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On general connections satisfying $\nabla I=\omega\otimes I$

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0. Introduction

The notion of general connections was initiated by T. OTSUKI in 1958 [10]. He obtained various result [11—21]. A. MOÓR studied Riemannian manifolds with general connections, he called them Otsuki spaces [2—7]. T. OTSUKI [21, 22] and H. NAGAYAMA [8] applied general connections to the theory of relativity. Recently N. ABE [1] defined general connections on arbitrary vector bundles and H. NEMOTO [9] studied the geometry of submanifolds in a Riemannian manifold with a general connection.

One of the appealing facts in the theory of general connection is the fact that the covariant derivative of the identity endomorphism does not necessarily vanish. In this paper, we will study the case where the identity endomorphism is recurrent.

1. Preliminaries

In this section we review the theory of general connections along [1, 9, 11]. Throughout this paper, we assume that all objects are smooth and all vector bundles are real. Let M be a manifold, TM the tangent bundle and $C(M)$ the ring of real-valued functions on M. Let V and W be vector bundles over M. The fibre of V at $p\in M$ will be denoted by V_p and the dual bundle of V is denoted by V^*. The space of cross-sections of V will be denoted by $\Gamma(V)$. By Hom (V, W) we will denote the vector bundle of which fibre Hom (V, W) at p is the vector space Hom (V_p, W_p) of linear maps from V_p to W_p. In particular, Hom (V, V) will be denoted by End (V). Let HOM (V, W) be the space of vector bundle homomorphisms from V to W. Especially HOM (V, V) will be denoted by END (V). Let I_V be the identity endomorphism of V. Note that HOM (V, W) can be naturally identified with the space

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$\Gamma(\mathrm{Hom}\,(V, W))$. We will generally use the same symbol to denote a vector bundle homomorphism and the induced linear map on the space of cross-sections.

For $s \in \Gamma(V)$, we will denote the 1-jet of s by $j^1(s)$ and the 1-jet at p by $j^1_p(s)$. Let $J^1(V)$ be the 1-jet bundle of V. Now we define two vector bundle homomorphisms. The vector bundle homomorphism ι: $TM^* \otimes V \to J^1(V)$ is defined to be

$$\iota((df)_p \otimes s(p)) := j^1_p((f-f(p))s) \quad \text{for} \quad f \in C(M), \quad s \in \Gamma(V).$$

The vector bundle homomorphism λ: $J^1(V) \to V$ is defined to be

$$\lambda(j^1_p(s)) := s(p) \quad \text{for} \quad s \in \Gamma(V).$$

Definition 1.1. A vector bundle homomorphism $\gamma \in \mathrm{HOM}\,(V, J^1(V))$ is called a general connection on V. The endomorphism $P^\gamma := \lambda \circ \gamma \in \mathrm{END}\,(V)$ is called the principal endomorphism of γ. The linear operator ∇^γ: $\Gamma(V) \to \Gamma(TM^* \otimes V)$, defined by

$$\nabla^\gamma s := \iota^{-1}(j^1(P^\gamma s) - \gamma(s)) \quad \text{for} \quad s \in \Gamma(V),$$

is called the covariant derivative of γ.

It is easily shown that the covariant derivative ∇^γ of a general connection γ with the principal endomorphism P^γ satisfies

$$\nabla^\gamma(fs) = (df) \otimes P^\gamma s + f \nabla^\gamma s \quad \text{for} \quad f \in C(M), \quad s \in \Gamma(V). \tag{1.1}$$

For $P \in \mathrm{END}\,(V)$, we will denote the set of linear operators on $\Gamma(V)$ into $\Gamma(TM^* \otimes V)$ satisfying (1.1) by $O(V; P)$. Then the following theorem is known [1]:

Theorem A. *If $\nabla \in O(V; P)$ for $P \in \mathrm{END}\,(V)$, then there exists a unique general connection γ such that $P^\gamma = P$ and $\nabla^\gamma = \nabla$.*

