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Generalized congruences and products of lattice varieties

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1. Introduction. If **V** and **W** are lattice varieties, their product $\mathbf{V}\circ\mathbf{W}$ consists of all lattices L for which there is a congruence relation Θ satisfying: (i) all Θ-classes of L are in **V**; (ii) L/Θ is in **W**. In general, $\mathbf{V}\circ\mathbf{W}$ is not a variety; however, $\mathbf{H}(\mathbf{V}\circ\mathbf{W})$ (the class of all homomorphic images of members of $\mathbf{V}\circ\mathbf{W}$) always is, see G. GRÄTZER and D. KELLY [4] and [5] for related results (e.g., $\mathbf{D}\circ\mathbf{D}$ is a variety).

If L is in $\mathbf{V}\circ\mathbf{W}$ (established by Θ), then L/Φ, where Φ is a congruence relation on L, is a typical member of $\mathbf{H}(\mathbf{V}\circ\mathbf{W})$. On L/Φ, Θ/Φ is a tolerance relation (a reflexive and symmetric binary relation with the substitution property, see e.g. G. CZÉDLI [1]). R. N. McKenzie conjectured that a lattice K belongs to the variety generated by $\mathbf{V}\circ\mathbf{W}$ iff there is a tolerance relation T on K satisfying (i) all T-blocks (maximal T-connected subsets) of L are in **V**; (ii) L/T (the lattice of T-blocks with the natural ordering) is in **W**.

In the paper [2] we disproved this conjecture.

In this paper we introduce a generalization of the concept of congruence relation, called *generalized congruence* (Definition 1). With this new concept, the analogue of McKenzie's conjecture can be proved.

A congruence relation or a tolerance relation is a special type of generalized congruence; each can be viewed in two ways: as a binary relation on a lattice or as a set of subsets of a lattice. A generalized congruence is introduced as a family of subsets of a lattice; this family is not, in general, derivable from a binary relation on the lattice.

While congruence relations and tolerance relations are sets of subsets, generalized congruences are *families* of subsets, that is, a subset may occur more than once. In a *slim generalized congruence* repetition of a subset is not allowed. The main result

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of this paper (Theorem 8) shows that slim generalized congruences do not, in general, describe the members of $\mathbf{H}(\mathbf{V}\circ\mathbf{W})$.

For the basic concepts and unexplained notations, the reader is referred to [3].

The authors would like to express their appreciation to the referee; his incisive, yet generous, comments greatly contributed to a better final version.

2. Generalized congruences. We start with the basic definition:

Definition 1. Let L be a lattice. A *generalized congruence* Θ is a lattice defined on a subset of $P(L)\times I$, where I is a nonempty set — called the *index-set* — and $P(L)$ is the set of all subsets of L, with the following properties:

(GC_1) For $\langle A, i\rangle\in\Theta$, the first component A is a nonempty subset of L (called a *Θ-class*) and the second component i is an element of I.

(GC_2) The union of all Θ-classes is L; moreover, every $i\in I$ occurs in an $\langle A, i\rangle\in\Theta$.

(GC_3) For $\langle A, i\rangle$, $\langle B, j\rangle$, $\langle C, k\rangle\in\Theta$, if $\langle A, i\rangle\wedge\langle B, j\rangle=\langle C, k\rangle$ in Θ, and $a\in A$, $b\in B$, then $a\wedge b\in C$; and dually.

Note that in (GC_3), the $\wedge$ in $\langle A, i\rangle\wedge\langle B, j\rangle$ is the meet in Θ, while the $\wedge$ in $a\wedge b$ is the meet in L.

Let us start with some examples. Obviously, every congruence relation Θ can be regarded as a generalized congruence: $I=\{1\}$, and we identify the congruence class A with $\langle A, 1\rangle$. It is obvious that Θ is a lattice (the quotient lattice L/Θ) and (GC_3) holds. Similarly, every tolerance relation Θ is a generalized congruence with $I=\{1\}$; the Θ-classes are the blocks (maximal Θ-connected subsets). In the first example, the Θ-blocks are pairwise disjoint, in the second they are not, but a Θ-block cannot be contained in another Θ-block.

Let L and K be arbitrary lattices. Let us define Θ on L as follows: the elements of Θ are $\langle L, k\rangle$ for all $k\in K$; we define

$$\langle L, k_1\rangle\wedge\langle L, k_2\rangle = \langle L, k_1\wedge k_2\rangle,$$

and dually. Obviously, Θ as a lattice is isomorphic to K. This shows that by allowing a Θ-class to be paired with more that one member of K, we loosened the bond between the lattice L and the lattice Θ. In particular, Θ does not have to belong to the variety generated by L.

Figure 1 shows another example of a generalized congruence: C is the three-element chain and Θ is M_3, the five-element modular, nondistributive lattice.

