On α_1^λ-products of automata

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1. Introduction

In [3] we introduced α_1^λ-products and gave an algebraic characterization of (homomorphically) complete classes of automata for the α_1^λ-product:

Theorem 1.1. *A class $\mathscr{K}$ of automata is complete for the α_1^λ-product if and only if for every simple group G there exists an $\mathbf{A}\in\mathbf{P}_{1\alpha_1}^\lambda(\mathscr{K})$ such that G is a divisor of the characteristic semigroup of $\mathbf{A}$, written $G|S(\mathbf{A})$.*

Further, we proved the following result.

Theorem 1.2. *Let $\mathscr{K}$ be a class of automata.*

(i) *If $\mathscr{K}$ contains a nonmonotone automaton, i.e. an automaton in $\mathscr{K}$ has a nontrivial cycle, then $\mathbf{A}\in\mathbf{HSP}_{\alpha_1}^\lambda(\mathscr{K})$ if and only if for every simple group G with $G|S(\mathbf{A})$ there exists an automaton $\mathbf{B}\in\mathbf{P}_{1\alpha_1}^\lambda(\mathscr{K})$ with $G|S(\mathbf{B})$.*

(ii) *If $\mathscr{K}$ consists of monotone automata one of which is not discrete, then $\mathbf{HSP}_{\alpha_1}^\lambda(\mathscr{K})$ is the class of all monotone automata.*

(iii) *If $\mathscr{K}$ consists of discrete automata one of which is not trivial then $\mathbf{HSP}_{\alpha_1}^\lambda(\mathscr{K})$ is the class of all discrete automata.*

(iv) *Otherwise, i.e. if $\mathscr{K}$ consists of trivial automata, then $\mathbf{HSP}_{\alpha_1}^\lambda(\mathscr{K})$ is the class of all trivial automata.*

The aim of this paper is to give a graph theoretic characterization of complete classes for the α_1^λ-product and to give a description of the classes of the form $\mathbf{HSP}_{\alpha_1}^\lambda(\mathscr{K})$ on the basis of graph theoretic terms. We believe this solution to be the final one as regards α_1^λ-products. The proofs are based on the fact that the symmetric group of degree $n-1$ ($n>1$) can be "realized" in a biconnected graph on n vertices. For recent results on α_0-products and α_1-products see [2] and [1].

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2. Notions and notations

An automaton is a system $\mathbf{A}=(A, X, \delta)$ with finite nonvoid sets A and X, the state set and input set, respectively, and transition $\delta\colon A\times X\to A$. The transition extends to a mapping $\delta\colon A\times X^*\to A$ in the usual way, where X^* is the free semigroup with unit element λ generated by X. The characteristic semigroup of $\mathbf{A}$, denoted $S(\mathbf{A})$, is the transformation semigroup on A consisting of all the mappings $\delta_u\colon A\to A$, $\delta_u(a)=\delta(a, u)$ $(a\in A,\ u\in X^*)$.

Given a system of automata $\mathbf{A}_t=(A_t, X_t, \delta_t)$ and a family of feedback functions

$$\varphi_t : A_1\times\ldots\times A_n\times X \to X_t\cup\{\lambda\},$$

$t=1, \ldots, n$, the g^λ-product of the $\mathbf{A}_t$'s with respect to X and φ is defined to be the automaton $\mathbf{A}$ with state set $A_1\times\ldots\times A_n$, input set X, and transition

$$\delta\big((a_1, \ldots, a_n), x\big) = \big(\delta_1(a_1, u_1), \ldots, \delta_n(a_n, u_n)\big)$$

where $(a_1, \ldots, a_n)\in A_1\times\ldots\times A_n$, $x\in X$ and

$$u_t = \varphi_t(a_1, \ldots, a_n, x),$$

$t=1, \ldots, n$. If none of the feedback functions $\varphi(a_1, \ldots, a_n, x)$ depends on the state variables a_s with $s>t$, we have an α_1^λ-product.

Given a (nonvoid) class $\mathcal{K}$ of automata, we set:

- $\mathbf{P}^\lambda_{\alpha_1}(\mathcal{K})$: all α_1^λ-products of automata from $\mathcal{K}$,
- $\mathbf{P}^\lambda_{1\alpha_1}(\mathcal{K})$: all α_1^λ-products with a single factor of automata from $\mathcal{K}$ (i.e. $n=1$ above),
- $\mathbf{S}(\mathcal{K})$: all subautomata of automata from $\mathcal{K}$,
- $\mathbf{H}(\mathcal{K})$: all homomorphic images of automata from $\mathcal{K}$.