Thus we may say that a pair (∇, P) of a linear operator ∇ and an endomorphism P satisfying (1.1) is a general connection on V. Given $v \in TM$ and $p \in M$, we define the linear map ∇_v: $\Gamma(V) \to V_p$ by $\nabla_v s := i_v(\nabla s)$ for $s \in \Gamma(V)$, where i_v is the inner product operator. Similarly, given $X \in \Gamma(TM)$, we define the linear operator ∇_X: $\Gamma(V) \to \Gamma(V)$ by $(\nabla_X s)(p) := \nabla_{X(p)} s$. We call ∇_X the covariant derivative along X. Then we have

$$\nabla_{fX} s = f \nabla_X s \quad \text{and} \quad \nabla_X(fs) = (Xf)Ps + f \nabla_X s \quad \text{for} \quad f \in C(M). \tag{1.1'}$$

When $P = I_V$, our general connection (∇, I_V) is nothing but a usual connection on V, that is, the linear operator ∇_X: $\Gamma(V) \to \Gamma(V)$ satisfies $\nabla_{fX} s = f \nabla_X s$ and $\nabla_X(fs) = (Xf)s + f \nabla_X s$.

Definition 1.2. A general connection (∇, P) on V is said to be regular if P is a regular endomorphism.

In the theory of general connection, we can define the product of $\nabla \in O(V; P)$ and $Q \in \mathrm{END}(V)$ as follows:

$$({}^{Q}\nabla)_X s := Q(\nabla_X s) \quad \text{and} \quad (\nabla^{Q})_X s := \nabla_X (Qs).$$

Then we have ${}^{Q}\nabla \in O(V; QP)$ and $\nabla^{Q} \in O(V; PQ)$. Hence, if a general connection (∇, P) is regular and Q is the inverse endomorphism of P, then the general connections ${}^{Q}\nabla$ and ∇^{Q} are usual connections. Furthermore we can naturally extend a general connection (∇, P) to general connections on the dual bundle and the tensor bundles. We will use the same symbol (∇, P) for the extensions. For instance, we present here the following formulas:

$$(\nabla_X \eta)(s) = X(\eta(Ps)) - \eta(\nabla_X s),$$

$$(\nabla_X \varphi)s = \nabla_X(\varphi Ps) - P\varphi(\nabla_X s),$$

$$(\nabla_X g)(s, s') = X\big(g(Ps, Ps')\big) - g(\nabla_X s, Ps') - g(Ps, \nabla_X s')$$

for $\eta \in \Gamma(V^*)$, $\varphi \in \Gamma(\mathrm{End}(V))$, $g \in \Gamma\big((V \otimes V)^*\big)$ and $s, s' \in \Gamma(V)$. In contrast to the case of usual connections, we must note that ∇I_V does not vanish in general.

Definition 1.3. Let $g \in \Gamma\big((V \otimes V)^*\big)$ be a fibre metric on V. A general connection (∇, P) on V is said to be metric if $\nabla g = 0$, that is,

$$(\nabla_X g)(s, s') = X\big(g(Ps, Ps')\big) - g(\nabla_X s, Ps') - g(Ps, \nabla_X s') = 0$$

for $s, s' \in \Gamma(V)$ and $X \in \Gamma(TM)$.

Definition 1.4. The element $R(\nabla) \in \mathrm{HOM}\big(\Lambda^2(TM), \mathrm{End}(V)\big)$ defined by

$$R(\nabla)_{X,Y} s := \nabla_X\big(\nabla_Y(Ps)\big) - \nabla_Y\big(\nabla_X(Ps)\big) - P\big(\nabla_{[X,Y]}(Ps)\big) - (\nabla_X I_V)\nabla_Y s + (\nabla_Y I_V)\nabla_X s$$

for $s \in \Gamma(V)$ and $X, Y \in \Gamma(TM)$, is called the curvature tensor field of the general connection (∇, P).

Remark. When the vector bundle is the tangent bundle TM, the curvature tensor field defined above coincides with the one defined by T. Otsuki [11].

