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On the joint Weyl spectrum. II

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Dedicated to Professor Satoshi Kotō in token of gratitude

1. Introduction

In [3], we studied the joint Weyl spectrum for a commuting pair. In this paper we shall show that the Weyl theorem holds for a commuting pair of normal operators.

Let H be a complex Hilbert space with the scalar product $(,)$ and the norm $\|\cdot\|$. Let $B(H)$ be the algebra of all bounded linear operators on H and $C(H)$ the algebra of all compact operators in $B(H)$.

Definition 1. Let $T=(T_1, T_2)\subset B(H)$ be a commuting pair. Taylor joint spectrum $\sigma(T)$ of T is defined by $\sigma(T)=\{z=(z_1, z_2)\in \mathbf{C}^2: \alpha(T-z)$ is not invertible on $H\oplus H\}$, where

$$\alpha(T-z) = \begin{pmatrix} T_1-z_1 & T_2-z_2 \\ -(T_2-z_2)^* & (T_1-z_1)^* \end{pmatrix}.$$

Definition 2. Let $T=(T_1, T_2)\subset B(H)$ be a commuting pair. The joint Weyl spectrum $\omega(T)$ of T is defined by

$$\omega(T) = \bigcap\{\sigma(T+C): C = (C_1, C_2) \subset C(H) \text{ and } T+C = (T_1+C_1, T_2+C_2)$$

is a commuting pair}.

$z=(z_1, z_2)$ in $\mathbf{C}^2$ is said to be joint eigenvalue of $T=(T_1, T_2)$ if there exists a non-zero vector x such that

$$T_i x = z_i x \quad (i = 1, 2).$$

$\sigma_p(T)$ is the set of joint eigenvalues of T.

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$z=(z_1, z_2)$ in $\mathbf{C}^2$ is said to be joint residual eigenvalue of $T=(T_1, T_2)$ if there exists a non-zero vector x such that

$$T_i^* x = \bar{z}_i x \quad (i = 1, 2).$$

$\sigma_r(T)$ is the set of joint residual eigenvalues of T.

For a commuting pair $T=(T_1, T_2)$, $\sigma_{pf}(T)$ is the set of joint eigenvalues of finite multiplicity, $\sigma_{rf}(T)$ is the set of joint residual eigenvalues of finite multiplicity, $\sigma_{pfi}(T)$ is the set of isolated points in $\sigma(T)$ which are joint eigenvalues of finite multiplicity and $\sigma_{rfi}(T)$ is the set of isolated points in $\sigma(T)$ which are joint residual eigenvalues of finite multiplicity.

For any operator S on H, we denote by $N(S)$ the null space of S.

2. Theorem

Theorem A (Vasilescu [6]). *For a commutating pair* $T=(T_1, T_2)$, $\alpha(T)$ *is invertible if and only if*

$$\beta(T) = \begin{pmatrix} T_1 & -T_2^* \\ T_2 & T_1^* \end{pmatrix}$$

is invertible on $H \oplus H$.

Theorem B (Chō and Takaguchi [3]). *For a commuting pair* $T=(T_1, T_2)$,

$$\sigma(T) - \omega(T) \subset \sigma_p(T) \cup \sigma_r(T).$$

Lemma 1. *Let* $T=(T_1, T_2)$ *be a commuting pair. Then*

$$\sigma(T) - \sigma_{pf}(T) \cup \sigma_{rf}(T) \subset \omega(T).$$

Proof. Let $z=(z_1, z_2)$ be a joint eigenvalue of infinite multiplicity. Let $C=(C_1, C_2)$ be in $C(H)$ such that $T+C=(T_1+C_1, T_2+C_2)$ is a commuting pair. For a infinite orthonormal sequence $\{x_n\}$ in $\{x\colon T_i x = z_i x\ (i=1, 2)\}$, we may assume that there exist vectors y_1 and y_2 such that $\lim C_i x_n = y_i$ $(i=1, 2)$. If

$$\beta(T+C-z) = \begin{pmatrix} T_1+C_1-z_1 & -(T_2+C_2-z_2)^* \\ T_2+C_2-z_2 & (T_1+C_1-z_1)^* \end{pmatrix}$$

is invertible, then

$$\lim (x_n \oplus 0) = \beta(T+C-z)^{-1}(y_1 \oplus y_2).$$

It is a contradiction to the choice of $\{x_n\}$. So it follows, by Theorem A, that $z \in \omega(T)$.

Let $z=(z_1, z_2)$ be a joint residual eigenvalue of infinite multiplicity. Then for an infinite orthonormal sequence $\{x_n\}$ in

$$\{x\colon T_i^* x = \bar{z}_i x \ (i = 1, 2)\},$$

we may assume that there exist vectors y_1 and y_2 such that

$$\lim C_i^* x_n = y_i \quad (i = 1, 2).$$

If $\beta(T+C-z)$ is invertible, then

$$\lim (0 \oplus x_n) = \beta(T+C-z)^{-1}(-y_2 \oplus y_1).$$

It is a contradiction. So it follows that $z \in \omega(T)$.