Recall that a class $\mathcal{K}$ is called (homomorphically) complete for the α_1^λ-product if and only if $\mathbf{HSP}^\lambda_{\alpha_1}(\mathcal{K})$ is the class of all automata.

By a semigroup (group) we shall mean a finite semigroup (group). We write $S_1|S_2$ for two semigroups S_1 and S_2 if S_1 is a homomorphic image of a subsemigroup of S_2. If S_1 is a group, this just means that S_1 is a homomorphic image of a subgroup of S_2. The following statement is known e.g. from [4]:

Proposition 2.1. *If $G|G_1\times\ldots\times G_n$ for a simple group G and a direct product of groups $G_1, \ldots, G_n$ $(n>0)$, then $G|G_i$ for some i.*

3. Some useful facts

To investigate α_1^λ-products of automata we introduce the (directed) graph $D(\mathbf{A})$ of an automaton $\mathbf{A}=(A, X, \delta)$ as follows. We put $D(\mathbf{A})=(V, E)$ where the vertex set V is just the state set A and

$$E = \{(a, b)\in A\times A \mid a \neq b, \ \exists x\in X \ \delta(a, x) = b\}.$$

We see that E does not contain loop edges, henceforth, by a (directed) graph we shall always mean a graph without loop edges.

Take a graph $D=(V, E)$. We say that D is connected if for every pair a, b of different vertices there is a (directed) path from a to b. A maximal connected subgraph of D is a connected graph $D'=(V', E')$ with $V'\subseteq V$, $E'\subseteq E$ and such that whenever $D''=(V'', E'')$ is a connected graph satisfying $V'\subseteq V''\subseteq V$ and $E'\subseteq E''\subseteq E$, we have $V'=V''$, $E'=E''$.

A cycle is a graph $D=(V, E)$ with $V=\{a_1, ..., a_n\}$, $n>1$, and $E=\{(a_1, a_2), ..., (a_{n-1}, a_n), (a_n, a_1)\}$. Thus, cycles are connected graphs. Connected graphs other than cycles and having at least two vertices will be referred to biconnected graphs.

Take a graph D with vertex set $V=\{a_1, ..., a_n\}$ and place a pebble p_i onto a_i for every $i=1, ..., n$. Suppose we are allowed to move the pebbles according to the following three rules:

R1: Each step, an arbitrary number of pebbles can be moved. (Thus, some pebbles may stay where they are.)

R2: Each step, a pebble on a vertex a can be moved to a vertex b only if (a, b) is an edge.

R3: Once two or more pebbles hit the same vertex, they cannot be separated, i.e. have to be moved jointly.

Suppose that after a (possibly zero) number of steps p_i is on vertex a_{j_i}, $i=1, ..., n$. To this sequence of transformations we assign the mapping $V\to V$ given by $a_i\to a_{j_i}$, $i=1, ..., n$. Denote by $S(D)$ the set of all mappings obtained in this way. Clearly, $S(D)$ is a transformation semigroup on V. We let $G(D)$ denote the group of all permutations in $S(D)$. The following observation easily comes from the definitions:

Fact 3.1. *Let* $\mathbf{A}$ *be an automaton and* $D=D(\mathbf{A})$. *Then, for every* $\mathbf{B}\in \mathbf{P}_{1\alpha_1}^\lambda(\{\mathbf{A}\})$, $S(\mathbf{B})$ *is a subsemigroup of* $S(D)$. *Further, there exists an automaton* $\mathbf{C}\in \mathbf{P}_{1\alpha_1}^\lambda(\{\mathbf{A}\})$ *with* $S(\mathbf{C})=S(D)$.

Our game can be further generalized. Take a graph $D=(V, E)$ and fix a nonvoid subset V' of V, say $V'=\{a_1, ..., a_n\}$. Put pebble p_i onto a_i, $i=1, ..., n$, and move the pebbles in the graph according to R1, R2 and R3. Suppose that after a (possibly zero) number of steps the pebbles get back to the vertices in V', i.e. for every i, p_i is located on a vertex a_{j_i} in V'. We obtain a mapping $V' \to V'$ that assigns a_{j_i} to a_i. The collection of all these mappings is a transformation semigroup on V', denoted $S(D, V')$. Put $G(D, V')$ for the group of all permutations in $S(D, V')$. The following statement is obvious.

