the first of which collapsing the two out-neighbours of u' and the second one collapsing the two in-neighbours of v'. It is easy to check that these two homomorphisms separate all vertices and non-edges.

Example 4.9. Let G be a directed cycle with a chord, that is, $V(G)=\{1, \ldots, m\}$ and $E(G)=\{(1, 2), \ldots, (m-1, m), (m, 1), (1, k)\}$ with $3 \leqq k \leqq m$. If $m-k+2$ does not divide m, then $\mathbf{V}(G)$ does not satisfy the descending chain condition for subvarieties.

Proof. Let $n=m-k+2$. It is sufficient to show by the previous example that $G(n, k-1, n) \in \mathbf{V}(G)$. Let us introduce the notation $u_1=1$, $u_2=k$, $u_3=k+1, \ldots,$ $u_n=m$, the induced subgraph on this set is isomorphic to C_n. We define the subset U of $G \times G$ to be $U_1 \cup U_2 \cup U_3$ with

$$U_1 = \{(u_1, u_1), (u_2, u_2), \ldots, (u_n, u_n)\},$$

$$U_2 = \{(1, u_1), (2, u_2), (3, u_3), \ldots, (k-1, u_{k-1}), (k, u_k)\},$$

$$U_3 = \{(u_2, u_k), (u_3, u_{k+1}), \ldots, (u_n, u_m), (u_1, u_{m+1})\},$$

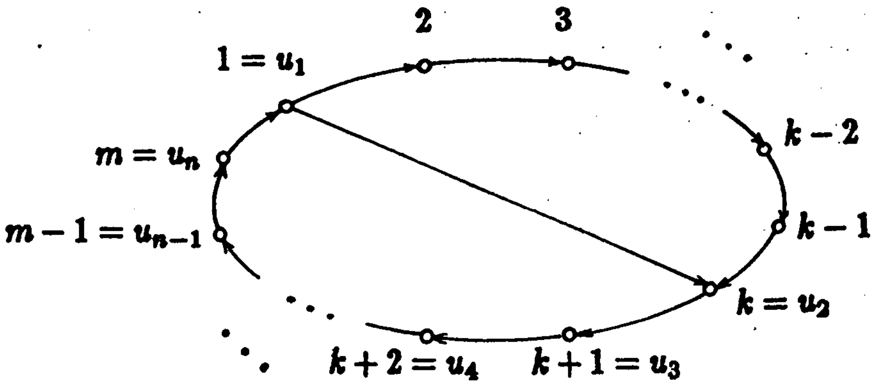

Figure 5

where the subscripts of u are understood modulo n. The condition that n does not divide m implies that U_1 and U_3 are disjoint. Since the vertices $2, \ldots, k-1$ of G occur only in U_2 we see that $U_1 \cup U_2 = \{(1, u_1)\}$ and $U_2 \cap U_3 = \{(k, u_k)\}$. It is clear that the induced graph on U_1 and U_3 is C_n and on U_2 is P_k. The union of these induced graphs is isomorphic to $G(n, k-1, n)$. To conclude the proof that $(G \times G) \restriction U$ is isomorphic to $G(n, k-1, n)$ therefore it is sufficient to check that there are no extra edges between elements of U. This follows easily from the fact that in the graph G every vertex has out-degree one, except $1 = u_1$, and that the endpoint of one of the edges starting from u_1, namely the vertex 2, occurs only in one of the pairs of U.

These examples point in the direction of characterizing those graphs G for which $\mathbf{V}(G)$ satisfies the descending chain condition for subvarieties. We have completed this task in the case of undirected graphs.

First we introduce names and recall some concepts. Let D_n be the undirected cycle (without loops) of length $n \geqq 3$, R_n the undirected (loopless) path of n edges $(n \geqq 0)$, and L_n the undirected path of $n+1$ vertices with a loop at every vertex $(n \geqq 0)$. Finally, G_0 denotes Murskii's graph (pictured on Figure 1), that is, the graph on $\{0, 1\}$ with $E(G_0) = \{(0,1), (1,0), (1,1)\}$.