In the case of $V = TM$, we can define a torsion tensor field of a general connection (∇, P) as follows:

Definition 1.5. Let $V = TM$. The element $\Psi \in \mathrm{HOM}(TM \otimes TM, TM)$ defined by

$$\Psi(X, Y) := \nabla_X Y - \nabla_Y X - P[X, Y]$$

for $X, Y \in \Gamma(TM)$, is called the torsion tensor field of the general connection (∇, P). If $\Psi = 0$, then the general connection on TM is said to be torsion free.

2. General connections of recurrent type

In a theory of general connection we noted that the covariant derivative of the identity endomorphism ∇I_V does not vanish in general. The case of $\nabla I_V=0$ was studied in [9]. The purpose of this paper is to study the case of $(\nabla_X I_V)=\omega(X)I_V$, where ω is some 1-form on M.

Definition 2.1. Let (∇, P) be a general connection on a vector bundle V over M. If the general connection (∇, P) satisfies

$$(2.1) \qquad (\nabla_X I_V)s = \omega(X)s$$

for some 1-form ω on M, then we call the general connection (∇, P) to be of recurrent type.

Example. For $\varrho\in C(M)$, we put $P:=\varrho I_V$. Let D be a usual connection on V. If we define a general connection (∇, P) by ${}^PD$, then it is easily seen that the general connection (∇, P) is of recurrent type whose recurrent 1-form ω is given by $\omega= =(1/2)d(\varrho^2)$. For the curvature tensor fields $R(\nabla)$ and $R(D)$, we can get the following formula:

$$R(\nabla) = \varrho^3 R(D),$$

which will be generalized in the folloving section. If ϱ does not vanish everywhere on M, the general connection (∇, P) is regular. Let g be a fibre metric on V and ϱ does not vanish everywhere on M. We define the fibre metric G which is conformal to g by $G:=\varrho^2 g$. Then we obtain that

$$\begin{aligned}(\nabla_X g)(s, s') &:= X(g(Ps, Ps'))-g(\nabla_X s, Ps')-g(Ps, \nabla_X s') = \\ &= X(g(\varrho s, \varrho s'))-g(\varrho D_X s, \varrho s')-g(\varrho s, \varrho D_X s') = \\ &= X(G(s, s'))-G(D_X s, s')-G(s, D_X s') = (D_X G)(s, s').\end{aligned}$$

Hence we know that the general connection (∇, P) is a metric general connection with respect to g if and only if the usual connection D is a metric connection with respect to G. Especially when $V=TM$, it is clear that the general connection (∇, P) is torsion free if and only if the usual connection D is torsion free. This type of general connections was treated by T. Otsuki [21] and H. Nagayama [8] when $V=TM$.

3. Regular general connections of recurrent type

In this section we study the case that the general connection (∇, P) is recurrent type and regular.

At first, we prepare several formulas for a regular general connection. Let Q be the inverse endomorphism of P, that is,

$$PQ = QP = I_V.$$

Thus the products ${}^{Q}\nabla$ and ∇^{Q} are usual connections on V and are denoted by D and D' respectively. The following equations were proved in [9].

$$(3.1) \qquad (\nabla_X I_V)s = P(D_X P)s = (D'_X P)(Ps),$$

$$(3.2) \qquad R(\nabla)_{X,Y}s = P^2 R(D)_{X,Y}(Ps) + P(D_X P)(D_Y P)s - P(D_Y P)(D_X P)s =$$
$$= PR(D')_{X,Y}(P^2 s) + (D'_X P)(D'_Y P)(Ps) - (D'_Y P)(D'_X P)(Ps).$$

Remark. When $V = TM$, these formulas are first proved by *T.* Otsuki in [11, 18].

Lemma 3.1. *Let* (∇, P) *be a regular general connection of recurrent type on* V. *Then we have the following equations:*

$$(3.3) \qquad (D_X P)s = (D'_X P)s = \omega(X)Qs,$$

$$(3.4) \qquad (D_X Q)s = (D'_X Q)s = -\omega(X)Q^3 s,$$

where ω *is the recurrent* 1*-form and* Q *is the inverse endomorphism of* P.

