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On the boundedness of solutions of nonautonomous differential equations

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Dedicated to L. Pintér on his 60th birthday

1. Introduction

In the study of existence of periodic solutions and almost periodic solutions as well as behavior of limiting sets of solutions of ordinary differential equations, the uniform boundedness and uniform ultimate boundedness of solutions are frequently needed [1—4, 9]. These properties of solutions can be regarded as either the instability of infinity or a special case of some kind of stability of a set. Therefore, there exists a close relation between Lyapunov's direct method and the boundedness of solutions. A typical result showing this relation is Theorem 10.4 in [3]. In this theorem the uniform ultimate boundedness is guaranteed by the existence of an appropriate Lyapunov function having a negative definite derivative along the solutions. However, in practice it is very difficult to construct such a Lyapunov function. For example, for mechanical systems the total mechanical energy, which is a typical Lyapunov function, never has a negative definite derivative along the motions with respect to the generalized coordinates.

The purpose of this paper is to study the boundedness and ultimate boundedness of solutions of nonautonomous differential equations by Lyapunov's direct method when the derivative of the Lyapunov function along the solutions is only semidefinite. The results generalize V. M. MATROSOV's theorem [5] on the asymptotic stability to the boundedness of solutions. An application is given to the boundedness of the motions of a holonomic scleronomic mechanical system of n degrees of freedom being under the action of potential, dissipative and gyroscopic forces.

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2. Notations and definitions

Consider the system

$$\dot{x} = X(t, x), \tag{2.1}$$

where $(t, x)\in \mathbf{R}^+\times\mathbf{R}^n$, $\mathbf{R}^+=[0, \infty)$ and $X\colon \mathbf{R}^+\times\mathbf{R}^n\to\mathbf{R}^n$ is continuous. Throughout this paper, for simplicity, we assume that for any $(t_0, x_0)\in\mathbf{R}^+\times\mathbf{R}^n$, there exists a unique solution $x(t; t_0, x_0)$ of (2.1) through (t_0, x_0) defined for all $t\geqq t_0$.

Definition 2.1 [3]. A solution $x(t; t_0, x_0)$ of (2.1) is *bounded*, if $\sup_{t\geqq t_0} |x(t; t_0, x_0)|<\infty$.

The solutions of (2.1) are *uniformly bounded* (U.B.) if for every $\alpha>0$ there exists a $\beta(\alpha)>0$ such that $[t_0\geqq 0, |x_0|<\alpha, t\geqq t_0]$ imply $|x(t; t_0, x_0)|<\beta(\alpha)$.

The solutions of (2.1) are *equiultimately bounded* (E.U.B.) for some bound B if for every $\alpha>0$ and $t_0\geqq 0$ there exists a $T(t_0, \alpha)>0$ such that $[|x_0|<\alpha, t\geqq t_0+T(t_0, \alpha)]$ imply $|x(t; t_0, x_0)|<B$.

The solutions of (2.1) are *uniformly ultimately bounded* (U.U.B.) for some bound B if for every $\alpha>0$ there exists a $T(\alpha)>0$ such that $[t_0\geqq 0, |x_0|<\alpha, t\geqq t_0+T(\alpha)]$ imply $|x(t; t_0, x_0)|<B$.

By a *pseudo wedge* W we mean a continuous and strictly increasing function $W\colon \mathbf{R}^+\to\mathbf{R}^+$ with $W(r)>0$ if $r>0$. A pseudo wedge W is called *unbounded* if $\lim_{r\to\infty} W(r)=+\infty$.

Denote by $[a]_+$ and $[a]_-$ the positive and negative part of the real number a, respectively, that is, $[a]_+=\max\{a, 0\}$, $[a]_-=\max\{-a, 0\}$.

Definition 2.2 [5]. A measurable function $\lambda\colon \mathbf{R}^+\to\mathbf{R}^+$ is said to be *integrally positive* if $\int_J \lambda(t)\,dt=\infty$ holds on every set $J=\bigcup_{m=1}^{\infty}[a_m, b_m]$ such that $a_m<b_m\leqq a_{m+1}$ and $b_m-a_m\geqq\delta>0$ $(m=1, 2, \ldots)$ for a constant $\delta>0$.

Definition 2.3 [7]. A measurable function $\lambda\colon \mathbf{R}^+\to\mathbf{R}^+$ is said to be *weakly integrally positive* if for every $\delta>0$, $\Delta>0$ and for every set $J=\bigcup_{m=1}^{\infty}[a_m, b_m]$ with $a_m+\delta\leqq b_m\leqq a_{m+1}<b_m+\Delta$ $(m=1, 2, \ldots)$ the relation $\int_J \lambda(t)\,dt=\infty$ holds.

Lemma 2.1. *If a measurable function $\lambda\colon \mathbf{R}^+\to\mathbf{R}^+$ is integrally positive, then for every $\alpha>0$ and $\delta>0$ there exists a positive integer $K(\alpha, \delta)$ such that for every set $J=\bigcup_{m=1}^{K}[a_m, b_m]$ with $a_m<a_m+\delta\leqq b_m\leqq a_{m+1}$ for $1\leqq m\leqq K-1$, we have $\int_J \lambda(t)\,dt\geqq\alpha$.*

Proof. It is easy to see that λ is integrally positive if and only if for every $\delta>0$ the inequality

$$\liminf_{t\to\infty} \int_t^{t+\delta} \lambda(s)\,ds > 0 \tag{2.2}$$

holds. Consequently, for any given $\delta>0$ there are $T=T(\delta)>0$ and $\mu(\delta)>0$ such that $t \geqq T(\delta)$ implies

$$\int_t^{t+\delta} \lambda(s)\,ds \geqq \mu(\delta).$$

Let $\alpha>0$ and $\delta>0$ be given, and define $K(\alpha, \delta)=[T(\delta)/\delta]+1+[\alpha/\mu(\delta)]+1$, where $[a]$ denotes the integer part of $a\in\mathbf{R}$, that is, $[a]=\max\{z\colon z \text{ is an integer with } z\leqq a\}$. Then the number $K(\alpha, \delta)$ has the property mentioned in the assertion.

The following assertion can be easily proved by making use of (2.2).

Lemma 2.2. *If a measurable function* $\lambda\colon \mathbf{R}^+\to\mathbf{R}^+$ *is integrally positive, then*

$$\lim_{T\to\infty} \int_{t_0}^{t_0+T} \lambda = \infty \tag{2.3}$$

uniformly with respect to $t_0\in\mathbf{R}^+$.

Remark 2.1. The property of weak integral positivity and property (2.3) are independent of one another. E.g. $\lambda(t)=1/(1+t)$ is weakly integrally positive, but it does not satisfy (2.3) and so it is not integrally positive. On the other hand, weak integral positivity and (2.3) together do not imply integral positivity. E.g., the function

$$\lambda(t) = \begin{cases} 1/(1+t) & n \leqq t \leqq n+1/2 \\ 1 & n+1/2 < t < n+1 \end{cases}$$

is weakly integrally positive and satisfy (2.3) but it is not integrally positive.

With a continuous function $V\colon \mathbf{R}^+\times\mathbf{R}^n\to\mathbf{R}$ we associate the function

$$\dot V_{(2.1)}(t, x) = \limsup_{h\to 0+} (1/h)\{V(t+h, x+hX(t, x))-V(t, x)\},$$

which called the derivative of V with respect to (2.1).