Fact 3.2. *$S(D, V')|S(D)$ and $G(D, V')|S(D)$.*

The next assertion is a reformulation of a well-known fact.

Fact 3.3. *If G is a subgroup of $S(D)$ then there is a nonvoid subset V' of the vertex set of D such that G is isomorphic to a subgroup of $G(D, V')$.*

Directly from Fact 3.3 and the observation that it is impossible to move a pebble back in a maximal connected subgraph if it has been moved out, we obtain:

Fact 3.4. *If G is a subgroup of $S(D)$ then G has maximal connected subgraphs $D_1, \ldots, D_n$ $(n>0)$ such that for some nonvoid subsets V_i of the vertex sets of the graphs D_i it holds that G is isomorphic to a subgroup of the direct product $G(D_1, V_1)\times \times\ldots\times G(D_n, V_n)$.*

Fact 3.5. *Let G be a simple group. Then $G|S(D)$ if and only if $G|G(D', V')$ for a maximal connected subgraph D' of D and a nonvoid subset V' of the vertex set of D'.*

Proof. Suppose that $G|S(D)$. There is a subgroup H of $S(D)$ which can be mapped homomorphically onto G. By Fact 3.4, H is isomorphic to a subgroup of a direct product $G(D_1, V_1)\times\ldots\times G(D_n, V_n)$ where the graphs D_i are maximal connected subgraphs of D and for every i, V_i is a nonvoid subset of the vertex set of D_i. Thus, $G|G(D_1, V_1)\times\ldots\times G(D_n, V_n)$. From Proposition 2.1, $G|G(D_i, V_i)$ for some i.

Conversely, $G|G(D', V')$ and $G(D', V')|S(D)$ yield $G|S(D)$.

Suppose we are given a graph $D=(V, E)$ with $V=\{a_0, \ldots, a_n\}$, $n\geqq 1$, i.e. D has at least two vertices. Set $V_i=V-\{a_i\}$, $i=0, \ldots, n$. Fix a pair of different integers $i, j\in\{0, \ldots, n\}$ and define the mapping $\psi_{i,j}\colon V_j\to V_i$ by

$$\psi_{i,j}(a_k) = \begin{cases} a_j & \text{if } i = k, \\ a_k & \text{otherwise.} \end{cases}$$

Let us say that $\psi_{i,j}$ has a realization in D if starting with pebble p_k located on a_k, $k=0, \ldots, n$, $k\neq j$, the placement that p_k is located on $\psi_{i,j}(a_k)$, $k=0, \ldots, n$, $k\neq j$, can be achieved by a sequence of moves according to R1, R2, R3. Obviously, if $\psi_{i,j}$ can be realized for every pair of different integers $i, j\in\{0, \ldots, n\}$, then for every $i\in\{0, \ldots, n\}$, $G(D, V_i)$ is the group of all permutations on V_i: to interchange two pebbles on a_{i_1} and a_{i_2} ($a_{i_1}, a_{i_2} \in V_i$, $a_{i_1} \neq a_{i_2}$), take a realization of $\psi_{i_1, i}$ followed by a realization of ψ_{i_2, i_1} and a realization of ψ_{i, i_2}.

Conversely, suppose that D is connected and for every $i \in \{0, ..., n\}$, $G(D, V_i)$ is the group of all permutations on V_i. It then follows that $\psi_{i,j}$ can be realized for every choise of i and j ($i, j \in \{0, ..., n\}$, $i \neq j$). Take a path $a_i = b_0, b_1, ..., b_t = a_j$ from a_i to a_j. If the length of this path is 1, i.e. $t=1$, just move the pebble on a_i to a_j, the others stand still. If $t>1$, since the permutation $(b_0 b_{t-1} ... b_1)$ is in $G(D, V_j)$, we can move the pebbles on $b_0, ..., b_{t-1}$ onto the vertices $b_{t-1}, b_0, ..., b_{t-2}$, respectively, so that the rest of the pebbles get back to their initial positions. To achive the final situation just move the pebbles on $b_0, ..., b_{t-1}$ one vertex forward along the path $b_0, ..., b_t$.