In the above examples, Θ-classes are always convex sublattices. This is true in general:

Corollary 2. *Let L be a lattice, and let Θ be a generalized congruence on L. Then every Θ-class is a convex sublattice.*

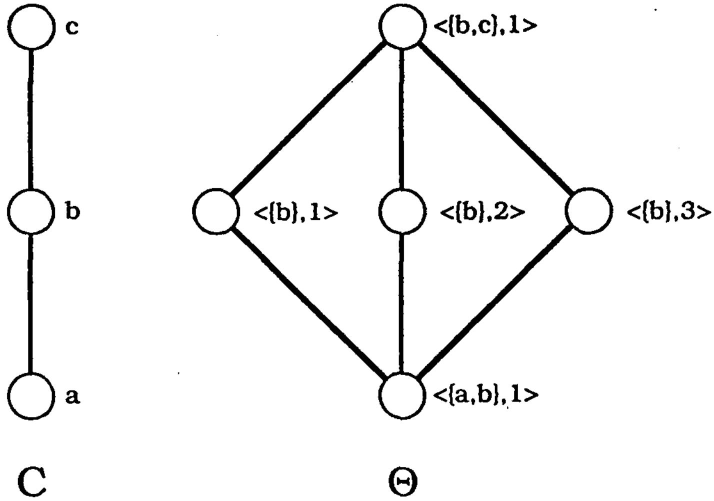


Figure 1

Proof. Let A be a Θ-class. Then $\langle A, i\rangle \in \Theta$ for some $i \in I$. Since Θ is a lattice, $\langle A, i\rangle \wedge \langle A, i\rangle = \langle A, i\rangle$. By ($GC_3$), this implies that A is a sublattice. Now let $a, b, c \in A$, $a < b$, and $c \in [a, b]$. By (GC_2), there is a $\langle C, j\rangle$ in Θ with $c \in C$. Since Θ is a lattice,

$$(\langle A, i\rangle \wedge \langle C, j\rangle) \vee \langle A, i\rangle = \langle A, i\rangle,$$

hence by (GC_3), $c = (b \wedge c) \vee a \in A$, finishing the proof.

Next we introduce a simple method of constructing generalized congruences:

Definition 3. Let L be a lattice, and let Θ and Φ be congruence relations on L. The generalized congruence Θ/Φ on L/Φ is defined as follows:

The index set is $I = L/\Theta$, that is, the Θ-classes. For every Θ-class A, define the pair: $\langle A/\Phi, A\rangle$, where A/Φ is the set of all Φ-classes not disjoint to A. The generalized congruence Θ/Φ on L/Φ is defined as the set of all such pairs; for two such pairs, $\langle A/\Phi, A\rangle$, $\langle B/\Phi, B\rangle$ we define the meet by:

$$\langle A/\Phi, A\rangle \wedge \langle B/\Phi, B\rangle = \langle (A \wedge B)/\Phi, A \wedge B\rangle$$

(where $A \wedge B$ is the meet of A and B in L/Θ), and dually for the join.

It is easy to verify that this defines a generalized congruence on L/Φ. Indeed, (GC_1) is obvious. Let A be an element of L/Φ, that is, A is a Φ-class on L. Let $a \in A$, and let B be a Θ-class containing a. Then $A \in B/\Phi$, and $\langle B/\Phi, B\rangle \in \Theta/\Phi$, hence the first clause of (GC_2) holds. The second clause of (GC_2) and (GC_3) are clear.

Now we prove the converse:

Lemma 4. *Let K be a lattice, and let Ψ be a generalized congruence on K. Then there exists a lattice L, and congruences Θ and Φ on L, such that K is isomorphic to L/Φ, and under this isomorphism the Ψ-classes of K correspond to the Θ/Φ-classes of L/Φ. In fact, L can be chosen as a subdirect product of K and the lattice Ψ.*

Proof. Let L be the set of all pairs $\langle a, \langle A, i\rangle\rangle$, where $a \in A$ and $\langle A, i\rangle \in \Psi$. Since a Ψ-block is nonempty and since Ψ satisfies (GC_2), it is easy to see that L is a subset of the direct product of K and Ψ, and the projection maps are onto. By (GC_3), L is a subdirect product. Let Φ be the kernel of the projection of L onto K, and let Θ be the kernel of the projection of L onto Ψ. An easy computation shows that the Θ/Φ-classes on L/Φ are the same as the Ψ-classes on K.

One could easily define formally when two generalized congruences on the lattice L are the "same"; basically, they must select the same convex sublattices, each the same number of times.

Now we use generalized congruences to describe members of the variety generated by the product of two varieties:

Theorem 5. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be lattice varieties. Then the lattice K belongs to the variety generated by* $\mathbf{V}\circ\mathbf{W}$ *iff there is a generalized congruence Ψ on K such that the Ψ-blocks are in* $\mathbf{V}$ *and Ψ as a lattice is in* $\mathbf{W}$.

Proof. By Definition 3 and Lemma 4.

Observe the following trivial, but useful, sharpening of Theorem 5. If K belongs to the variety generated by $\mathbf{V}\circ\mathbf{W}$, then there is a lattice L in $\mathbf{V}\circ\mathbf{W}$, and there are congruences Θ and Φ on L such that Θ establishes that L belongs to $\mathbf{V}\circ\mathbf{W}$, K is (isomorphic to) L/Φ, and $\Theta\wedge\Phi=\omega$. In other words, the Ψ of Theorem 5 can always be chosen to be in the form Θ/Φ with $\Theta\wedge\Phi=\omega$.