Proof. From (2.1) and (3.1), we obtain

$$P(D_X P)s = (D'_X P)(Ps) = (\nabla_X I_V)s = \omega(X)s,$$

from which we get (3.3). Since D is a usual connection, we have

$$D_X s = D_X(PQs) = (D_X P)Qs + PD_X(Qs) =$$
$$= (D_X P)Qs + P\{(D_X Q)s + QD_X s\} = (D_X P)Qs + P(D_X Q)s + D_X s.$$

Hence we find by (3.3) that

$$(D_X Q)s = -Q(D_X P)Qs = -\omega(X)Q^3 s.$$

Similarly we get $(3.4)_2$.

As a regular general connection (∇, P) can be naturally related to usual connections $D := {}^{Q}\nabla$ and $D' := \nabla^{Q}$, we give a relation among the curvature tensor fields of $R(\nabla)$, $R(D)$ and $R(D')$.

Theorem 3.2. *Let* (∇, P) *be a regular general connection of recurrent type on* V. *Then we have the following equations:*

(3.5) $R(\nabla)_{X,Y}s = P^3R(D)_{X,Y}s+2d\omega(X,Y)Ps = P^3R(D')_{X,Y}s+4d\omega(X,Y)Ps,$

where

$$2d\omega(X,Y) = X(\omega(Y))-Y(\omega(X))-\omega([X,Y]).$$

Proof. At first, substituting (3.3) into (3.2), we have

$$R(\nabla)_{X,Y}s = P^2R(D)_{X,Y}(Ps).$$

Using (3.3) and (3.4), we calculate $D_XD_Y(Ps)$ and $D_{[X,Y]}(Ps)$ as follows:

$$D_XD_Y(Ps) = D_X\{(D_YP)s+PD_Xs\} = D_X\{\omega(Y)Qs+PD_Ys\} =$$
$$= X(\omega(Y))Qs+\omega(Y)(D_XQ)s+\omega(Y)QD_Xs+(D_XP)D_Ys+PD_XD_Ys =$$
$$= X(\omega(Y)Qs-\omega(X)\omega(Y)Q^3s+\omega(Y)QD_Xs+\omega(X)QD_Ys+PD_XD_Ys,$$
$$D_{[X,Y]}(Ps) = (D_{[X,Y]}P)s+PD_{[X,Y]}s = \omega([X,Y])Qs+PD_{[X,Y]}s.$$

Hence we obtain

$$R(D)_{X,Y}(Ps) = PR(D)_{X,Y}s+\{X(\omega(Y))-Y(\omega(X))-\omega([X,Y])\}Qs,$$

from which we get $(3.5)_1$. By similar calculations we get $(3.5)_2$.

4. Regular metric general connections of recurrent type

In this section we will deal with a regular metric general connection (∇, P) of recurrent type.

Let g be a fibre metric on V and P be regular. Now we define the new metric G by

(4.1) $$G(s,s') := g(Ps, Ps').$$

It is known that when $V=TM$ and g is a Riemannian metric, G is also a Riemannian metric. Furthermore if the regular metric general connection is torsion free, the product ${}^{Q}\nabla$ is the Levi—Civita connection with respect to G [9].

Lemma 4.1. *Let* (∇, P) *be a regular metric general connection of recurrent type on* V. *Then we obtain*

(4.2) $$(D_Xg)(s,s') = -\omega(X)g(Ts,s'),$$

where we put

$$T:=Q^2+Q^{*2}$$

and Q^* *is defined by* $g(Q^*s,s'):=g(s,Qs')$.