4. The main results

In this section we give a graph theoretic characterization of complete classes for the α_1^λ-product. Further, we give a complete description of the classes of the form $\mathbf{HSP}^\lambda_{\alpha_1}(\mathcal{K})$.

We start with two lemmas. In these lemmas the following designations will be used. Given a path $a_0, ..., a_n$, $n \geqq 1$, so that a_n is free and for each $i=0, ..., n-1$ there is a pebble on a_i, by moving the pebbles along the path $a_0, ..., a_n$ we shall mean the transformation that, in a single step, we move each pebble on a_i to a_{i+1}, $i=0, ..., n-1$. This definition extends to the case $n=0$: the placement of the pebbles remains unchanged. Given a cycle $a_0, ..., a_{n-1}$ ($n \geqq 2$) with at most one pebble on a_i, $i=0, ..., n-1$, by rotating the pebbles around the cycle we shall mean the transformation obtained by moving the pebble on a_i to $a_{i+1 \bmod n}$ for every i, provided that there was a pebble on a_i.

Lemma 4.1. *Let $D=(V, E)$ be a graph with $D=\{a_0, ..., a_{n+m}\}$, $n, m \geqq 1$, $E=\{(a_0, a_1), ..., (a_{n+m-1}, a_{n+m}), (a_{n+m}, a_0), (a_n, a_0)\}$. Then for every pair i, j of different integers in $\{0, ..., n+m\}$, $\psi_{i,j}$ can be realized in D.*

Proof. Fix an integer $i \in \{0, ..., n+m\}$. We shall show that $G(D, V_i)$ is the group of all permutations on V_i. Since $a_0, ..., a_{n+m}$ is a cycle in D, we may restrict ourselves to $i=n+1$. To see that $G(D, V_{n+1})$ is the group of all permutations on V_{n+1} if suffices to prove that the cyclic permutation $(a_0 ... a_n a_{n+2}, ..., a_{n+m})$ and the transposition $(a_{n-1} a_n)$ are in $G(D, V_{n+1})$.

Place pebble p_i onto a_i, $i=0, ..., n, n+2, ..., n+m$. Move p_n from a_n to a_{n+1}, then rotate the pebbles around the cycle $a_0, ..., a_{n+m}$. We see that $(a_0 ... a_n a_{n+2} ... a_{n+m}) \in G(D, V_{n+1})$. For the transposition $(a_{n-1} a_n)$, apply the following procedure:

Step 1. Move p_n from a_n to a_{n+1}.

Step 2. Check if p_n is located on a_{n+m}, if so, go to Step 3. Move the pebbles along the path $a_{n+m}, a_0, \ldots, a_n$. (It is garanteed that a_n is free when this transformation applies.) Next, rotate the pebbles n times around the cycle $a_0, \ldots, a_n$, and after that, move the pebbles along the path $a_n, \ldots, a_{n+m}$ and go back to Step 2.

Step 3. Before this step applies, the placement of the pebbles is this: for every $i \in \{0, \ldots, n-1\}$, p_i is located on a_i; a_n is free; for every $i \in \{n+2, \ldots, n+m\}$, p_i is on a_{i-1}; p_n is on a_{n+m}. Move p_{n-1} from a_{n-1} to a_n and then rotate the pebbles around the cycle $a_0, \ldots, a_n$ until a_0 gets free, we see that a_0 is free, p_{n-1} is located on a_1, and for every $i \in \{0, \ldots, n-2\}$, p_i is on a_{2+i}. Now move p_n from a_{n+m} to a_0, rotate the pebbles $n-1$ times around the cycle $a_0, \ldots, a_n$, and move the pebbles along the path $a_{n+1}, \ldots, a_{n+m}$.