3. Slim generalized congruences. The use of the index set is the most fundamental difference between congruence relations and tolerance relations on the one hand and generalized congruences on the other. If every subset can occur only once in a generalized congruence, then we get a concept much closer to those of congruence relations and tolerance relations.

Another difference is the way the meet (and the join) of two classes is found. For congruences, if A and B are two classes, $a \in A$ and $b \in B$, then we take the unique class C containing $a\wedge b$, and C is $A\wedge B$. For tolerances, the situation is somewhat more complicated. $a\wedge b$ may be contained in more than one block. However, it was pointed out in [1] and [6], that

$$\{a\wedge b \mid a\in A,\ b\in B\}$$

is always contained in a unique block, namely, in $A\wedge B$.

This suggests that one could define two special types of generalized congruences:

Definition 6. Let L be a lattice and let Θ be a generalized congruence on L. We call Θ *slim* iff the index set is a singleton. We call Θ *conservative* iff for Θ-blocks A and B, the set $\{a \wedge b \mid a \in A,\ b \in B\}$ uniquely defines the $\langle C, i\rangle \in \Theta$ where C contains this set, and dually.

By (GC_3), if the generalized congruence Θ is conservative, then the lattice operations on L are uniquely defined on (preserved by) Θ.

Corollary 7. *A conservative generalized congruence is slim.*

Proof. Let Θ be a conservative generalized congruence on L. If Θ is not slim then there is a Θ-class A, and there are $i, j \in I$, such that $\langle A, i\rangle$ and $\langle A, j\rangle \in \Theta$. Now observe that $\{a \wedge b | a \in A, b \in A\} = A$ is contained by the Θ-class in both $\langle A, i\rangle$ and $\langle A, j\rangle$, a contradiction.

It would be highly desirable to prove Theorem 5 for slim generalized congruences. The main result of this paper shows that this cannot be done:

Theorem 8. *There are lattice varieties* $\mathbf{V}$ *and* $\mathbf{W}$ *and there is a lattice* K *in the variety generated by* $\mathbf{V} \circ \mathbf{W}$ *such that there is no slim generalized congruence* Ψ *on* K *with the properties: all* Ψ*-classes are in* $\mathbf{V}$ *and* Ψ *as a lattice is in* $\mathbf{W}$.

For $\mathbf{W}$, we can take $\mathbf{D}$ the variety of all distributive lattices. The variety $\mathbf{V}$ and the lattice K will be constructed in Section 4. The proof of Theorem 8 will be presented in Section 5.

4. The variety V. The lattice variety $\mathbf{V}$ is generated by the lattice F of Figure 2 and its dual. Note that F is a subdirectly irreducible lattice. The crucial property of this variety is stated in

Lemma 9. *The lattices* $M_{3,3}$ *and* N_8 *of Figure* 3 *are not in* $\mathbf{V}$.

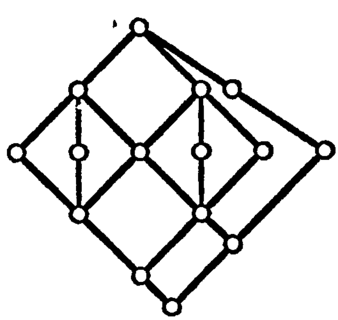
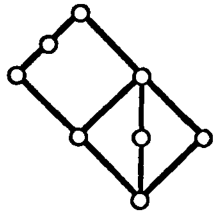
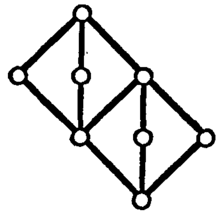


Figure 2



N_8



$M_{3,3}$

Figure 3

Proof. By B. Jónsson's Lemma (see, e.g. [3]), since F and its dual are finite, the subdirectly irreducible members of $\mathbf{V}$ are homomorphic images of sublattices of F or of its dual. If M_3 is a homomorphic image of a finite lattice, then the largest inverse image of its zero is the pairwise meet of three pairwise incomparable elements. This easily implies both the statements of this lemma.

Corollary 10. *The dual of N_8 is not in* $\mathbf{V}$.

Proof. Indeed, $\mathbf{V}$ is selfdual.

The rest of the paper deals with the lattice K of Figure 4.