So the proof is complete by Theorem B.

Next following Baxley we consider the following condition $\mathscr{C}_1$: If $\{z_n\}$ is an infinite sequence of distinct points in $\sigma_{pf}(T) \cup \sigma_{rf}(T)$ and $\{x_n\}$ is any sequence of corresponding normalized joint eigenvectors, then the sequence $\{x_n\}$ does not converge.

Lemma 2. *If a commuting pair $T=(T_1, T_2)$ satisfies $\mathscr{C}_1$, then*

$$\sigma(T) - (\sigma_{pfi}(T) \cup \sigma_{rfi}(T)) \subset \omega(T).$$

Proof. We have the identity

$$\sigma(T) - (\sigma_{pfi}(T) \cup \sigma_{rfi}(T)) =$$
$$(\sigma(T) - (\sigma_{pf}(T) \cup \sigma_{rf}(T))) \cup ((\sigma_{pf}(T) \cup \sigma_{rf}(T)) - (\sigma_{pfi}(T) \cup \sigma_{rfi}(T))).$$

So, by the above lemma, it will be sufficient to prove that $z=(z_1, z_2)$ is in $(\sigma_{pf}(T) \cup \sigma_{rf}(T)) - (\sigma_{pfi}(T) \cup \sigma_{rfi}(T))$ and not in the closure of $(\sigma(T) - (\sigma_{pf}(T) \cup \sigma_{rf}(T)))$, then z is in $\sigma(T+C)$ for every $C=(C_1, C_2)$ such that $T+C=(T_1+C_1, T_2+C_2)$ is a commuting pair.

Assume that z is in $(\sigma_{pf}(T) \cup \sigma_{rf}(T)) - (\sigma_{pfi}(T) \cup \sigma_{rfi}(T))$. Then there exist $z_n = (z_1^n, z_2^n)$ $(n=1, 2, \ldots)$ in $\sigma_{pf}(T)$ or in $\sigma_{rf}(T)$ such that $z_n \neq z_m$ $(n \neq m)$ and $\lim z_n = z$. Suppose that the z_n's are in $\sigma_{pf}(T)$, then we can consider a sequence $\{x_n\}$ of unit vectors such that $T_i x_n = z_i^n x_n$ $(i=1, 2)$ for every n. Of course, we can suppose, without loss of generality, that there exist vectors y_1 and y_2 such that $\lim C_i x_n = y_i$ $(i=1, 2)$. If, for $T+C-z=(T_1+C_1-z_1, T_2+C_2-z_2)$, $\beta(T+C-z)$ is invertible, then

$$\lim (x_n \oplus 0) = \beta(T+C-z)^{-1}(y_1 \oplus y_2).$$

It is a contradiction to the condition $\mathscr{C}_1$. So it follows that $z \in \omega(T)$.

When $\{z_n\}$ is in $\sigma_{rf}(T)$, then we can prove that z belongs to $\omega(T)$ in a similar way (see the proof of the lemma above).

So the proof is complete.

Next we shall show that the Weyl theorem holds for a commuting pair of normal operators. We need the following theorem. An easy computation shows that the theorem holds:

Theorem C. *Let $T=(T_1, T_2)$ be a commuting pair of normal operators. Then $\alpha(T)$ is invertible if and only if $T_1^* T_1 + T_2^* T_2$ is invertible.*

Theorem. *Let $T=(T_1, T_2)$ be a commuting pair of normal operators. Then the Weyl theorem holds, that is,*

$$\sigma(T)-\sigma_{pfi}(T) = \omega(T).$$

Proof. Since $T=(T_1, T_2)$ is a commuting pair of normal operators, T satisfies the condition $\mathscr{C}_1$. So, by Lemma 2, it suffices to prove that

$$\sigma(T)-\sigma_{pfi}(T) \supset \omega(T).$$

Let us consider a point in $\sigma_{pfi}(T)$. We may assume without loss of generality that this is (0, 0). We define $N=N(T_1^*T_1+T_2^*T_2)$, then $\dim(N)<\infty$. Let P denote the orthogonal projection of H onto N. Then P is a compact operator and $T_iP= =PT_i=0$ $(i=1, 2)$. Hence

$$T+Q = (T_1+(1/\sqrt{2})P,\ T_2+(1/\sqrt{2})P)$$

is a commuting pair of normal operators. Since (0, 0) is an isolated point of $\sigma(T)$, so using Theorem C for T_1-z_1 and T_2-z_2 in place of T_1 and T_2, respectively, by continuity arguments we obtain that 0 is an isolated point in the spectrum of $T_1^*T_1+ +T_2^*T_2$. It follows that

$$(T_1+(1/\sqrt{2})P)^*(T_1+(1/\sqrt{2})P)+(T_2+(1/\sqrt{2})P)^*(T_2+(1/\sqrt{2})P) = T_1^*T_1+T_2^*T_2+P$$

is invertible. So, by Theorem C, $(0, 0)\notin\sigma(T+Q)$ and thus $(0, 0)\notin\omega(T)$.

So the proof is complete.

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