Proof. As D is a usual connection, we get

$$X(g(Ps, Ps')) = D_X(g(Ps, Ps')) =$$
$$= (D_X g)(Ps, Ps') + g((D_X P)s, Ps') + g(Ps, (D_X P)s') + g(PD_X s, Ps') + g(Ps, PD_X s') =$$
$$= (D_X g)(Ps, Ps') + g((D_X P)s, Ps') + g(Ps, (D_X P)s') + g(\nabla_X s, Ps') + g(Ps, \nabla_X s'),$$

where we used ${}^P D_X s = \nabla_X s$. Therefore, substituting (3.3) and $X(g(Ps, Ps')) = g(\nabla_X s, Ps') + g(Ps, \nabla_X s')$ into the above equation, we obtain

$$(D_X g)(Ps, Ps') = -g((D_X P)s, Ps') - g(Ps, (D_X P)s') =$$
$$= -\omega(X)\{g(Qs, Ps') + g(Ps, Qs')\}.$$

Changing s to Qs and s' to Qs', we find (4.2).

Theorem 4.2. *Let* (∇, P) *be a regular metric general connection of recurrent type on* V. *If* $G(s, s') = g(s, s')$, *that is* $g(Ps, Ps') = g(s, s')$, *then the recurrent* 1-*form* ω *vanishes identically.*

Proof. At first, we note that $g(Ps, s') = g(s, Qs')$ because of $g(Ps, Ps') = g(s, s')$. Moreover, by virtue of Lemma 4.1 and $Dg = DG = 0$, we obtain

$$\omega(X)\{g(Q^2 s, s') + g(s, Q^2 s')\} = 0,$$

for any $X \in \Gamma(TM)$ and $s, s' \in \Gamma(V)$. Suppose that there is a point $p \in M$ such that $\omega \neq 0$ at p, then $\omega \neq 0$ on some open neighborhood U of p. Thus, on U, we have

$$g(Q^2 s, s') + g(s, Q^2 s') = 0,$$

from which we have

$$P^4 = -I_V.$$

Then from (3.3), we can easily get the following equation:

$$-D_X s = D_X(-s) = D_X(P^4 s) = 4\omega(X) P^2 s - D_X s,$$

which yields that

$$4\omega(X) P^2 s = 0.$$

Since P is regular, this implies that $\omega = 0$ on U. This is a contradiction. Therefore, there are no points $p \in M$ such that $\omega \neq 0$ at p.

5. Regular metric general connections of recurrent type on TM

In Section 4, we mentioned that if the general connection (∇, P) on TM is torsion free, regular and metric with respect to g, then D is the Levi—Civita connection with respect to G. On the other hand, there is the Levi—Civita connection $\bar{D}$ with respect to the original metric g. From now on, we study the relation between D and $\bar{D}$.

From the definition of $\bar{D}$, we have

$$(5.1) \qquad X(g(Y,Z)) = g(\bar{D}_X Y, Z) + g(Y, \bar{D}_X Z), \quad \bar{D}_X Y - \bar{D}_Y X = [X,Y].$$

On the other hand, by Lemma 4.1, we also obtain

$$(D_X g)(Y,Z) = X(g(Y,Z)) - g(D_X Y, Z) - g(Y, D_X Z) = -\omega(X) g(TY, Z).$$

Substituting (5.1) into above equation, we have

$$(5.2) \qquad g(D_X Y - \bar{D}_X Y, Z) + g(Y, D_X Z - \bar{D}_X Z) = \omega(X) g(TY, Z).$$

Since both D and $\bar{D}$ are torsion free, we get

$$g(D_X Y - \bar{D}_X Y, Z) + g(Y, D_X Z - \bar{D}_X Z) + g(D_Y Z - \bar{D}_Y Z, X) +$$
$$+ g(Z, D_Y X - \bar{D}_Y X) - g(D_Z X - \bar{D}_Z X, Y) - g(X, D_Z Y - \bar{D}_Z Y) = 2g(D_X Y - \bar{D}_X Y, Z).$$