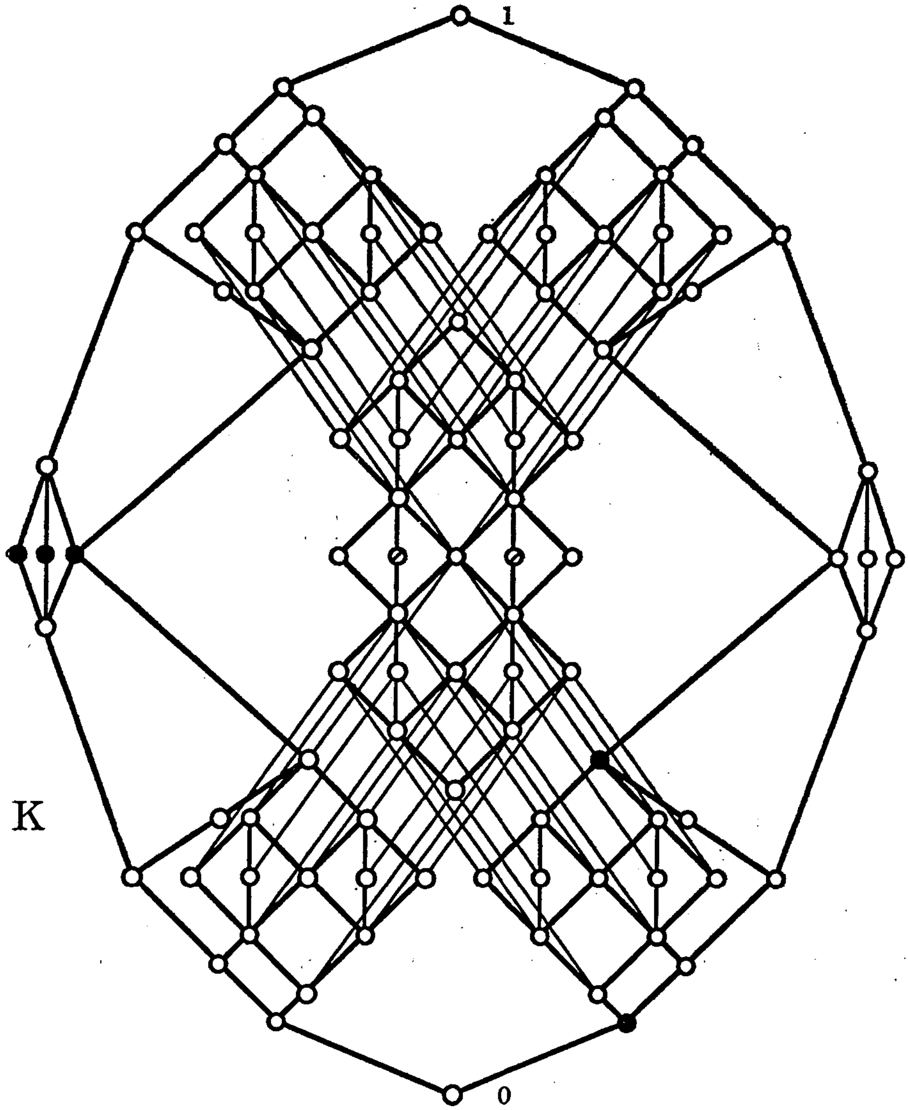


Figure 4

Lemma 11. *K is a lattice.*

Proof. K has 97 elements. By inputting the elements and the covering relations into a computer program H. Lakser verified that K is a lattice. Alternatively, one can apply the following two lattice theoretic trivialities to build K up from smaller parts.

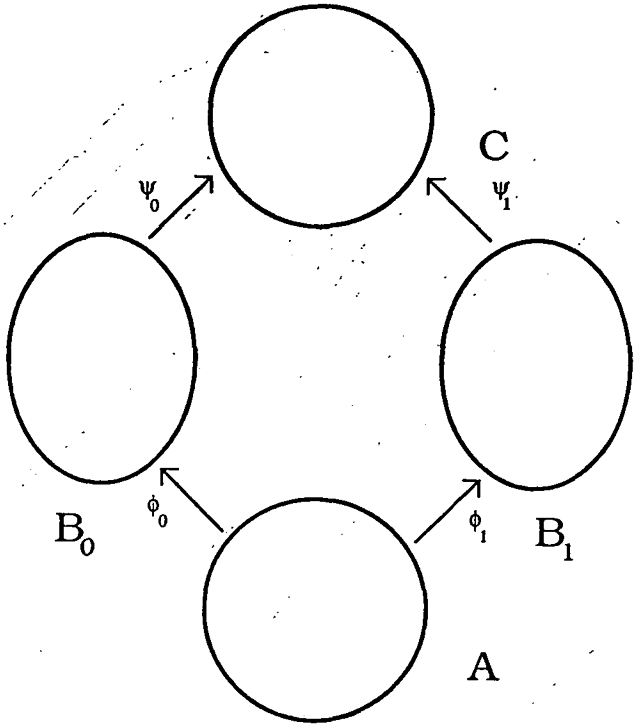


Figure 5

Lemma 12. *Let L be a partially ordered set made up of four pairwise disjoint convex sublattices: A, B_0, B_1, C arranged as in Figure 5. Let us assume that for each $a \in A$ and $i=0, 1$, there is a smallest upper bound of a in B_i, denoted by $a\varphi_i$, and for every $b \in B_i$ and $i=0, 1$, there is a smallest upper bound of b in C, denoted by $b\psi_i$; and dually. Finally, assume that for $a \in A$, $a\varphi_0\psi_0 = a\varphi_1\psi_1$, and dually. Then L is a lattice.*