It can be proved (see [3], p. 3) that if V is locally Lipschitz, then for an arbitrary solution $x(t)$ of (2.1) we have

$$V(t_2, x(t_2))-V(t_1, x(t_1)) = \int_{t_1}^{t_2} \dot V(t, x(t))\,dt, \quad (t_1, t_2\in\mathbf{R}^+).$$

3. The theorems and their proofs

Theorem 3.1. *Suppose that there exist nonnegative constants B and D, nonnegative locally Lipschitz functions $V(t, x)$, $P(t, x)$ and continuous $K(t, x)$ defined for $t \geqq 0$, $|x| \geqq B$ satisfying the following conditions:*

(i) $W_1(|x|) \leqq V(t, x) \leqq W_2(|x|)$, *where W_1 and W_2 are unbounded pseudo wedges;*

(ii) *the derivative of V with respect to* (2.1) *satisfies the inequality*

$$\dot{V}_{(2.1)}(t, x) \leqq -K(t, x) \quad for \quad t \geqq 0, \quad |x| \geqq B; \tag{3.1}$$

(iii) *for each $M > B$ there are $k = k(M) > 0$ and $H = H(M) \geqq 0$ such that $[t \geqq 0, B \leqq |x| \leqq M, P(t, x) \geqq H]$ imply $K(t, x) \geqq k$;*

(iv) *for each $M > B$ there exists an $L(M) > 0$ such that $[t \geqq 0, B \leqq |x| \leqq M, H(M) \leqq P(t, x) \leqq 2H(M)]$ imply $\dot{P}_{(2.1)}(t, x) \leqq L(M)$;*

(v) *for each $M > B$ there is a $T(M) > 0$ such that for any solution $x(t)$ of* (2.1) *with $B \leqq |x(t)| \leqq M$ and $P(t, x(t)) \leqq 2H(M)$ for $t_0 \leqq t \leqq t_0 + T(M)$ there exists $s \in [t_0, t_0 + T(M)]$ with $|x(s)| < D$.*

Then the solutions of (2.1) *are U.B. and U.U.B.*

Proof. For any $\alpha > 0$, define $\beta(\alpha) = W_1^{-1}(W_2(\max\{B, \alpha\}))$. It is easy to prove that $[t_0 \geqq 0, |x_0| \leqq \alpha]$ imply $|x(t; t_0, x_0)| \leqq \beta(\alpha)$ for $t \geqq t_0$. Therefore, the solutions of (2.1) are U.B. Throughout the remainder of this proof we use the notations $x(t) = x(t; t_0, x_0)$, $V(t) = V(t, x(t))$ and $\dot{V}(t) = \dot{V}_{(2.1)}(t, x(t))$.

To prove the uniform ultimate boundedness, we consider the following two cases:

(a) there exists a $t_2 \geqq t_0$ with $|x(t_2)| \leqq B$;

(b) $|x(t)| \geqq B$ for all $t \geqq t_0$.

In case (a) $|x(t)| \leqq \beta(B)$ for $t \geqq t_2$.

In case (b) we have $\dot{V}(t) \leqq -K(t, x(t))$ for all $t \geqq t_0$. By (iii) there exist $k = k(\beta(\alpha)) > 0$ and $H = H(\beta(\alpha)) > 0$ such that $P(t, x(t)) \geqq H$ implies $K(t, x(t)) \geqq k$. Let $\bar{t} \geqq t_0$ be fixed, and choose a constant $S = S(\alpha) > W_2(\beta(\alpha))/k$. Then by (3.1) the nonnegativeness of V implies the existence of a $t_3 \in [\bar{t}, \bar{t} + S(\alpha)]$ such that $P(t_3, x(t_3)) < H$. By (v), there exists $T = T(\beta(\alpha)) > 0$ such that if $P(t, x(t)) < 2H$ for $t \in [t_3, t_3 + T]$, then there is an $s \in [t_3, t_3 + T]$ with $|x(s)| < D$, which implies $|x(t)| < \beta(D)$ for $t \geqq t_3 + T$, especially, for $t \geqq \bar{t} + S + T$.

Therefore, only two cases may occur:

(b_1) $P(t, x(t)) < 2H$ for all $t \in [t_3, t_3 + T]$.

In this case, $|x(t)| < \beta(D)$ for $t \geqq \bar{t} + T + S$.

(b_2) there exists $t_4 \in [t_3, t_3 + T]$ with $P(t_4, x(t_4)) \geqq 2H$.

In this case, there are t_5, t_6 such that $t_3 < t_5 < t_6 \leqq t_4$, $P(t_5, x(t_5)) = H$, $P(t_6, x(t_6)) =$

$=2H$ and $H<P(t,x(t))<2H$ for $t_5<t<t_6$. By (iv), we get $t_6-t_5\geqq H/L(\beta(\alpha))$. On the other hand, by $\dot{V}(t)\leqq -K(t,x(t))\leqq -k$ for $t\in[t_5,t_6]$ we obtain

$$(3.2)\qquad V(t_6)\leqq V(t_5)-kH/L(\beta(\alpha)).$$

Since in case (b) $\dot{V}(t)\leqq -K(t,x(t))\leqq 0$ for all $t\geqq t_0$, we get $V(\bar{t}+S+T)\leqq \leqq V(\bar{t})-kH/L(\beta(\alpha))$. Let $\bar{t}=t_0+m(S+T)$, where m is a nonnegative integer. Then from the argument above we get either

$$(\mathrm{c}_m)\qquad |x(t)|\leqq \max\{\beta(B),\beta(D)\}\quad\text{for}\quad t\geqq t_0+(m+1)(S+T),$$

or

$$(\mathrm{d}_m)\qquad V(t_0+(m+1)(S+T))\leqq V(t_0+m(S+T))-kH/L(\beta(\alpha)).$$

Choose a positive integer $N=N(\alpha)$ such that

$$(3.3)\qquad N(\alpha)kH/L(\beta(\alpha))>W_2(\beta(\alpha)).$$

Then by the nonnegativeness of V, (d_m) holds for at most $m=0,1,\dots,N-1$, and thus $|x(t)|<\max\{\beta(B),\beta(D)\}$ for $t\geqq t_0+N(S+T)$. This completes the proof.

Remark 3.1. Using the same argument as one above, the comparison method and Lemma 2.1, we can prove the following assertion:

If conditions (i), (iii)—(v) of Theorem 3.1 are satisfied and if for each $M>B$ there exists a weakly integrally positive function λ_M: $\mathbf{R}^+\to\mathbf{R}^+$ such that

$$\dot{V}_{(2.1)}(t,x)\leqq -\lambda_M(t)K(t,x)+F(t,V(t,x))\quad\text{for}\quad t\geqq 0$$

and $B\leqq|x|\leqq M$, where F: $\mathbf{R}^+\times\mathbf{R}^+\to\mathbf{R}^+$ is continuous, the solutions of $\dot{z}=F(t,z)$ are uniformly bounded, and $\int_0^\infty \sup_{0\leqq z\leqq r} F(t,z)\,dt<\infty$ for $r\geqq 0$, then the solutions of (2.1) are U.B. and E.U.B. If, in addition, λ_M is integrally positive, then the solutions of (2.1) are U.B. and U.U.B.

Remark 3.2. If conditions (i), (iii) and (v) of Theorem 3.1 are satisfied and if

(a) $\dot{V}_{(2.1)}(t,x)\leqq -\lambda(t)K(t,x)+F(t,V(t,x))$ for $t\geqq 0$ and $|x|\geqq B$, where λ: $\mathbf{R}^+\to\mathbf{R}^+$ is measurable and satisfies condition (2.3), and F is of the same kind as in Remark 3.1;

(b) for any $M>0$ there exists a $\mu=\mu(M)>0$ such that $[B\leqq|x|\leqq M,\ H(M)\leqq \leqq P(t,x)\leqq 2H(M)]$ imply

$$\dot{V}_{(2.1)}(t,x)\leqq -\mu\dot{P}_{(2.1)}(t,x)+F(t,V(t,x)),$$

then the solutions of (2.1) are U.B. and U.U.B.

To prove this remark it is sufficient to replace (3.2) and (3.3) in the proof of Theorem 3.1 by

$$V(t_6) \leqq V(t_5) - \mu(\beta(\alpha))H(\beta(\alpha)) + \int_{t_5}^{t_6} \max\{F(t, z): 0 \leqq z \leqq W_2(\beta(\alpha))\}\, dt$$

and

$$N\mu(\beta(\alpha))H(\beta(\alpha)) > W_2(\beta(\alpha)) + \int_0^\infty \max\{F(t, z): 0 \leqq z \leqq W_2(\beta(\alpha))\}\, dt,$$

respectively.