From (5.2), the left hand side of the above equation equals

$$\omega(X) g(TY, Z) + \omega(Y) g(TZ, X) - \omega(Z) g(TX, Y).$$

Therefore, we have

$$(5.3) \qquad 2(D_X Y - \bar{D}_X Y) = \omega(X) TY + \omega(Y) TX - g(TX, Y) W,$$

where W is the vector field defined by $g(W, X) := \omega(X)$ and we used $g(TX, Y) = g(X, TY)$. For brevity, we set

$$(5.4) \qquad S(X,Y) := (1/2)\{\omega(X) TY + \omega(Y) TX - g(TX, Y) W\}.$$

Then (5.3) is rewritten as

$$(5.5) \qquad D_X Y = \bar{D}_X Y + S(X,Y).$$

Now, we consider the relation between the curvature tensor fields $R(D)$ and $R(\bar{D})$. Using (5.5) twice, we have

$$D_X D_Y Z = \bar{D}_X \bar{D}_Y Z + (\bar{D}_X S)(Y,Z) + S(\bar{D}_X Y, Z) +$$
$$+ S(Y, \bar{D}_X Z) + S(X, \bar{D}_Y Z) + S(X, S(Y,Z)).$$
$$D_{[X,Y]} Z = \bar{D}_{[X,Y]} Z + S([X,Y], Z).$$

Hence it follows from above equations and $\bar{D}_X Y - \bar{D}_Y X = [X, Y]$ that

(5.6)

$$R(D)_{X,Y} Z = R(\bar{D})_{X,Y} Z + (\bar{D}_X S)(Y,Z) - (\bar{D}_Y S)(X,Z) + S(X, S(Y,Z)) - S(Y, S(X,Z)).$$

To express the right hand side of (5.6) more precisely, we prepare several formulas. At first, we put

$$(5.7) \qquad U := Q^2 \quad \text{and} \quad U^* := Q^{*2}.$$

Then we have

$$T = U + U^*.$$

From $(3.4)_1$, we easily get

(5.8) $$(D_X U)Y = -2\omega(X)U^2 Y.$$

Let us calculate $(D_X U^*)Y$.

$$g((D_X U^*)Y, Z) = g(D_X(U^*Y), Z) - g(U^* D_X Y, Z) =$$
$$= X(g(U^*Y, Z)) - (D_X g)(U^*Y, Z) - g(U^*Y, D_X Z) - g(U^* D_X Y, Z) =$$
$$= (D_X g)(Y, UZ) + g(D_X Y, UZ) + g(Y, (D_X U)Z) + g(Y, U D_X Z) +$$
$$+ \omega(X) g(TU^*Y, Z) - g(Y, U D_X Z) - g(D_X Y, UZ) =$$
$$= -\omega(X) g(TY, UZ) - 2\omega(X) g(Y, U^2 Z) + \omega(X) g(TU^*Y, Z).$$

Therefore we find that

(5.9) $$(D_X U^*)Y = -\omega(X)[U^*TY - TU^*Y + 2U^{*2}Y].$$

Using (5.8) and (5.9), we compute $(D_X T)Y$.

(5.10) $$(D_X T)Y = (D_X U)Y + (D_X U^*)Y = -\omega(X)[2U^2 Y + 2U^{*2}Y + U^*TY - TU^*Y].$$

Next, we compute $(\bar{D}_X T)Y$ by the aids of (5.4), (5.5) and (5.10).

(5.11) $$(\bar{D}_X T)Y = \bar{D}_X(TY) - T\bar{D}_X Y = (D_X T)Y - S(X, TY) + TS(X, Y) =$$
$$= -\omega(X)[2U^2 Y + 2U^{*2}Y + U^*TY - TU^*Y] -$$
$$-(1/2)[\omega(TY)TX - \omega(Y)T^2 X - g(TX, TY)W + g(TX, Y)TW].$$

By virtue of these equations, we can get the following:

(5.12) $$(\bar{D}_X S)(Y, Z) - (\bar{D}_Y S)(X, Z) =$$
$$= (1/2)\{[(\bar{D}_X \omega)(Y) - (\bar{D}_Y \omega)(X)]TZ + [(\bar{D}_X \omega)(Z)TY - (\bar{D}_Y \omega)(Z)TX] -$$
$$-[g(TY, Z)\bar{D}_X W - g(TX, Z)\bar{D}_Y W] - (1/2)\omega(TZ)[\omega(Y)TX - \omega(X)TY] +$$
$$+(1/2)\omega(Z)[\omega(Y)T^2 X - \omega(X)T^2 Y] - (1/2)[\omega(Y)g(TX, Z) - \omega(X)g(TY, Z)]TW -$$
$$-\omega(Z)[2\omega(X)U^2 Y - 2\omega(Y)U^2 X + 2\omega(X)U^{*2}Y - 2\omega(Y)U^{*2}X +$$
$$+\omega(X)U^*TY - \omega(Y)U^*TX - \omega(X)TU^*Y + \omega(Y)TU^*X] -$$
$$-(1/2)\omega(Z)[\omega(TY)TX - \omega(TX)TY - \omega(Y)T^2 X + \omega(X)T^2 Y] +$$
$$+[2\omega(X)g(U^2 Y, Z) - 2\omega(Y)g(U^2 X, Z) +$$
$$+2\omega(X)g(U^{*2}Y, Z) - 2\omega(Y)g(U^{*2}X, Z) + \omega(Y)g(TU^*X, Z) -$$
$$-\omega(X)g(TU^*Y, Z) - \omega(Y)g(U^*TX, Z) + \omega(X)g(U^*TY, Z)]W +$$
$$+(1/2)[\omega(TY)g(TX, Z) - \omega(TX)g(TY, Z)]W\}.$$

The following equation follows from (5.4) and (5.12).

$$(\bar{D}_X S)(Y,Z)-(\bar{D}_Y S)(X,Z)+S(X,S(Y,Z))-S(Y,S(X,Z)) =$$
$$= d\omega(X,Y)TZ+(1/2)\{[(\bar{D}_X\omega)(Z)TY-(\bar{D}_Y\omega)(Z)TX]-$$
$$-[g(TY,Z)\bar{D}_X W-g(TX,Z)\bar{D}_Y W]\}-(1/4)\{|\omega|^2[g(TY,Z)TX-g(TX,Z)TY]+$$
$$+\omega(X)\omega(Z)[4U^2Y+4U^{*2}Y+2U^*TY-2TU^*Y+T^2Y]-$$
$$-\omega(Y)\omega(Z)[4U^2X+4U^{*2}X+2U^*TX-2TU^*X+T^2X]-$$
$$-\omega(X)g(4U^2Y+4U^{*2}Y+2U^*TY-2TU^*Y+T^2Y,Z)W+$$
$$+\omega(Y)g(4U^2X+4U^{*2}X+2U^*TX-2TU^*X+T^2X,Z)W.$$

Therefore, we obtain the following theorem:

Theorem 5.1. *Let (∇, P) be a torsion free regular metric general connection of recurrent type on TM, D the product ${}^Q\nabla$ and $\bar{D}$ the Levi—Civita connection with respect to G. Then the curvature tensor fields $R(D)$ and $R(\bar{D})$ satisfy the following equation.*

(5.13)
$$R(D)_{X,Y}Z = R(\bar{D})_{X,Y}Z+d\omega(X,Y)TZ+$$
$$+(1/2)\{[(\bar{D}_X\omega)(Z)TY-(\bar{D}_Y\omega)(Z)TX]-[g(TY,Z)\bar{D}_X W-g(TX,Z)\bar{D}_Y W]\}-$$
$$-(1/4)\{|\omega|^2[g(TY,Z)TX-g(TX,Z)TY]+$$
$$+\omega(X)\omega(Z)AY-\omega(Y)\omega(Z)AX-\omega(X)g(AY,Z)W+\omega(Y)g(AX,Z)W]\},$$

where we put

$$A = 4U^2+4U^{*2}+2U^*T-2TU^*+T^2.$$

6. Regular metric general connections of recurrent type whose principal endomorphism is symmetric

In this section, we study the case that the principal endomorphism P is symmetric with respect to g, that is,

(6.1) $$g(PX,Y) = g(X,PY).$$

As a consequence of this, we easily get the following:

(6.2) $$Q^* = Q \quad \text{and} \quad U = U^*,$$

(6.3) $$T = 2U,$$

(6.4) $$A = 12U^2.$$

Then the equation (5.13) is rewritten as

$$(6.5)\quad R(D)_{X,Y}Z = R(\bar{D})_{X,Y}Z+2d\omega(X,Y)UZ+ +[(\bar{D}_X\omega)(Z)UY-(\bar{D}_Y\omega)(Z)UX]-[g(UY,Z)\bar{D}_XW-g(UX,Z)\bar{D}_YW]- -|\omega|^2[g(UY,Z)UX-g(UX,Z)UY]+3[\omega(Y)\omega(Z)U^2X-\omega(X)\omega(Z)U^2Y+ +\omega(X)g(U^2Y,Z)W-\omega(Y)g(U^2X,Z)W].$$

Proposition 6.1. *Let* (∇, P) *be a torsion free regular metric general connection of recurrent type on TM. If the general connection* (∇, P) *satisfies* $g(PX, Y)=g(X, PY)$, *then the* 1*-form* ω *is closed.*

Proof. Let $\{e_i\}$ be a local orthonormal frame field with respect to g and $\{f^i\}$ the dual frame of $\{e_i\}$. Then, from (6.5), we have

$$(6.6)\quad f^i(R(D)_{e_i,Y}Z)=f^i(R(\bar{D})_{e_i,Y}Z)+2d\omega(e_i,Y)f^i(UZ)+ +[(\bar{D}_{e_i}\omega)(Z)f^i(UY)-(\bar{D}_Y\omega)(Z)f^i(Ue_i)]- -[g(UY,Z)f^i(\bar{D}_{e_i}W)-g(Ue_i,Z)f^i(\bar{D}_YW)]- -|\omega|^2[g(UY,Z)f^i(Ue_i)-g(Ue_i,Z)f^i(UY)]+ +3[\omega(Y)\omega(Z)f^i(U^2e_i)-\omega(e_i)\omega(Z)f^i(U^2Y)+ +\omega(e_i)g(U^2Y,Z)f^i(W)-\omega(Y)g(U^2e_i,Z)f^i(W)],$$

where we used the summation convention. Thus we get

$$(6.7)\quad K(D)(Y,Z) = K(\bar{D})(Y,Z)+2d\omega(UZ,Y)+(\bar{D}_{UY}\omega)(Z)-(\bar{D}_Y\omega)(Z)\operatorname{Tr}U- -g(UY,Z)f^i(\bar{D}_{e_i}W)+g(U\bar{D}_YW,Z)-|\omega|^2(g(UY,Z)\operatorname{Tr}U-g(U^2Y,Z))+ +3(\omega(Y)\omega(Z)\operatorname{Tr}U^2-\omega(U^2Y)\omega(Z)+\omega(W)g(U^2Y,Z)-\omega(Y)g(U^2W,Z))$$

where $K(D)(Y,Z)$, $K(\bar{D})(Y,Z)$ denote the Ricci curvature tensor fields with respect to G and g respectively. Changing Y and Z in (6.7) and subtracting this from (6.7), we obtain

$$(6.8)\quad 2d\omega(Y,Z)\operatorname{Tr}U=0,$$

since $K(D)(Y,Z)$ and $K(\bar{D})(Y,Z)$ are symmetric. As $\operatorname{Tr}U=\operatorname{Tr}Q^2=|Q|^2\neq 0$, we have

$$d\omega=0.$$

This proves our proposition.

In this case, (6.5) reduces to

$$(6.9)\quad K(D)_{X,Y}Z = K(D)_{X,Y}Z+[(\bar{D}_X\omega)(Z)UY-(\bar{D}_Y\omega)(Z)UX]- -[g(UY,Z)\bar{D}_XW-g(UX,Z)\bar{D}_YW]-|\omega|^2[g(UY,Z)UX-g(UX,Z)UY]+ +3[\omega(Y)\omega(Z)U^2X-\omega(X)\omega(Z)U^2Y+\omega(X)g(U^2Y,Z)W-\omega(Y)g(U^2X,Z)W].$$

Remark. Excepting Proposition 6.1, our results are true in the case that the metrics are pseudo-Riemannian.

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