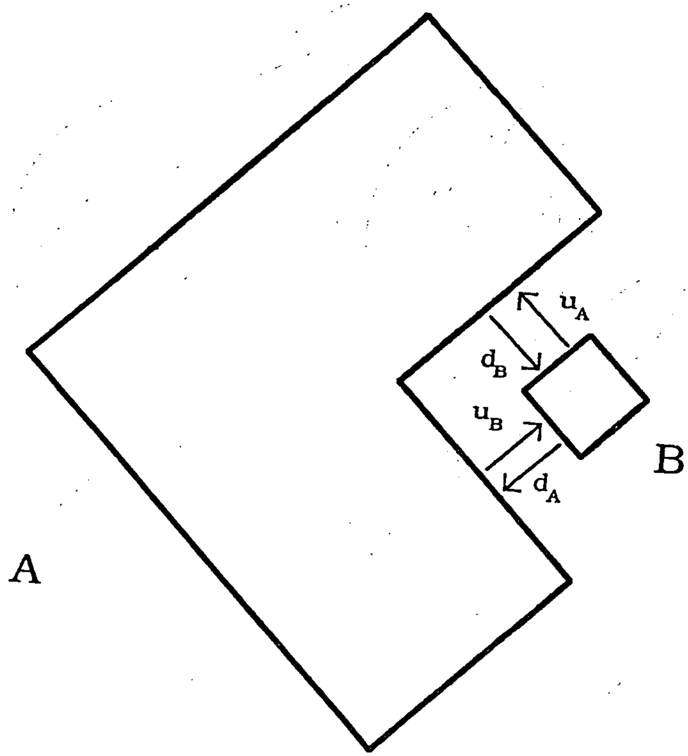


Figure 6

Lemma 13. *Let L be a partially ordered set made up of two disjoint lattices A and B as arranged in Figure 6. Let us assume that for each $a \in A$ that has an upper bound in B, there is a smallest upper bound of a in B, denoted by au_B, and dually (the element denoted by ad_B), and for every $b \in B$ there is a smallest upper bound bu_A in A, and dually (the element denoted by bd_A). Then L is a lattice.*

Now to verify that K is a lattice, look at K as depicted in Figure 7.

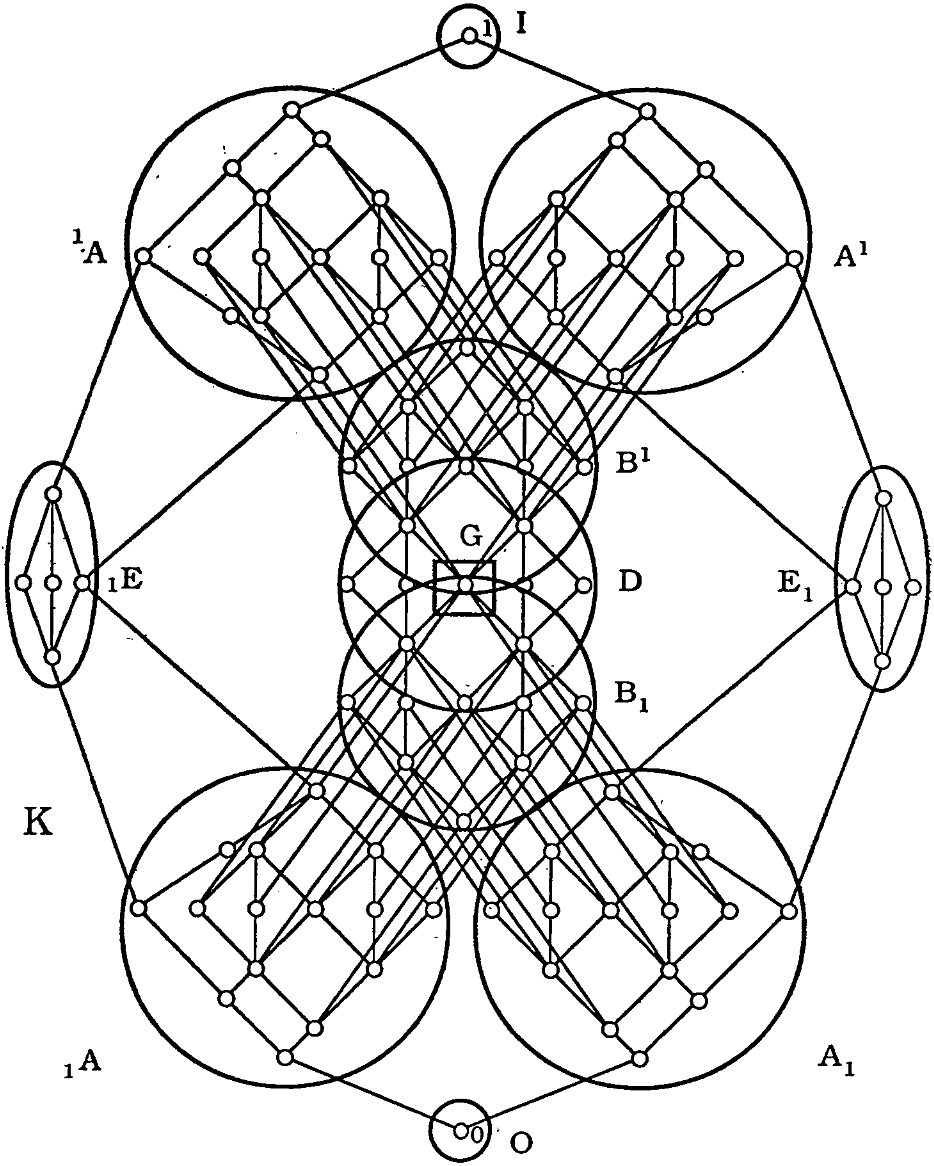


Figure 7

Applying Lemma 12, we get that $O \cup A_1 \cup {}_1A \cup B_1$ is a lattice. Gluing this together with the lattice $B_1 \cup D \cup B^1$, then with $I \cup A^1 \cup {}^1A \cup B^1$, we obtain the lattice A of Lemma 13. We then apply Lemma 13 with $B = E_1$, and renaming the resulting lattice A, with $B = {}_1E$, obtaining K. This shows that K is a lattice.