Remark 3.3. Condition (iv) in Theorem 3.1 can be weakened as follows: for any $M > B$ there exists a continuous function L_M: $\mathbf{R}^+ \to \mathbf{R}^+$ such that $\int_0^t L_M$ is uniformly continuous on $[0, \infty)$ and either

$$[\dot{P}_{(2.1)}(t, x)]_+ \leqq L_M(t) \quad \text{for} \quad t \geqq 0, \quad B \leqq |x| \leqq M \quad \text{and} \quad H(M) \leqq P(t, x) \leqq 2H(M),$$

or

$$[\dot{P}_{(2.1)}(t, x)]_- \leqq L_M(t) \quad \text{for} \quad t \geqq 0, \quad B \leqq |x| \leqq M \quad \text{and} \quad H(M) \leqq P(t, x) \leqq 2H(M).$$

Remark 3.4. Condition (i) in Theorem 3.1 can be replaced by $0 \leqq V(t, x) \leqq \leqq W_2(|x|)$ if the solutions of (2.1) are U.B.

Example 3.1. Consider a Liénard equation with forcing term

$$\ddot{x} + f(x)\dot{x} + g(t, x) = e(t), \tag{3.4}$$

where $f(x)$, $g(t, x)$, $\partial g(t, x)/\partial t$ and $e(t)$ are continuous for $(t, x) \in \mathbf{R}^+ \times \mathbf{R}$ and $\int_0^\infty |e(s)|\, ds < \infty$. Besides, we assume that there exist unbounded pseudo wedges W_1, W_2, a continuous W_3: $\mathbf{R}^+ \to \mathbf{R}^+$ with $W_3(r) > 0$ for $r > 0$ and an integrally positive function λ: $\mathbf{R}^+ \to \mathbf{R}^+$ such that

$$W_1(|x|) \leqq \int_0^x g(t, x)\, dx \leqq W_2(|x|),$$

$$g(t, x)F(x) - \int_0^x (\partial g(t, r)/\partial t)\, dr \geqq \lambda(t) W_3(|x|),$$

where $F(x) = \int_0^x f(s)\, ds$. Obviously, (3.4) is equivalent to

$$\dot{x} = y - F(x), \quad \dot{y} = -g(t, x) + e(t). \tag{3.5}$$

Let $V(t, x, y)=\left[y^2+2\int_0^x g(t, r)\,dr\right]^{1/2}+\int_t^\infty |e(s)|\,ds$, then

$$[y^2+2W_1(|x|)]^{1/2} \leqq V(t, x, y) \leqq [y^2+2W_2(|x|)]^{1/2}+\int_0^\infty |e(s)|\,ds$$

$$\dot V_{(3.5)}(t, x, y) \leqq -\lambda(t)W_3(|x|)[y^2+2W_2(|x|)]^{-1/2}.$$

Let $K(t, x, y)=W_3(|x|)[y^2+2W_2(|x|)]^{-1/2}$, $P(t, x, y)=|x|$, $B=1$ and $H=1$. Then for each $M>1$ and for $t\geqq 0$, $1\leqq|x|+|y|\leqq M$ and $|x|\geqq 1$, we have $K(t, x, y)\geqq \geqq \min\{W_3(r)\colon 1\leqq r\leqq M\}\ (M^2+2W_2(M))^{-1/2}$. Therefore, conditions (i)—(iv) of Theorem 3.1 hold (see also Remark 3.1). Now we check condition (v).

Let $E=\max\{|F(x)|+1\colon |x|\leqq 2\}$, $D=E+2$, and for $M>1$ define $T(M)= =2M+1$. Suppose that $(x(t), y(t))$ is a solution of (3.5) with $1\leqq|x(t)|+|y(t)|\leqq M$ and $|x(t)|\leqq 2$ for $t\in[t_0, t_0+T(M)]$. If $|x(t)|+|y(t)|\geqq E+2$ for all $t\in[t_0, t_0+T(M)]$, then $|y(t)|\geqq E$, e.g. $y(t)\geqq E$, and consequently $\dot x(t)=y(t)-F(x(t))\geqq E-\max_{|x|\leqq 2} F(x)\geqq \geqq 1$. Hence we obtain the inequality $2M\geqq|x(t_0+T(M))-x(t_0)|\geqq T(M)=2M+1$, which is a contradiction. Therefore, there is an $s\in[t_0, t_0+T(M)]$ with $|x(s)|+ +|y(s)|<D=E+2$, i.e. condition (v) in Theorem 3.1 holds.

Consequently, under our conditions the solutions of (3.5) are U.B. and U.U.B.

Notice that if $P(t, x)=|x|$, then condition (iv) in Theorem 3.1 can be dropped. (Indeed, if condition (i)—(iii), (v) are satisfied for $P(t, x)=|x|$, then all the conditions of the theorem are satisfied for the new auxiliary function $\tilde P(t, x)=V(t, x)$. If, in addition, H in (iii) is constant, then (v) obviously holds. This special case initiates the following generalization of T. Yoshizawa's theorem ([3], Theorem 10.4):

Theorem 3.2. *Suppose that there exist a constant $B\geqq 0$, a locally Lipschitz function $V(t, x)$ and a continuous function $K(t, x)$ defined for $t\geqq 0$ and $|x|\geqq B$ satisfying the following conditions:*

(i) $W_1(|x|)\leqq V(t, x)\leqq W_2(|x|)$, *where W_1 and W_2 are unbounded pseudo wedges;*

(ii) $\dot V_{(2.1)}(t, x)\leqq -\lambda(t)K(t, x)$ *for $t\geqq 0$ and $|x|\geqq B$, where $\lambda\colon \mathbf{R}^+\to\mathbf{R}^+$ is measurable with $\lim_{t\to\infty}\int_{t_0}^t \lambda(s)\,ds=\infty$ for any $t_0\geqq 0$;*

(iii) *for each $M>B$ there exists $k(M)>0$ such that $B\leqq|x|\leqq M$ implies $K(t, x)\geqq k(M)$.*

Then the solutions of (2.1) *are U.B. and E.U.B. If, in addition, λ satisfies condition* (2.3), *then the solutions of* (2.1) *are U.B. and U.U.B.*

Proof. For any $\alpha>0$, define $\beta(\alpha)=W_1^{-1}(W_2(\max\{B, \alpha\}))$. Let $x(t; t_0, x_0)$ be a solution of (2.1) with $|x_0|<\alpha$. Then $|x(t; t_0, x_0)|<\beta(\alpha)$ for all $t\geqq t_0$, i.e. the solutions are U.B.

For a given $t_0 \geqq 0$ choose $T(t_0, \alpha) > 0$ such that

$$\int_{t_0}^{t_0+T(t_0,\alpha)} \lambda(s)\,ds > W_2(\beta(\alpha))/k(\beta(\alpha)).$$

It is easy to prove that $|x(t; t_0, x_0)| < \beta(B)$ for all $t \geqq t_0 + T(t_0, \alpha)$.

The second conclusion can be proved similarly.

The following theorem is a generalization of V. M. MATROSOV's stability theorem [5] to the boundedness of solutions.

Theorem 3.3. *Suppose that there exist a constant $B \geqq 0$ and nonnegative locally Lipschitz functions $V(t, x)$, $W(t, x)$, $P(t, x)$, a continuous function $F(t, u)$ defined for $t \geqq 0$, $|x| \geqq B$, $u \geqq 0$ and such that*

(i) *$W_1(|x|) \leqq V(t, x) \leqq W_2(|x|)$, where W_1 and W_2 are unbounded pseudo wedges;*

(ii) *for every $M > B$ there is a measurable function $\lambda_M\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that*

$$\dot V_{(2.1)}(t, x) \leqq -\lambda_M(t) P(t, x) + F(t, V(t, x)) \quad \textit{for} \quad t \geqq 0 \quad \textit{and} \quad B \leqq |x| \leqq M,$$

where

(a) *λ_M is weakly integrally positive;*

(b) *the solutions of the equation $\dot z = F(t, z)$ are U.B., and $\int_0^\infty [\sup_{0 \leqq z \leqq r} F(t, z)]\,dt < \infty$ for every $r > 0$;*

(iii) *for every $M > B$ there exists a continuous function $L_M\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that $\int_0^t L_M$ is uniformly continuous on $\mathbf{R}^+$ and either $[\dot P_{(2.1)}(t, x)]_+ \leqq L_M(t)$ or $[\dot P_{(2.1)}(t, x)]_- \leqq L_M(t)$ for $t \geqq 0$, $B \leqq |x| \leqq M$;*

(iv) *for every $M > B$ there exists a constant $A(M) > 0$ such that $|W(t, x)| \leqq A(M)$ for $t \geqq 0$ and $B \leqq |x| \leqq M$;*

(v) *there exists a constant $D \geqq B$ and for any $M > B$ there exists a continuous function $W_3\colon \mathbf{R}^+ \to \mathbf{R}^+$ with $W_3(r) > 0$ for $r \geqq D$ such that*

$$\max\{P(t, x), |\dot W_{(2.1)}(t, x)|\} \geqq W_3(|x|) \quad \textit{for} \quad t \geqq 0 \quad \textit{and} \quad D \leqq |x| \leqq M.$$

Then the solutions of (2.1) are U.B. and E.U.B. If, in addition, $\lambda_M(t)$ is integrally positive, then the solutions of (2.1) are U.B. and U.U.B.