A graph G is called *bipartite,* if $V(G)$ can be decomposed into the union of two disjoint subsets X and Y such that $E(G) \subseteqq (X \times Y) \cup (Y \times X)$. If equality holds instead of inclusion, then G is called *bipartite complete.* A graph is a *linear one-factor,* if every connected component is a (loopless, undirected) path of at most two vertices. Finally, a *discrete* graph is one without edges that are not loops.

First we formulate a result that belongs to the folklore of graph theory.

Lemma 4.10. *If a finite, undirected, loopless graph contains a cycle of odd length, then it contains an induced subgraph, which is a cycle of odd length. The graphs containing no cycle of odd length are exactly the two colorable, or bipartite graphs.*

Proof. Consider an odd cycle of minimal length. If the induced subgraph on this set is not a cycle, then this cycle has a chord that decomposes it into two smaller cycles, one of which must have odd length. The second statement is found in any standard textbook on graphs.

Example 4.11. (S. Oates-Williams [8]). Let $n\geqq 3$. Then $D_{n+2}\mathrm{V}(D_n)$, and if n is odd, then $D_n\notin \mathrm{V}(D_{n+2})$. Hence if $n\geqq 3$ is odd, then $\mathrm{V}(D_n)$ does not satisfy the descending chain condition for subvarieties.

Proof. Our methods enable to present a quick proof. First observe that an undirected cycle of length four can be collapsed to an undirected path of length two, so it may happen that a homomorphic image of a cycle does not contain a cycle at all. However, every homomorphic image (with or without loops) of an undirected cycle of odd length contains an undirected cycle of odd length (possibly a loop, which is a cycle of length one). Indeed, the edge set of this homomorphic image can be decomposed into the union of disjoint cycles, some of which have length two. These degenerate cycles, however, swallow up an even number of edges. Therefore this homomorphic image has a cycle of odd length. This argument shows that if n is odd, then there is no homomorphism from D_n to D_{n+2}, hence $D_n\notin \mathrm{V}(D_{n+2})$. On the other hand, consider the cycle $\{1, 2, ..., n+2\}$ and identify i and $i+2$ as well as $i+1$ and $i+3$. This yields a homomorphism into D_n. Doing this for all vertices i we obtain a separating family (actually, two of these homomorphisms will suffice).

Some of the following three examples are well-known (see for example, the paper of G. McNulty, C. Shallon [5]), and all of them have a straightforward proof based on Lemma 4.6 (1) and Lemma 4.5 (4).

Example 4.12. The variety $\mathrm{V}(G_0)$ contains all loopless, undirected graphs.

Example 4.13. The variety $\mathrm{V}(L_2)$ contains all looped, undirected graphs. The variety $\mathrm{V}(L_1)$ consists of all disjoint unions of complete looped graphs. Finally, $\mathrm{V}(L_0)$ contains exactly the discrete looped graphs.

Example 4.14. The variety $\mathscr{B}=\mathrm{V}(R_3)$ is the class of all (loopless, undirected) bipartite graphs, and $\mathrm{V}(R_2)=\mathrm{V}(D_4)$ is the class of all (loopless, undirected) bipartite complete graphs. The variety $\mathrm{V}(R_1)$ consists of the loopless graphs that are linear one-factors, finally $\mathrm{V}(R_0)$ consists of the loopless discrete graphs. Therefore all these classes of graphs are graph varieties.

Corollary 4.15. *Let $\mathscr{V}$ be a variety of undirected graphs satisfying the descending chain condition on subvarieties, and let G be a finite, connected member of $\mathscr{V}$. Then either G is loopless and bipartite, or it has a loop at every vertex. Hence* $\mathscr{V}\subseteqq \mathrm{V}\{L_2, R_3\}$.

Proof. The graph G cannot contain G_0 as an induced subgraph by 4.12 and 4.11, and since it is connected, it is either loopless or looped. In the first case it is bipartite by Lemma 4.10 and Example 4.11.

Example 4.16. The strongly critical members of the variety $\mathcal{V}_0 = \mathbf{V}\{L_2, R_3\}$ are exactly $L_0, L_1, L_2, R_0, R_1, R_2, R_3$. Hence $\mathcal{V}_0$ has finitely many subvarieties.