Theorem 14. *K is a lattice in the variety generated by* $\mathbf{V} \circ \mathbf{D}$.

Proof. By Theorem 5, we have to find a generalized congruence Ψ such that Ψ is a distributive lattice and the Ψ-blocks are in $\mathbf{V}$.

First, we define 24 convex sublattices on K. Figure 7 defines 12 of them. A_1 contains two more sublattices: A_2 and A_3 as shown in Figure 8; A_2 is the sublattice over the lower dividing line, it has 10 elements, while A_3 is the three-element chain. Similarly, A^1, ${}_1A$, and ${}^1A$ contain two more sublattices each. Finally, B_1 contains two more sublattices: ${}_1C$ and C_1, as shown in Figure 9, and similarly, B^1 contains ${}^1C$ and C^1.

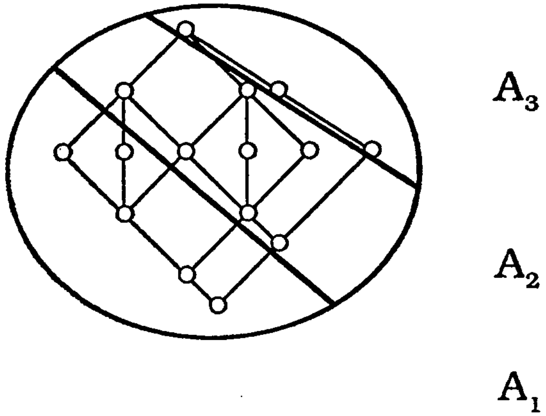


Figure 8

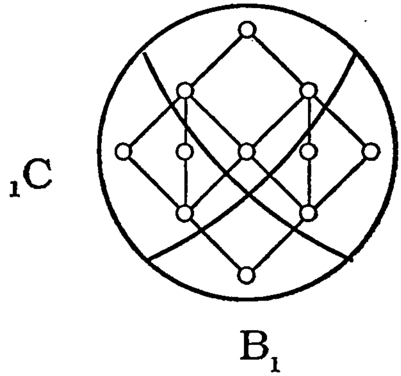


Figure 9

Ψ is shown in Figure 10. For the index set we choose $I=\{1, 2\}$. For a convex sublattice X of K, Figure 10 shows X for $\langle X, 1\rangle$. Only G occurs twice in Ψ: $\langle G, 1\rangle$ and $\langle G, 2\rangle$. Ψ is not slim, but it is very close to being slim.

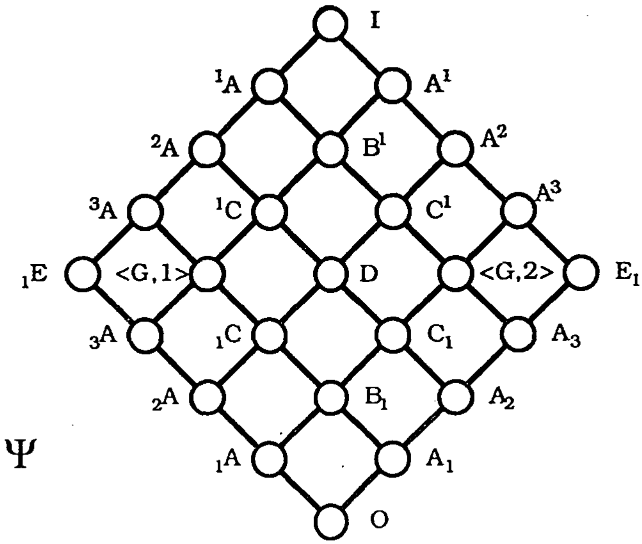


Figure 10

To show that Ψ is a generalized congruence one has to check (GC_1), (GC_2), and (GC_3). (GC_1) and (GC_2) are trivial; (GC_3) is tedious but easy.

Ψ as a lattice is isomorphic to $(C_5)^2$; so it is in **D**. The Ψ-classes are all isomorphic to a sublattice of F or of its dual, so they are all in **V**. This completes the proof of Theorem 14.

Theorem 14, combined with Lemma 4, gives us a lattice L in $\mathbf{V} \circ \mathbf{D}$ and congruence relations Θ and Φ on L such that L/Φ is isomorphic to K, L/Θ is distributive and the Ψ-blocks on K agree with the Θ/Φ-blocks on L/Φ. This lattice L has 191 elements.

5. Generalized congruences on K. Now we come to the crucial part of this paper: we have to show that K is not a homomorphic image of some member of $\mathbf{V} \circ \mathbf{D}$, such that the resulting Θ/Φ is slim. We had a similar problem in [2]: how to show the nonexistence of a tolerance relation with some properties on a certain lattice. The problem in [2] was much easier. The tolerance relations on a lattice form a lattice; the elements of the tolerance lattice can be described.