Proof. First we show that under the assumptions of the theorem condition (v) in Theorem 3.1 is satisfied.

For any $M > D$, choose $H(M) > 0$ such that $2H < \alpha(M) = \min_{D \leqq r \leqq M} W_3(r)$ and define $T(M) = [2A(M) + 1]/\alpha$. Let $x(t)$ be a solution of (2.1) with $B \leqq |x(t)| \leqq M$ and $P(t, x(t)) \leqq 2H(M)$ for $t \in [t_0, t_0 + T(M)]$. If $|x(t)| \geqq D$ for all $t \in [t_0, t_0 + T(M)]$ then according to condition (v) we get $|\dot W_{(2.1)}(t, x(t))| \geqq \alpha$, hence $2A(M) \geqq$

$|W(t_0+T(M), x(t_0+T(M)))-W(t_0, x(t_0))|\geqq \alpha T(M)=2A(M)+1$, which is a contradiction. Therefore, condition (v) of Theorem 3.1 holds.

An application of Theorem 3.1, Remark 3.1 and Remark 3.3 completes the proof.

Remark 3.5. Condition (v) of Theorem 3.3 can be weakened by asking there is a constant $D\geqq B$ such that for every $M>D$ there are $B_2(M)>0$ and a continuous function μ_M: $\mathbf{R}^+\to\mathbf{R}^+$ with property (2.3) and such that $[t\geqq 0,\ D\leqq|x|\leqq M,\ P(t, x)\leqq B_2]$ imply $|\dot{W}_{(2.1)}(t, x)|\geqq \mu_M(t)$.

An application of this theorem to a holonomic scleronomic mechanical system will be given in Section 4.

As we have seen so far, the key step in the application of Theorem 3.1 is to check condition (v). Now we establish a sufficient condition for this property by Lyapunov's direct method.

Lemma 3.1. *Suppose that there exist* $H_0>0$, $D>B$ *and a locally Lipschitz function* $Q(t, x)$ *defined on the set* $\{(t, x)\colon t\geqq 0,\ |x|\geqq D,\ P(t, x)\leqq 2H_0\}$ *such that*

(i) *for each* $M>D$ *there are continuous functions* $\gamma, g\colon \mathbf{R}^+\to\mathbf{R}$ *and a number* $H\in(0, H_0]$ *such that* γ *has property* (2.3), *the function* $\int_0^t [g(s)]_+\,ds$ *is bounded on* $\mathbf{R}^+$, *and* $[t\geqq 0,\ D\leqq|x|\leqq M,\ P(t, x)\leqq 2H]$ *imply* $\dot{Q}_{(2.1)}(t, x)\leqq -\gamma(t)+g(t)$;

(ii) *for each* $M>D$ *there exists* $L(M)>0$ *with* $|Q(t, x)|\leqq L(M)$ *for* $t\geqq 0$ *and* $D\leqq|x|\leqq M$.

Then condition (v) *of Theorem* 3.1 *holds with these numbers H and D.*

Proof. Let $M>D$ be given and let a solution $x(t)$ of (2.1) satisfy $B\leqq|x(t)|\leqq M$ and $P(t, x(t))\leqq 2H(M)$ for $t\in[t_0, t_0+T(M)]$, where $T(M)>0$ is a constant such that

$$\int_{t_0}^{t_0+T(M)} \gamma(s)\,ds > 2L(M)+\int_0^\infty [g(s)]_+\,ds \quad \text{for all } t_0\geqq 0.$$

If $|x(t)|\geqq D$ for $t\in[t_0, t_0+T(M)]$, then we get

$$-L(M)\leqq Q(t_0+T(M), x(t_0+T(M)))\leqq L(M)-\int_{t_0}^{t_0+T(M)}\gamma(t)\,dt+\int_0^\infty [g(s)]_+\,ds$$

which yields a contradiction to the choice of $T(M)$. Consequently, there is $s\in\in[t_0, t_0+T(M)]$ with $|x(s)|<D$, and the proof is complete.

Example 3.2. Consider the equation

$$\ddot{x}+a(t)\dot{x}+f(x)=e(t) \tag{3.6}$$

and suppose that the continuous functions a, e: $\mathbf{R}^+ \to \mathbf{R}$, f: $\mathbf{R} \to \mathbf{R}$ satisfy the following conditions:

(i) $a(t) \geqq 0$ for $t \in \mathbf{R}^+$, a is weakly integrally positive, and there exist constant $\tilde{a}>0$, $T>0$ such that $[t_0 \geqq 0,\ t \geqq T]$ imply $(1/t) \int_{t_0}^{t_0+t} a(s)\,ds \leqq \tilde{a}$;

(ii) $e \in L^1[0, \infty)$;

(iii) there is an $r_0>0$ such that $xf(x)>0$, $|f(x)|>0$ provided $|x|>r_0$, and $F(x)=\int_0^x f(s)\,ds \to \infty$, as $|x| \to \infty$.

Then the solutions of equation (3.6) and their derivatives are U.B. and E.U.B. If, in addition, the function $a(t)$ is integrally positive, then the solutions and their derivatives are U.B. and U.U.B.

Equation (3.6) is equivalent to the system

$$\dot{x} = y, \quad \dot{y} = -f(x) - a(t)y + e(t). \tag{3.7}$$

Define $V(t, x, y)=[y^2+2F(x)]^{1/2}+\int_t^\infty |e(s)|\,ds$. Then

$$\dot{V}_{(3.7)}(t, x, y) \leqq -a(t)y^2[y^2+2F(x)]^{-1/2}.$$

Choose $K(t, x, y)=y^2[y^2+2F(x)]^{-1/2}$, $P(t, x, y)=y^2$. Then

$$[\dot{P}_{(3.7)}(t, x, y)]_+ = [-f(x)y - a(t)y^2 + e(t)y]_+ \leqq |f(x)|\,|y| + |e(t)|\,|y|.$$

Let $B>0$ be fixed arbitrarily. For $M>B$ let $K_M = \max\{|f(x)|:\ 0 \leqq |x| \leqq M\}$ and suppose $B \leqq |x|+|y| \leqq M$. Then $[\dot{P}_{(3.7)}(t, x, y)]_+ \leqq [K_M+|e(t)|]M$ and $\int_0^t (K_M+|e(s)|)M\,ds$ is uniformly continuous in $\mathbf{R}^+$. Consequently, conditions (i)—(iv) of Theorem 3.1 (see also Remark 3.3) are met with arbitrary $H>0$, and the solutions are U.B.

Now define $D=r_0+1$, $H_0=1/2$, and

$$Q(t, x, y) = \begin{cases} y & \text{if } x \geqq r_0, \\ -y & \text{if } x \leqq -r_0, \end{cases}$$

whose derivative is

$$\dot{Q}_{(3.7)}(t, x, y) = \begin{cases} -f(x) - a(t)y + e(t) & \text{if } x \geqq r_0, \\ f(x) + a(t)y - e(t) & \text{if } x \leqq -r_0. \end{cases}$$

For a given $M>D$ introduce the notation $m(M)=\min\{|f(x)|:\ r_0 \leqq |x| \leqq M\}$. By the conditions, $m(M)>0$, and $[t \geqq 0,\ D \leqq |x|+|y| \leqq M,\ y^2 \leqq 2H]$ imply the inequality

$$\dot{Q}_{(3.7)}(t, x, y) \leqq -m(M) + a(t)[2H]^{1/2} + e(t).$$

Let $H=\min\left\{\frac{1}{2}[m(M)/(\tilde{a}+1)]^2, \frac{1}{2}\right\}$, $\gamma(t)=m(M)-(2H)^{1/2}a(t)$ and $g(t)=|e(t)|$. Then $\dot{Q}_{(3.7)}(t, x, y)\leqq -\gamma(t)+g(t)$ and for sufficiently large $T>0$,

$$\int_{t_0}^{t_0+T} \gamma(t)\,dt = m(M)T-(2H)^{1/2}\int_{t_0}^{t_0+T} a(t)\,dt \geqq m(M)\tilde{a}/(\tilde{a}+1)T\to\infty$$

as $T\to\infty$ uniformly with respect to $t_0\geqq 0$, and so all the conditions of Lemma 3.1 are satisfied.