Lemma 4.2. *Let* $G=(V, E)$ *be a graph with* $V=\{a_0, \ldots, a_{n+m+l}\}$, $n \geqq 0$, $m, l \geqq 1$, *and* $E=\{(a_0, a_1), \ldots, (a_{n+m-1}, a_{n+m}), (a_{n+m}, a_0), (a_n, a_{n+m+1}), \ldots, \ldots, (a_{n+m+l-1}, a_{n+m+l}), (a_{n+m+l}, a_0)\}$. *Then, for every pair of different integers* $i, k \in \{0, \ldots, n+m+l\}$, $\psi_{i,k}$ *can be realized in D.*

Proof. Place p_t onto a_t, $t=0, \ldots, n+m+l$, $t \neq k$. First we show that we may restrict the consideration to the case that $k=n$. Either $k \in \{0, \ldots, n+m\}$ or $k \in \in \{0, \ldots, n, n+m+1, \ldots, n+m+l\}$. If $k \in \{0, \ldots, n+m\}$ rotate the pebbles around the cycle $a_0, \ldots, a_{n+m}$ until a_n gets free, then move p_i to a_n so that the rest of the pebbles get back to the position they were after the rotations. Finally, rotate the pebbles around the cycle $a_0, \ldots, a_{n+m}$ so that p_i gets onto a_k. The pebbles p_t other than p_i get back to a_t, respectively. Similar procedure applies when $k \in \{0, \ldots, n+m+1, \ldots, \ldots, n+m+l\}$.

Let $k=n$. Because the assumptions $i \in \{0, \ldots, n+m\}$ and $i \in \{0, \ldots, n, n+m+1, \ldots, \ldots, n+m+l\}$ are symmetrical, we may suppose $i \in \{0, \ldots, n+m\}$. We shall realize $\psi_{i,n}$ in five steps.

Step 1. Rotate the pebbles once around the cycle $a_0, \ldots, a_n, a_{n+m+1}, \ldots, a_{n+m+l}$. Observe that a_{n+m+1} becomes free and p_{n+m+l} gets onto a_0.

Step 2. Rotate the pebbles around the cycle $a_0, \ldots, a_{n+m}$ until p_i hits a_n. Then move p_i from a_n to a_{n+m+1}, so that a_n becomes free.

Step 3. When this step applies, one of the vertices $a_0, \ldots, a_{n+m}$ is free, and exactly one of $p_{n+m+1}, \ldots, p_{n+m+l}$, say p_t, is in the cycle $a_0, \ldots, a_{n+m}$ (p_{n+m+l} for the first time). Check if p_i is on a_{n+m+l}, if so, go to Step 4. Otherwise rotate the pebbles around the cycle $a_0, \ldots, a_{n+m}$ until p_t gets onto a_n, and rotate the pebbles once around the cycle $a_0, \ldots, a_n, a_{n+m+1}, \ldots, a_{n+m+l}$. Go to Step 3.

Step 4. Observe that the placement of the pebbles is this. The cycle $a_0, \ldots, a_{n+m}$ contains p_{n+m+1} and the pebbles p_j with $j \in \{0, \ldots, n+m\}$, $j \neq i$, $j \neq n$. Thus, one of

$a_0, \ldots, a_{n+m}$ is free. The relative order of the pebbles p_j ($j \in \{0, \ldots, n+m\}$, $j \neq i$, $j \neq n$) is their original order. Further, p_i is on a_{n+m+l}, p_{n+m+2} is on $a_{n+m+1}, \ldots, \ldots, p_{n+m+l}$ is on $a_{n+m+l-1}$. It is now clear that the pebbles in the cycle $a_0, \ldots, a_{n+m}$ can be arranged in such a way that a_0 gets free and after moving the pebbles along the path $a_{n+m+1}, \ldots, a_{n+m+l}, a_0$ (so that p_i gets onto a_0), the relative order of the pebbles p_j, $j \in \{0, \ldots, n+m\}$, $j \neq n$, in the cycle $a_0, \ldots, a_{n+m}$ will be just as desired.

Step 5. We have p_{n+m+1} free. The pebbles $p_{n+m+2}, \ldots, p_{n+m+l}$ are back on $a_{n+m+2}, \ldots, a_{n+m+l}$, respectively. Further, the cycle $a_0, \ldots, a_{n+m}$ contains the pebbles p_j $j \in \{0, \ldots, n+m\}$, $j \neq n$, and the pebble p_{n+m+1}. The relative order of the pebbles p_j ($j \in \{0, \ldots, n+m\}$, $j \neq n$) is just as desired. Rotate the pebbles around the cycle $a_0, \ldots, a_{n+m}$ until p_{n+m+1} gets onto a_n then move p_{n+m+1} from a_n to a_{n+m+1}. The pebbles $p_{n+m+1}, \ldots, p_{n+m+l}$ are now back on $a_{n+m+1}, \ldots, a_{n+m+l}$, respectively. Further, it is clear that the pebbles in the cycle $a_0, \ldots, a_{n+m}$ can be arranged so that p_i is on a_n, and for $j \in \{0, \ldots, n+m\}$, $j \neq i$, $j \neq n$, p_j is on a_j.