The generalized congruences on a lattice do not form a lattice. One cannot show the nonexistence of a generalized congruence by enumeration. Our proof will be presented in many steps, as a sequence of propositions.

In this section the lattice K is the lattice of Figure 4, and Z is a slim generalized congruence on K with the properties that the Z-classes are in **V** and Z as a lattice is distributive. Since Z is slim, we can ignore the index set: the element $\langle X, i\rangle$ of Z will be identified with the Z-class X. So Z is a distributive lattice of nonempty convex subsets of K; the union of these sublattices is K by (GC_2); if $A \wedge B = C$ in Z, $a \in A$ $b \in B$, then $a \wedge b \in C$, and dually, by (GC_3).

For a sublattice X of K, $0(X)$ and $1(X)$ will denote the smallest and largest elements of X, respectively.

Proposition 15. *Let m and M be the lower and upper median polynomials. If $a_0, a_1, a_2 \in K$, then there exists a Z-class X containing $m(a_0, a_1, a_2)$ and $M(a_0, a_1, a_2)$.*

Proof. By (GC_2), there are $A_j \in Z$ such that $a_j \in A_j$ for $j=0, 1, 2$. Let $m(A_0, A_1, A_2)=X$. By the distributivity of Z,

$$m(A_0, A_1, A_2) = X = M(A_0, A_1, A_2),$$

hence, by (GC_3), $m(a_0, a_1, a_2)$, $M(a_0, a_1, a_2) \in X$, as claimed.

Note that Proposition 15 claims that $m(a_0, a_1, a_2)$ and $M(a_0, a_1, a_2)$ are contained in some Z-class; other Z-classes may separate the two elements; examples of this can be found in the lattice K.

Let us call a Z-class *maximal* if there is no Z-class properly containing it.

Proposition 16. *${}_1E$ is a Z-class, in fact, it is a maximal Z-class.*

Proof. Apply Proposition 15 to the three black-filled elements on the left of K in Figure 4. Then m and M give us $0({}_1E)$ and $1({}_1E)$. Therefore, there is a Z-class X containing ${}_1E$. If X properly contained ${}_1E$, then it would contain N_8 or its dual as a sublattice, contradicting that the Z-classes are in $\mathbf{V}$ and Lemma 9.

Proposition 17. *E_1 is a Z-class, in fact, it is a maximal Z-class.*

Proof. By symmetry.

Proposition 18. *There is no Z-class containing* 0 *and* $0({}_1A)$.

Proof. Let X be a Z-class containing 0 and $0({}_1A)$. Then $X \vee E_1$ properly contains E_1; this contradicts Proposition 16.

Proposition 19. *There is no Z-class containing x, y with $x \in E_1$ and $y \notin E_1$.*

Proof. Let X be a Z-class containing x and y as in this proposition. Then $Y = X \vee E_1$ contains $1(E_1)$ and its unique cover. Let U be any Z-class containing $0({}_1A)$; then $Y \wedge U$ is a Z-class containing 0 and $0({}_1A)$, contradicting Proposition 18.

Proposition 20. *There is no Z-class containing x, y with $x \in B_1 \cup D \cup B^1$ and $y \notin B_1 \cup D \cup B^1$.*

Proof. Let X be a Z-class containing x and y. By meeting or joining X with ${}_1E$ or E_1 (depending on where y is), we get a contradiction with Proposition 19.

Proposition 21. $A_1, A^1, {}_1A$ and ${}^1A$ are Z-classes, in fact, they are maximal Z-classes.

Proof. Take a black-filled element on the left and the two dot-filled elements on the right in Figure 4, and apply Proposition 15. We conclude that there is a Z-class X containing $0(A_1)$ and $1(A_1)$. If X properly contains A_1, we get a contradiction with Proposition 17 (if $y \in E_1$), Proposition 18 (if $y=0$ or $y \in {}_1A$), or Proposition 20 (if $y \in B_1 \cup D \cup B^1$). The other cases are similar.

Proposition 22. *B_1 and B^1 are Z-classes, in fact, they are maximal Z-classes.*

Proof. ${}_1A \vee A_1 = X$ is a Z-class containing B_1. If X has an element x not in B_1, then we obtain a contradiction: if $x \in A_1 \cup {}_1A$ or $x=0$, this contradicts Proposition 20; if $x \in E_1 \cup {}_1E$, this contradicts Proposition 19; finally, if x is anywhere else, then by the convexity of X, we can assume that $x \in D$, and we find in X a sublattice that has a homomorphism onto $M_{3,3}$, contradicting $X \in \mathbf{V}$ (see Lemma 9). Thus B_1 is a maximal Z-class. Similarly, B^1 is a maximal Z-class.

Proposition 23. *$B_1 \wedge E_1 = A_1$, and similarly for ${}_1B$ and ${}_1E$, etc.*

Proof. By (GC_3), if $B_1 \wedge E_1 = X$, then

$$0(B_1) \wedge 0(E_1) = 0(A_1) \in X$$

similarly, $1(A_1) \in X$, hence X contains A_1. By Proposition 21, $X = A_1$.