Proof. Examples 4.13 and 4.14 show that the graphs listed are indeed strongly critical (since they generate different varieties). Conversely, let G be a strongly critical member of $\mathcal{V}_0$. Then G is connected and is an induced subgraph of a direct product of copies of L_2 and R_3. If R_3 occurs in none of the factors, then G is looped, otherwise it is loopless and bipartite, since it has a homomorphism to the bipartite graph R_3. In the first case, G either contains L_2 as an induced subgraph, and as G is strongly critical, $G = L_2$ by Example 4.13, or G is complete. In the latter case either $G = L_0$ or G is in the variety generated by L_1 and hence, by criticality, $G = L_1$.

The other possibility is that G is loopless and bipartite, so it is in the variety $\mathcal{B}$. If G has no edges at all, then $G = R_0$. If G does not contain a path of length two, but contains an edge, then $G = R_1$. Otherwise, R_2 is a subgraph of G. As G is bipartite, it has no triangles, hence the induced subgraph on this path of length $G = R_2$. If not, then R_3 is an induced subgraph of G, hence $G = R_3$.

The proof of Theorem 1.7. Let $\mathcal{V}$ be a variety of undirected graphs. If $\mathcal{V} \subseteq \mathcal{V}_0$, then by Example 4.16, $\mathcal{V}$ has only finitely many subvarieties. If $\mathcal{V} \nsubseteq \mathcal{V}_0$, then $\mathcal{V}$ does not satisfy the descending chain condition on subvarieties by Corollary 4.15.

The lattice of subvarieties of $\mathcal{V}_0$ has seventeen elements, and is shown on Figure 6. The correctness of this picture follows from the following assertion, where the sign $\subset$ means: contained and not equal. The simple proofs are based on Lemma 4.6 (1), and are left to the reader.

Proposition 4.17. *The following relations hold.*

(1) $\mathbf{V}(R_0) \subset \mathbf{V}(R_1) \subset \mathbf{V}(R_2) \subset \mathbf{V}(R_3)$.

(2) $\mathbf{V}(L_0) \subset \mathbf{V}(L_1) \subset \mathbf{V}(L_2)$.

(3) *If* $L_i \in \mathbf{V}\{L_j, R_k\}$, *then* $L_i \in \mathbf{V}\{L_j\}$.

(4) *If* $R_i \in \mathbf{V}\{L_0, R_k\}$, *then* $R_i \in \mathbf{V}\{R_k\}$.

(5) $R_2 \in \mathbf{V}\{L_1, R_1\}$, $R_3 \notin \mathbf{V}\{L_1, R_2\}$, $R_1 \notin \mathbf{V}\{L_2, R_0\}$, $R_3 \in \mathbf{V}\{L_2, R_1\}$.

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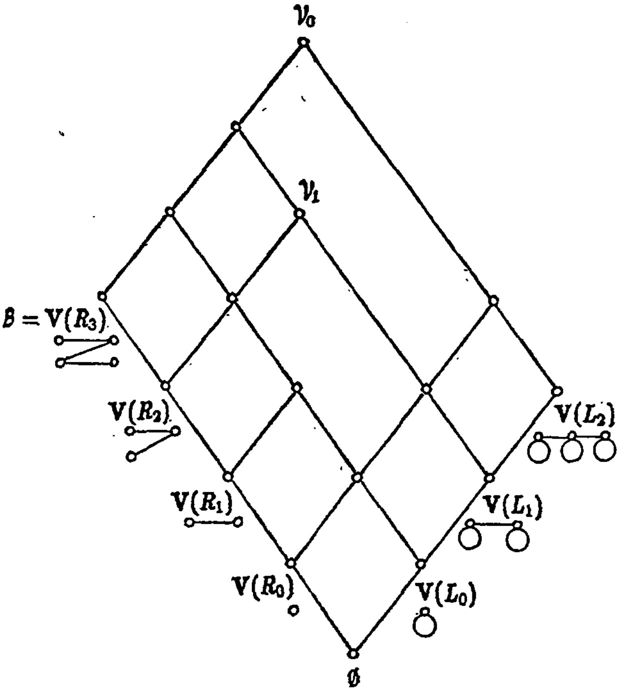

Figure 6

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