Theorem 4.3. *$S_n | S(D)$ for every biconnected graph D on $n+1$ vertices.*

Proof. Let $D = (V, E)$ with $V = \{a_0, \ldots, a_n\}$. We are going to show that $\psi_{i,j}$ can be realized in D for every possible pair of different integers i, j. Consequently, $G(D, V_i)$ is the group of all permutations on V_i for every i ($0 \leqq i \leqq n$). Hence the result follows by Fact 3.2.

Put pebble p_t onto a_t for every $t \in \{0, \ldots, n\}$, $t \neq j$. Take a path

$$a_i = b_0, b_1, \ldots, b_k = a_j$$

from a_i to a_j. If $k=1$, $\psi_{i,j}$ can be realized obviously. We proceed by induction on k. Assume $k>1$. There are an $m \in \{0, \ldots, k-1\}$ and a path

$$a_j = b_k, b_{k+1}, \ldots, b_{k+l} = b_m$$

with $\{b_0, \ldots, b_k\} \cap \{b_{k+1}, \ldots, b_{k+l-1}\} = \emptyset$. We distinguish two cases.

Case $m \neq 0$. Let us rotate the pebbles l times around the cycle $b_m, \ldots, b_k, b_{k+1}, \ldots, b_{k+l-1}$. We see that b_m is free now. By induction hypothesis, p_1 can be moved from a_i to b_m in such a way that meanwhile all the other pebbles get back to the vertex they were before. Finally, rotate the pebbles $k-m$ times around the cycle $b_m, \ldots, b_k, b_{k+1}, \ldots, b_{k+l-1}$. Obviously, we obtained a realization of $\psi_{i,j}$.

Case $m=0$. We have a cycle

$$b_0, b_1, \ldots, b_k, b_{k+1}, \ldots, b_{k+l-1}.$$

Two subcases arise according to whether this cycle contains all the vertices of D or not.

Subcase $V=\{b_0, ..., b_{k+l-1}\}$. Since D is biconnected, there is at least one edge in E other than the edges $(b_0, b_1), ..., (b_{k+l-2}, b_{k+l-1}), (b_{k+l-1}, b_0)$. The result follows by Lemma 4.1.

Subcase $V \neq \{b_0, ..., b_{k+l-1}\}$. Take a vertex $c \in V-\{b_0, ..., b_{k+l-1}\}$ closest to the cycle $b_0, ..., b_{k+l-1}$. We then have paths $b_t=c_0, c_1, ..., c_u=c$ and $c=d_0, ..., d_v=b_s$ for $t, s \in \{0, ..., k+l-1\}$ such that the sets $\{b_0, ..., b_{k+l-1}\}$, $\{c_1, ..., c_u\}$ and $\{d_1, ..., d_{v-1}\}$ are pairwise disjoint. The result follows by Lemma 4.2.

Theorem 4.4. *Let $D=(V, E)$ be a cycle with n vertices. Then for every group G, $G|S(D)$ if and only if $G|Z_m$ for some $m \leqq n$.*

Proof. It suffices to show that a group is isomorphic to a subgroup of $S(D)$ if and only if it is isomorphic to a subgroup of Z_m with $m \leqq n$.

Suppose that H is isomorphic to a subgroup of $S(D)$. From Fact 3.3, there is a subset V' of the vertex set of D such that H is isomorphic to a subgroup of $G(D, V')$. Let m be the cardinality of V'. We prove that $G(D, V')$ is a cyclic group of order m.

Set $V=\{a_1, ..., a_n\}$ and $V'=\{a_{i_1}, ..., a_{i_m}\}$ so that $a_1, ..., a_n$ is a cycle and $i_1 < ... < i_m$. Place pebble p_j onto a_{i_j}, $j=1, ..., m$. Rotate the pebbles once around the cycle $a_1, ..., a_n$. If each of the pebbles p_j is on the vertex $a_{i_{j+1}}$, or on a_{i_1} if $j=m$, we see that the cyclic permutation $(a_{i_1} ... a_{i_m})$ is in $G(D, V')$. Otherwise, rotate those pebbles around the cycle $a_1, ..., a_n$ for which it does not hold. In a finite number of steps we obtain a realization of the cyclic permutation $(a_{i_1} ... a_{i_m})$. Thus, $(a_{i_1} ... a_{i_m}) \in G(D, V')$. On the other hand, since by our rules and the structure of D the pebbles can never pass each other, every permutation in $G(D, V')$ is a power of the cyclic permutation $(a_{i_1} ... a_{i_m})$.