Proposition 24. *O and I are Z-classes, in fact, they are maximal Z-classes.*

Proof. There is a Z-class containing O by (GC_2). This class must be O by Proposition 18.

Proposition 25. *${}_1E \wedge E_1 = O$ and ${}_1E \vee E_1 = I$.*

Proof. By Proposition 24.

Proposition 26. *D is a Z-class, in fact, it is a maximal Z-class.*

Proof. Apply Proposition 15 to the following elements in Figure 4: the leftmost black-filled element and the two elements with a right-slanted bar. We get that there is a Z-class X containing $0(D)$ and $1(D)$; thus X contains D. If there is an x in X but not in $\mathbf{D}$, this would lead to a contradiction, just as in the proof of Proposition 25.

Proposition 27. *$D \wedge {}_1E = {}_2A$, $D \wedge E_1 = A_2$, and dually.*

Proof. Since Z is distributive, $m({}_1E, D, E_1)=M({}_1E, D, E_1)$. Let X be this Z-class. Compute:

$$m(0({}_1E), 0(D), 0(E_1)) = 0(D) \quad \text{and} \quad M(1({}_1E), 1(D), 1(E_1)) = 1(D).$$

So X must contain D. By Proposition 26, $X=D$. Hence,

$$D = (D\wedge{}_1E)\vee(D\wedge E_1)\vee(E_1\wedge{}_1E).$$

Obviously, $D\wedge{}_1E$ is ${}_1A$ or ${}_2A$, and $D\wedge E_1$ is A_1 or A_2. If, say, $D\wedge E_1=A_1$, then the last equation implies that

$$0({}_2A)\vee 0(A_1)\vee 0\in D,$$

a contradiction.

Proposition 28. *${}_3A, A_3, {}^3A,$ and A^3 are Z-classes.*

Proof. ${}_1E\wedge A^1={}_2A$ or ${}_3A$. But ${}_1E\wedge A^1={}_2A$ contradicts distributivity.

Proposition 29. *$G={}_3A\vee A_1$ and $G={}^3A\wedge A^1$.*

Proof. Let us assume that ${}_3A\vee A_1<{}^3A\wedge A^1$. Then ${}_1E$, ${}_3A\vee A_1$, and ${}^3A\wedge A^1$ generate N_5, the five-element nonmodular lattice. Therefore, ${}_3A\vee A_1={}^3A\wedge A^1$. Since $0({}_3A)\vee 0(A_1)=1({}^3A)\wedge 1(A^1)\in G$, it follows that ${}_3A\vee A_1$ includes G as a subset. Now $({}_3A\vee A_1)\vee E_1=A^3$, ${}_3A\wedge({}_3A\vee A_1)={}_3A$, with their duals and symmetric counterparts yield by (GC_3) that G is not a proper subset of ${}_3A\vee A_1={}^3A\wedge A^1$.

Proposition 30. *Z is not distributive.*

Proof. Using the distributivity of Z, compute: $G={}_3A\vee A_1=A_3\vee{}_1A$. Hence

$$G = ({}_3A\vee A_1)\wedge(A_3\vee{}_1A) = ({}_3A\wedge A_3)\vee({}_3A\wedge{}_1A)\vee(A_1\wedge A_3)\vee(A_1\wedge{}_1A) =$$
$$= O\vee{}_1A\vee A_1\vee O = B_1,$$

a contradiction.

The proof of Theorem 8 is now complete.

6. Concluding comments. It would be interesting to find the smallest lattice variety **V** for which Theorem 8 holds with **W**=**D**. We suspect that the variety generated by M_3 would do; we have even constructed the lattice that corresponds to K. However, we cannot prove that K works.

One could conjecture that Theorem 8 holds for any pair of lattice varieties **V** and **W**, for which **V**∘**W** is not a variety. The proof of this conjecture is well beyond the capabilities of the methods used in this paper.

References

[1] G. Czédli, Factor lattices by tolerances, *Acta Sci. Math.*, **44** (1982), 35—42.

[2] E. Fried and G. Grätzer, Notes on tolerance relations of lattices: On a conjecture of R. McKenzie, *Acta Sci. Math.*, to appear.

[3] G. Grätzer, General Lattice Theory, Pure and Applied Mathematics Series, Academic Press (New York, N. Y.); Mathematische Reihe, Band 52, Birkhäuser Verlag (Basel); Akademie Verlag (Berlin, 1978).

[4] G. Grätzer and D. Kelly, The lattice variety $\mathbf{D} \circ \mathbf{D}$, *Acta Sci. Math.*, **51** (1987), 73—80. Addendum: *Acta Sci. Math.*, **52** (1988), 465.

[5] G. Grätzer and D. Kelly, A survey of products of lattice varieties, in: *Contributions to Lattice Theory* (Proc. Conf. Szeged, Hungary, 1980), Colloq. Math. Soc. János Bolyai 33, North-Holland (Amsterdam, 1983), pp. 457—472.

[6] G. Grätzer and G. H. Wenzel, Notes on tolerance relations of lattices, *Acta Sci. Math.*, to appear.

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