This completes the proof.

Consider now the system

$$\dot{x} = X(t, x, y), \quad \dot{y} = Y(t, x, y) \tag{3.8}$$

where $x\in\mathbf{R}^m$, $y\in\mathbf{R}^k$; X: $\mathbf{R}^+\times\mathbf{R}^{m+k}\to\mathbf{R}^m$ and Y: $\mathbf{R}^+\times\mathbf{R}^{m+k}\to\mathbf{R}^k$ are continuous. The following theorem shows that the function Q in Lemma 3.1 can be constructed from the reduced subsystem

$$\dot{y} = Y(t, 0, y). \tag{3.9}$$

Theorem 3.4. *Suppose that*

(i) *There exist constants B, $H\geqq 0$ and a locally Lipschitz function $V(t, x, y)$ defined for $t\geqq 0$ and $|x|+|y|\geqq B$ such that*

(a) *$W_1(|x|+|y|)\leqq V(t, x, y)\leqq W_2(|x|+|y|)$, where W_1 and W_2 are unbounded pseudo wedges;*

(b) *$\dot{V}_{(3.8)}(t, x, y)\leqq -\lambda(t)K(x, y)$ for $t\geqq 0$ and $|x|+|y|\geqq B$, where $\lambda(t)$ is weakly integrally positive, $K(x, y)\geqq 0$ for $|x|+|y|\geqq B$, and for any $M>B$ there exists $k(M)>0$ such that $K(x, y)\geqq k(M)$ for $H\leqq|x|$, $B\leqq|x|+|y|\leqq M$;*

(ii) *there exist a constant $B_1>0$, a continuous N: $\mathbf{R}^+\to\mathbf{R}^+$ with $N(s)>0$ for $s\geqq B_1$ and a locally Lipschitz function $Q(t, y)$ defined for $t\geqq 0$ and $|y|\geqq B_1$ such that*

(c) *$0\leqq Q(t, y)\leqq W_3(|y|)$, where W_3 is a pseudo wedge;*

(d) *$\dot{Q}_{(3.9)}(t, y)\leqq -W_4(|y|)$ for $|y|\geqq B_1$, where W_4 is a pseudo wedge;*

(e) *$|Q(t, y)-Q(t, \bar{y})|\leqq N(\max\{|y|, |\bar{y}|\})|y-\bar{y}|$;*

(iii) *for any $M>0$ there exists $L(M)>0$ such that $|X(t, x, y)|\leqq L(M)$ if $|x|+|y|\leqq M$;*

(iv) *there exist continuous P_1, P_2: $\mathbf{R}^+\to\mathbf{R}^+$ with $P_1(s)>0$ for $s\geqq B_1$ such that $|Y(t, x, y)-Y(t, 0, y)|\leqq P_1(|y|)P_2(|x|)$;*

(v) *$\lim_{r\to\infty} W_4(r)/(P_1(r)N(r))=\infty$.*

Then the solutions of (3.8) *are U.B. and E.U.B. If, in addition, λ is integrally positive, then the solutions of* (3.8) *are U.B. and U.U.B.*

Proof. Obviously, (i)—(iv) of Theorem 3.1 hold with $P(t, x, y)=|x|$. Choose $D>0$ such that $D-2H\geqq B_1$, $W_4(r)/N(r)P_1(r)\geqq\max\{P_2(s): |s|\leqq 2H\}+1$

for $s \geqq D-2H$. Then if $D \leqq |x|+|y| \leqq M$, $|x| \leqq 2H$, then $|y| \geqq D-2H \geqq B_1$, and thus

$$\dot{Q}_{(3.8)}(t, y) \leqq \dot{Q}_{(3.9)}(t, y)+N(|y|)|Y(t, x, y)-Y(t, 0, y)| \leqq -W_4(|y|)+$$

$$+N(|y|)P_1(|y|)P_2(|x|) \leqq -N(|y|)P_1(|y|)\left[\frac{W_4(|y|)}{N(|y|)P_1(|y|)}-P_2(|x|)\right]-$$

$$-N(|y|)P_1(|y|) \leqq -\inf\{N(r)P_1(r)\colon B_1 \leqq r \leqq M\}.$$

Therefore, condition (v) of Theorem 3.1 holds by Lemma 3.1, and so the proof is complete.

Example 3.3. Consider now the system

$$\dot{x} = f_1(t, x)+by, \quad \dot{y} = f_2(t, x)+dy+e(t), \tag{3.10}$$

where f_1, $f_2 \in C(\mathbf{R}^+ \times \mathbf{R}, \mathbf{R})$ with $f_1(t, 0)=0$, $f_2(t, 0)=0$, $e(t)$ is a bounded continuous function on $\mathbf{R}^+$ with $e \in L^1[0, \infty)$, b, d are constants with $db \neq 0$. Besides, we assume

(i) $\sup\{|f_1(t, x)|+|f_2(t, x)|\colon t \geqq 0, |x| \leqq M\} < \infty$ for any $M>0$;

(ii) $[df_1(t, x)-bf_2(t, x)]/x \geqq \alpha(x)>0$ for $t \geqq 0$ and $x \neq 0$, where α is continuous and $\lim_{|x|\to\infty} \int_0^x \alpha(r)r\,dr=\infty$;

(iii) $[f_1(t, x)+dx][bf_2(t, x)-df_1(t, x)]-\int_0^x [(d\partial f_1(t, r)/\partial t)-(b\partial f_2(t, r)/\partial t)]\,dr \geqq$ $\geqq \lambda(t)\beta(x)$, where $\lambda(t)$ is integrally positive, β is continuous with $\beta(x)>0$ if $x \neq 0$.

Under these conditions the solutions of (3.10) are U.B. and U.U.B.

Indeed, let

$$V(t, x, y)=\left[(dx-by)^2+2\int_0^x [df_1(t, r)-bf_2(t, r)]\,dr\right]^{1/2}+b\int_t^\infty |e(s)|\,ds.$$

Then

$$\dot{V}_{(3.10)}(t, x, y) \leqq$$

$$\leqq \frac{-[bf_2(t, x)-df_1(t, x)][f_1(t, x)+dx]+\int_0^x \left[d\frac{\partial}{\partial t}f_1(t, r)-b\frac{\partial}{\partial t}f_2(t, r)\right]dr}{\left[(dx-by)^2+2\int_0^x [df_1(t, r)-bf_2(t, r)]\,dr\right]^{1/2}} \leqq$$

$$\leqq -\lambda(t)K(x, y),$$

where

$$K(x, y)=\beta(x)\left[(dx-by)^2+2\sup_{t\geqq 0}\int_0^x [df_1(t, r)-bf_2(t, r)]\,dr\right]^{-1/2}.$$

It is easy to prove that for any $M>0$ there exists $k=k(M)>0$ such that $[|x|+|y|\leqq M,\ |x|\geqq H]$ imply $K(x,y)>k(M)$. Therefore, (i) of Theorem 3.4 holds.

On the other hand, for the subsystem

$$\dot{y} = dy+e(t) \tag{3.11}$$

and for $Q(t,y)=y^2/2$, $N(r)=r$, we have

$$\dot{Q}_{(3.11)}(t,y) \leqq d|y|\big[|y|+(1/d)\sup_{t\geqq 0}|e(t)|\big] \leqq (1/2)\,dy^2 \quad \text{for} \quad |y| \geqq -(2/d)\sup_{t\geqq 0}|e(t)|.$$

Therefore, after making the choice $P_1(r)=1$, $P_2(r)=\sup\{|f_2(t,x)|:\ t\geqq 0,\ |x|\leqq r\}$ all the conditions of Theorem 3.4 are met, and our assertion is true.