Conversely, it is clear from the above proof that if H is isomorphic to a subgroup of a cyclic group Z_m with $m \leqq n$ then H is isomorphic to a subgroup of $G(D, V')$ for every subset V' of V with m elements. Thus, Fact 4.2 yields $G|S(D)$.

Let $\mathcal{K}$ be a class of automata. Set $D(\mathcal{K})=\{D \mid \exists \mathbf{A} \in \mathcal{K}\ D$ is a subgraph of $D(\mathbf{A})\}$, where the notion of a subgraph of a graph is used in the usual sense. With the concept of $D(\mathcal{K})$ and that of a biconnected graph we are able to characterize complete classes for the α_1^λ-product:

Theorem 4.5. *A class $\mathcal{K}$ is complete for the α_1^λ-product if and only if for every positive integer n, $D(\mathcal{K})$ contains a biconnected graph on at least n vertices.*

Proof. If $D(\mathcal{K})$ does not contain biconnected graphs then, by Theorem 4.4, Fact 3.5 and Fact 3.1, every simple group dividing $S(\mathbf{A})$ for some $\mathbf{A} \in \mathbf{P}_{1\alpha_1}^\lambda(\mathcal{K})$ is commutative. If n is the highest integer such that $D(\mathcal{K})$ contains a biconnected graph on n vertices then, again by Theorem 4.4, Fact 3.5 and Fact 3.1, every simple group dividing $S(\mathbf{A})$ for an $\mathbf{A} \in \mathbf{P}_{1\alpha_1}^\lambda(\mathcal{K})$ is either commutative or a divisor of S_n. In either case, $\mathcal{K}$ cannot be complete for the α_1^λ-product by Theorem 1.1.

For the converse, suppose that for every positive integer n there exists a biconnected graph in $D(\mathcal{K})$ having at least n vertices. Take a simple group G. There is a positive integer n with $G|S_n$. By Theorem 4.3, Fact 3.2 and Fact 3.1, it is easy to see that $S_n|S(\mathbf{A})$ for some $\mathbf{A}\in\mathbf{P}^\lambda_{1\alpha_1}(\mathcal{K})$. Thus, $\mathcal{K}$ is complete for the α_1^λ-product by Theorem 1.1.

In exactly the same way we obtain the following result:

Theorem 4.6. *Let $\mathcal{K}$ be a class of automata. If $\mathcal{K}$ is not complete for the α_1^λ-product then three cases arise.*

(i) *There is a highest integer n such that $D(\mathcal{K})$ contains a biconnected graph on n vertices. Then $\mathbf{A}\in\mathbf{HSP}^\lambda_{\alpha_1}(\mathcal{K})$ if and only if for every simple group G with $G|S(\mathbf{A})$, either $G|S_{n-1}$ or $G|G(D)$ for a biconnected graph $D\in D(\mathcal{K})$ on n vertices or G is a prime group of order p and $D(\mathcal{K})$ contains a cycle of length at least p.*

(ii) *$D(\mathcal{K})$ does not contain biconnected graphs but there is at least one cycle in $D(\mathcal{K})$. Then $\mathbf{A}\in\mathbf{HSP}^\lambda_{\alpha_1}(\mathcal{K})$ if and only if for every simple group with $G|S(\mathbf{A})$, G is a prime group of order p such that $D(\mathcal{K})$ contains a cycle of length at least p.*

(iii) *Otherwise, i.e. if there is no cycle in $D(\mathcal{K})$, then $\mathbf{HSP}^\lambda_{\alpha_1}(\mathcal{K})$ is the class of all monotone automata or the class of all discrete automata or the class of all trivial automata, just as in Theorem* 1.2.

Corollary 4.7. *There are a countable number of classes of automata of the form $\mathbf{HSP}^\lambda_{\alpha_1}(\mathcal{K})$.*

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