Theorem 3.5. *For system* (3.8), *suppose that*

(i) *there exist continuous functions* $P_1, P_2\colon \mathbf{R}^+\to\mathbf{R}^+$ *with* $P_1(s)>0$ *for* $s>0$ *such that* $|Y(t,x,y)-Y(t,0,y)|\leqq P_1(|y|)P_2(|x|)$;

(ii) *there exist a constant* $B_1>0$ *and a locally Lipschitz function* $V_1(t,x,y)$ *defined for* $t\geqq 0$, $|x|\geqq B_1$ *and* $y\in\mathbf{R}^k$ *such that*

$$W_1(|x|) \leqq V_1(t,x,y) \leqq W_2(|x|),$$

$$\dot{V}_{1(3.8)}(t,x,y) \leqq -W_3(|x|) \quad \textit{for} \quad t\geqq 0,\quad |x|\geqq B_1 \quad \textit{and} \quad y\in\mathbf{R}^k,$$

where W_1 *and* W_2 *are unbounded pseudo wedges and* $W_3\colon \mathbf{R}^+\to\mathbf{R}^+$ *is continuous with* $W_3(r)>0$ *for* $r\geqq B_1$;

(iii) *there exist a constant* $B_2>0$, *a locally Lipschitz function* $V_2(t,y)$ *defined for* $t\geqq 0$ *and* $|y|\geqq B_2$, *and a positive continuous function* $N\colon \mathbf{R}^+\to\mathbf{R}^+$ *with* $N(r)>0$ *for* $r\geqq B_2$ *and such that*

$$W_4(|y|) \leqq V_2(t,y) \leqq W_5(|y|),$$

$$\dot{V}_{2(3.9)}(t,y) \leqq -W_6(|y|) \quad \textit{for} \quad |y|\geqq B_2,$$

$$|V_2(t,y)-V_2(t,\tilde{y})| \leqq N\big(\max\{|y|,|\tilde{y}|\}\big)|y-\tilde{y}|,$$

where W_4, W_5 *are unbounded pseudo wedges,* W_6 *is nonnegative and continuous with* $\lim_{r\to\infty} W_6(r)/(N(r)P_1(r))=\infty$.

Then the solutions of (3.8) *are U.B. and U.U.B.*

Proof. First, we shall prove the uniform boundedness. For any $\alpha>\max\{B_1,B_2\}$, there exist $\beta(\alpha)$, $\beta_1(\alpha)$ and $\beta_2(\alpha)>0$ such that $W_1(\beta(\alpha))>W_2(\alpha)$, $\beta_2(\alpha)>\beta_1(\alpha)>\alpha$, $W_6(s)/N(s)P_1(s)-\max_{r\leqq\beta(\alpha)}P_2(r)\geqq 1$ for $s\geqq\beta_1(\alpha)$, and $W_4(\beta_2(\alpha))>W_5(\beta_1(\alpha))$. Then for any solution $(x(t),y(t))$ with $|x(t_0)|<\alpha$, and $|y(t_0)|<\alpha$, we have $x(t)<\beta(\alpha)$ and $|y(t)|<\beta_2(\alpha)$ for $t\geqq t_0$.

If this is not true, then only two cases may occur:

Case 1. There exist $t_2>t_1>t_0$ with $|y(t_1)|=\beta_1(\alpha)$, $|y(t_2)|=\beta_2(\alpha)$, $\beta_1(\alpha)<|y(t)|<\beta_2(\alpha)$ for $t\in(t_1, t_2)$ and $|x(t)|<\beta(\alpha)$ for $t\in[t_0, t_2)$.

Case 2. There exist $t_4>t_3>t_0$ such that $|x(t_3)|=\alpha$, $|x(t_4)|=\beta(\alpha)$, $\alpha<|x(t)|<\beta(\alpha)$ for $t\in(t_3, t_4)$ and $|y(t)|\leqq\beta_2(\alpha)$ for $t\in[t_3, t_4)$.

In Case 1, for $t\in[t_1, t_2]$, we have

$$\dot V_{2(3.8)}(t, y(t)) \leqq -W_6(|y(t)|)+N(|y(t)|)P_1(|y(t)|)P_2(|x(t)|) \leqq$$
$$\leqq -N(|y(t)|)P_1(|y(t)|)[W_6(|y(t)|)/(N(|y(t)|))P_1(|y(t)|)-P_2(|x(t)|)] \leqq$$
$$\leqq -N(|y(t)|)P_1(|y(t)|) \leqq 0.$$

Therefore, $W_4(\beta_2(\alpha))\leqq V_2(t_2, y(t_2))\leqq V_2(t_1, y(t_1))\leqq W_5(\beta_1(\alpha))$. This contradicts $W_4(\beta_2(\alpha))>W_5(\beta_1(\alpha))$.

In Case 2, for $t\in[t_3, t_4]$, we have $\dot V_1(t, x(t), y(t))\leqq 0$, thus

$$W_1(\beta(\alpha)) \leqq V_1(t_4, x(t_4), y(t_4)) \leqq V_1(t_3, x(t_3), y(t_3)) \leqq W_2(\alpha),$$

which contradicts $W_1(\beta(\alpha))>W_2(\alpha)$.

Therefore, $|x(t; t_0, x_0, y_0)|<\beta(\alpha)$ and $|y(t; t_0, x_0, y_0)|<\beta_2(\alpha)$ for $t\geqq t_0$ if $|x_0|<\alpha$ and $|y_0|<\alpha$. This completes the proof of uniform boundedness.

Let $v_1(\alpha)=\min\{W_3(r)\colon B_1+1\leqq r\leqq\beta(\alpha)\}$ and $T_1(\alpha)=W_2(\alpha)/v_1(\alpha)$. If $|x(t)|\geqq B_1+1$ holds for $t\in[t_0, \bar t]$ $(\bar t>t_0+T_1(\alpha))$ then

$$W_1(B_1+1) \leqq V_1(\bar t, x(\bar t), y(\bar t)) \leqq V_1(t_0, x(t_0), y(t_0))-v_1(\alpha)(\bar t-t_0) <$$
$$< W_2(\alpha)-v_1(\alpha)W_2(\alpha)/v_1(\alpha) = 0,$$

which yields a contradiction. Therefore, there exists $t_5\in[t_0, t_0+T_1(\alpha)]$ with $|x(t_5)|\leqq B_1+1$. Following the same argument as in the proof of uniform boundedness, we get $|x(t)|<\beta(B_1+1)$ for $t\geqq t_5$, especially for $t\geqq t_0+T_1(\alpha)$.

Choose $B_3>B_2$ with $W_6(s)/N(s)P_1(s)-\max\{P_2(r)\colon |r|<\beta(B_1+1)\}\geqq 1$ for $s\geqq B_3$. If $|y(t)|\geqq B_3$ for $t\geqq t_0+T_1(\alpha)$, then there exists $v_2(\alpha)>0$ such that $P_1(|y(t)|)N(|y(t)|)\geqq v_2(\alpha)$, and so

$$\dot V_{2(3.8)}(t, y(t)) \leqq -P_1(|y(t)|)N(|y(t)|)[W_6(|y(t)|)/N(|y(t)|)P_1(|y(t)|)-P_2(|x(t)|)] \leqq$$
$$\leqq -N(|y(t)|)P_1(|y(t)|) \leqq -v_2(\alpha).$$

Therefore, if $|y(t)|\geqq B_3$ for $t\in[t_0+T_1(\alpha), t_0+T_1(\alpha)+\bar t]$, then

$$V_2(t_0+T_1(\alpha)+\bar t, y(t_0+T_1(\alpha)+\bar t)) \leqq$$
$$\leqq V_2(t_0+T_1(\alpha), y(t_0+T_1(\alpha)))-v_2(\alpha)\bar t \leqq W_5(\beta(\alpha))-v_2(\alpha)\bar t.$$

If $\bar t\geqq T_2(\alpha)$, where $T_2(\alpha)=(W_5(\beta_2(\alpha))-W_4(B_3))/v_2(\alpha)$, then

$$W_4(B_3) \leqq V_2(t_0+T_1(\alpha)+\bar t, y(t_0+T_1(\alpha)+\bar t)) < W_5(\beta_2(\alpha))-v_2(\alpha)T_2(\alpha) \leqq W_4(B_3),$$

which yields a contradiction. Therefore, there exists $t_6 \in [t_0+T_1(\alpha),\ t_0+T_1(\alpha)+T_2(\alpha)]$ with $|y(t_6)|<B_3$, and thus $|x(t_6)|<B_4$ and $|y(t_6)|<B_4$, where $B_4=\max\{B_3, \beta(B_1+1)\}$. This implies $|x(t)|<\beta(B_4)$ and $|y(t)|<\beta_2(B_4)$ for $t \geqq t_0+T_1(\alpha)+T_2(\alpha)$. This completes the proof.

Sometimes in practice it is very difficult to find a Lyapunov function satisfying the condition $V_1(t, x, y) \leqq W_2(|x|)$ (see Example 3.4). Now we give a modification of Theorem 3.5 asking the much milder property $V_1(t, x, y) \leqq W_2(|x|+|y|)$.

Theorem 3.6. *Suppose that*

(i) *conditions* (i), (iii) *of Theorem* 3.5 *hold;*

(ii) *there exist a constant* $B_1>0$ *and a continuous function* $V_1(t, x, y)$ *defined for* $t \geqq 0$, $(x, y) \in \mathbf{R}^{m+k}$ *and such that*

$$W_1(|x|) \leqq V_1(t, x, y) \leqq W_2(|x|+|y|),$$

$$\dot{V}_{1(3.8)}(t, x, y) \leqq -W_3(x, y),$$

where W_1 *and* W_2 *are unbounded pseudo wedges, and* $W_3\colon \mathbf{R}^{m+k} \to \mathbf{R}^+$ *is continuous and* $|x| \geqq B_1$ *implies* $W_3(x, y)>0$;

(iii) *for any* $M>0$ *there exists* $L(M)>0$ *such that* $[t \geqq 0,\ |x|+|y| \leqq M]$ *imply* $|X(t, x, y)| \leqq L(M)$;

Then the solutions of (3.8) *are U.B. and U.U.B.*

Proof. Obviously, by (ii) for any $\alpha>0$, if $|x_0|+|y_0|<\alpha$, then $|x(t; t_0, x_0, y_0)< <W_1^{-1}(W_2(\alpha))=\beta(\alpha)$ provided that $(x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$ exists. Following the same argument as in the proof of Theorem 3.5, there exists $\beta_2(\alpha)>0$ such that $|y(t; t_0, x_0, y_0)|<\beta_2(\alpha)$ provided that $|x_0|+|y_0|<\alpha$ and $(x(t; t_0, x_0, y_0), y(t; t_0, x_0, y_0))$ exists. Then the solutions of (3.8) are U.B. Throughout the remainder of the proof denote $x(t)=x(t; t_0, x_0, y_0)$, $y(t)=y(t; t_0, x_0, y_0)$.

Let $T_1(\alpha)=W_2(\beta(\alpha)+\beta_2(\alpha))/\min\{W_3(x, y)\colon B_1+1 \leqq |x| \leqq \beta(\alpha),\ |y| \leqq \beta_2(\alpha)\}$. Then by (ii), for any $\bar{t} \geqq t_0$ there is a $t_1 \in [\bar{t}, \bar{t}+T_1(\alpha)]$ with $|x(t_1)|<B_1+1$.

Suppose that for all $t \in [t_1, \bar{t}+T_1(\alpha)+t^*]$ we have $|x(t)|<B_1+2$ and $|y(t)| \geqq B_3$, where $B_3=B_2$ is a fixed constant such that

$$W_6(r)/N(r)P_1(r)-\max\{P_2(s)\colon 0 \leqq s \leqq B_1+2\} \geqq 1 \quad \text{for} \quad r \geqq B_3.$$

Then from

$$\dot{V}_{2(3.8)}(t, y(t)) \leqq -N(|y(t)|)P_1(|y(t)|)\left[\frac{W_6(|y(t)|)}{N(|y(t)|)P_1(|y(t)|)}-P_2(|x(t)|)\right] \leqq$$

$$\leqq -\min\{N(r)P_1(r)\colon B_3 \leqq r \leqq \beta_2(\alpha)\} = -m$$

we get

$$0 \leqq V_2(\bar{t}+T_1(\alpha)+t^*,\ y(\bar{t}+T_1(\alpha)+t^*)) \leqq$$

$$\leqq V_2(t_1, y(t_1))-m[t^*+T_1(\alpha)+\bar{t}-t_1] \leqq W_5(\beta_2(\alpha))-m[t^*+T_1(\alpha)+\bar{t}-t_1].$$

Therefore, $t^* < T_2(\alpha) = [W_5(\beta_2(\alpha)) + 1]/m$. This shows only two cases may occur:

Case 1. $|x(t)| < B_1 + 2$ for all $t \in [t_1, \bar{t} + T_1(\alpha) + T_2(\alpha)]$ and there exists $t_2 \in \in [t_1, \bar{t} + T_1(\alpha) + T_2(\alpha)]$ with $|y(t_2)| < B_3$. In this case, $|x(t)| < \beta(B_1 + B_3 + 2)$ and $|y(t)| < \beta_2(B_1 + B_3 + 2)$ for $t \geqq \bar{t} + T_1(\alpha) + T_2(\alpha)$.

Case 2. There exists $t_3 \in [t_1, \bar{t} + T_1(\alpha) + T_2(\alpha)]$ such that $|x(t_3)| \geqq B_1 + 2$. In this case, there exist $t_4, t_5 \in [t_1, t_3]$ with $|x(t_4)| = B_1 + 1$ and $|x(t_5)| = B_1 + 2$ and $B_1 + 1 < < |x(t)| < B_1 + 2$ for $t \in (t_4, t_5)$. By condition (iii) $t_5 - t_4 \geqq 1/L(\beta(\alpha) + \beta_2(\alpha))$, and (ii) implies $V_1(\bar{t} + T_1(\alpha) + T_2(\alpha)) \leqq V_1(t_5) \leqq V_1(t_4) - (t_5 - t_4)m(\alpha) \leqq V_1(\bar{t}) - v(\alpha)$, where $V_1(t) = V_1(t, x(t), y(t))$, $v(\alpha) = [L(\beta(\alpha) + \beta_2(\alpha))]^{-1} m(\alpha)$, and $m(\alpha) = \min\{W_3(x, y)\colon B_1 + 1 \leqq |x| \leqq \beta(\alpha),\ |y| \leqq \beta_2(\alpha)\}$. Making the choice $\bar{t} = t_m = t_0 + m[T_1(\alpha) + T_2(\alpha)]$ $(m = 0, 1, 2, \ldots)$ we get that either $|x(t)| < \beta(B_1 + B_3 + 2)$ and $|y(t)| < \beta_2(B_1 + B_3 + 2)$ for $t \geqq t_{m+1}$, or

$$V_1(t_{m+1}) \leqq V_1(t_m) - v(\alpha). \tag{3.12}$$

On the other hand, $0 \leqq V_1(t) \leqq W_2(\beta(\alpha) + \beta_2(\alpha))$ for $t \geqq t_0$, and so (3.12) can not be true for $m = 0, 1, \ldots, N$, where $N = N(\alpha)$ is a positive integer such that $N(\alpha)v(\alpha) > > W_2(\beta(\alpha) + \beta_2(\alpha))$. Therefore, $|x(t)| < \beta(B_1 + B_3 + 2)$ and $|y(t)| < \beta_2(B_1 + B_3 + 2)$ for $t \geqq t_0 + [N(\alpha) + 1][T_1(\alpha) + T_2(\alpha)]$. This completes the proof.

Example 3.4. Consider the Liénard equation with forcing term

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t), \tag{3.13}$$

where $f(x)$ and $g(x)$ are continuous for $x \in \mathbf{R}$ and $p(t)$ is continuous for $t \geqq 0$. Besides, we assume that

(i) $f(x) > 1$;
(ii) $x\{g(x) - x[f(x) - 1]\} \geqq 0$;
(iii) $\int_0^\infty |p(s)|\, ds < \infty$.

Then the solutions of (3.13) are U.B. and U.U.B.

Proof. System (3.13) is equivalent to

$$\dot{x} = -x + y, \quad \dot{y} = -\{g(x) - x[f(x) - 1]\} - [f(x) - 1]y + p(t). \tag{3.14}$$

Let $V(t, x, y) = \left[y^2 + 2\int_0^x \{g(r) - r[f(r) - 1]\}\, dr\right]^{1/2} + \int_t^\infty |p(s)|\, ds$.

Then

$$\dot{V}_{(3.14)}(t, x, y) \leqq \frac{-[f(x) - 1]y^2 - x\{g(x) - x[f(x) - 1]\}}{\left[y^2 + 2\int_0^x \{g(r) - r[f(r) - 1]\}\, dr\right]^{1/2}} = -W(x, y).$$

Then $|y|>0$ implies $W(x,y)>0$. On the other hand, for the subsystem $\dot{x}=-x$ the auxiliary function $V_2(t,x)=x^2$, $N(r)=2r$ and $W_6(r)=2r^2$ satisfy condition (iii) of Theorem 3.5 and so the solutions of (3.13) are U.B. and U.U.B. by Theorem 3.6.

4. An application to a holonomic scleronomic mechanical system

Consider a holonomic scleronomic mechanical system of n degrees of freedom being under the action of potential, disspative and gyroscopic forces. The motions such a system can be described by the Langrangian equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}-\frac{\partial T}{\partial q}=-\frac{\partial \pi}{\partial q}-B\dot{q}+G\dot{q}, \tag{4.1}$$

where $q, \dot{q}\in \mathbf{R}^n$ are the vectors of the generalized coordinates and velocities, respectively, $\pi=\pi(t,q)$ is the potential energy, $T=T(q,\dot{q})=(1/2)\dot{q}^TA(q)\dot{q}$ is the kinetic energy where $A(q)$ is a symmetric $n\times n$ matrix function (v^T denotes the transposed of $v\in\mathbf{R}^n$); $B=B(t,q)$ is the symmetric positive semi-definite $n\times n$ matrix function of dissipation, and $G=G(t,q)$ is the antisymmetric $n\times n$ matrix of the gyroscopic coefficients.

By the Hamiltonian variables $q, p=A(q)\dot{q}$ system (4.1) can be rewritten into the form

$$\dot{q}=\frac{\partial H}{\partial p}, \quad \dot{p}=-\frac{\partial H}{\partial q}+(G-B)\frac{\partial H}{\partial p}, \tag{4.2}$$

where $H=H(t,p,q)$ is the total mechanical energy:

$$H=H(t,q,p)=T+\pi=(1/2)p^TA^{-1}(q)p+\pi(t,q).$$

Choose the auxiliary functions $V=H(t,p,q)$, $W=p^Tq$. Their derivatives with respect to (4.2) read as follows:

$$\dot{H}=\left(\frac{\partial H}{\partial p}\right)^T(G-B)\frac{\partial H}{\partial p}+\frac{\partial \pi}{\partial t}=-p^TA^{-1}(q)B(t,q)A^{-1}(q)p+\frac{\partial \pi(t,q)}{\partial t}\leqq$$

$$\leqq -\beta(t,q)\Lambda^{-1}(q)p^TA^{-1}(q)p+\left[\frac{\partial \pi(t,q)}{\partial t}\right]_+$$

where $\beta(t,q)$ denotes the smallest eigenvalue of the matrix $B(t,q)$; $\Lambda(q)$ denotes the largest eigenvalue of $A(q)$. It is known from the mechanics that the kinetic energy is a positive definite quadratic form of the velocities, consequently $\Lambda(q)>0$ for all $q\in\mathbf{R}^n$.

Let

$$A^{-1}(q) = (a_{ij}^{-1}(q))_{nxn},$$

$$d_{ij} = \left(\frac{\partial a_{ij}^{-1}(q)}{\partial q_1}, \ldots, \frac{\partial a_{ij}^{-1}(q)}{\partial q_n}\right) A^{-1}(q)p, \quad D = (d_{ij})_{nxn},$$

$$e_k = \sum_{i,j=1}^{n} \frac{\partial a_{ij}^{-1}(q)}{\partial q_k} p_i p_j, \quad e = (e_1, \ldots, e_n)^T.$$

Then for $P := p^T A^{-1}(q)p$, its derivative with respect to (4.2) is

$$\dot{P} = \left[-\frac{\partial \pi}{\partial q} - \frac{1}{2}\frac{\partial}{\partial q} p^T A^{-1}(q)p + (G-B)A^{-1}(q)p\right]^T A^{-1}(q)p +$$

$$+p^T A^{-1}(q)\left[-\frac{\partial \pi}{\partial q} - \frac{1}{2}\frac{\partial}{\partial q} p^T A^{-1}(q)p + (G-B)A^{-1}(q)p\right] + p^T D p =$$

$$= -2\left[\frac{\partial \pi(t, q)}{\partial q}\right]^T A^{-1}(q)p + p^T A^{-1}(q)[(G-B)^T + (G-B)]A^{-1}(q)p -$$

$$-p^T A^{-1}(q)\frac{\partial}{\partial q}[p^T A^{-1}(q)p] + p^T D p = -2\left[\frac{\partial \pi(t, q)}{\partial q}\right]^T A^{-1}(q)p -$$

$$-2p^T A^{-1}(q)BA^{-1}(q)p - p^T A^{-1}(q)e + p^T D p;$$

$$[\dot{P}]_+ \leqq \left|\frac{\partial}{\partial q}\pi(t, q)\right| F_2(q, p) + F_3(q, p),$$

where

$$F_2(q, p) = 2|A^{-1}(q)p|, \quad F_3(q, p) = |p|\,|A^{-1}(q)|\,|e| + |D|p^2.$$

Similarly,

$$\dot{W} = \dot{p}^T q + p^T \dot{q} = -\left[\frac{\partial \pi(t, q)}{\partial q}\right]^T q + \frac{1}{2}e^T q + p^T A^{-1}(q)(G-B)^T q + p^T A^{-1}(q)p,$$

$$|\dot{W}| \geqq \left|q^T \frac{\partial \pi(t, q)}{\partial q}\right| - |G(t, q) - B(t, q)| F_5(q, p) - F_4(q, p),$$

where

$$F_4(q, p) = \frac{1}{2}|e|\,|q| + |A^{-1}(q)|p^2, \quad F_5(q, p) = |A^{-1}(q)|\,|q|\,|p|.$$

It is easy to prove that $F_i(q, p)$ are continuous for $p, q \in \mathbf{R}^n$, and for every $M > 0$, $\lim_{p\to 0} \sup_{|q| \leqq M} F_i(q, p) = 0$ for $i = 2, \ldots, 5$. Therefore, from Theorem 3.3 and Remark 3.5, we get the following

Corollary 4.1. *Suppose that there are $B \geqq 0$ and unbounded pseudo wedges W_1, W_2 such that*

(i) $W_1(|q|) \leqq \pi(t, q) \leqq W_2(|q|)$ *for* $t > 0$ *and* $q \in \mathbf{R}^n$;

(ii) *for every* $M > 0$ *the function* $\beta_M(t) = \min \{\beta(t, q)\colon 0 \leqq |q| \leqq M\}$ *is weakly integrally positive;*

(iii) *there is a continuous function* $r\colon \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $r(t, u)$ *is increasing with respect to* u *for every* $t \in \mathbf{R}^+$ *and* $[\partial\pi(t, q)/\partial t]_+ \leqq r(t, \pi(t, q))$ *for* $t \in \mathbf{R}^+$ *and* $q \in \mathbf{R}^n$;

(iv) *for every* $u_0 > 0$ *there is a* $u_1 > u_0$ *with* $\int_0^\infty r(s, u_1)\, ds < u_1 - u_0$;

(v) *for every* $M > 0$ *the function* $|\partial\pi(t, q)/\partial q|$ *is bounded for* $t \geqq 0$ *and* $|q| \leqq M$;

(vi) *for every* $M > B$ *there are* $\mu_M > 0$ *and* $K_M > 0$ *such that* $|q^T \partial\pi(t, q)/\partial q| \geqq \mu_M$ $|G(t, q) - B(t, q)| \leqq K_M$ *for* $t \geqq 0$ *and* $B \leqq |q| \leqq M$.

Then the motions are U.B. and E.U.B.

If, in addition, $\beta_M(t)$ *is integrally positive, then the motions are U.B. and U.U.B.*

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