Acta Sci. Math., 53 (1989), 217—223

On composition of idempotent functions

A. KISIELEWICZ

The general problem of composition of functions was raised by W. SIERPIŃSKI [13]. Since then the problem has been extensively investigated in many-valued logic, synthesis of automata, and recently, in universal algebra (cf. [11], [1], [3], [4]). There are some results and problems showing that idempotent clones play here a special role (cf. [2], [3], [8], [10,] [12], [14], see also [7]). In this paper some further special properties of idempotent clones are established, and examples are provided to show that our theorems do not hold in the general (nonidempotent) case.

The results are stated in Section 3. Before we introduce some definitions (Section 1) and give background information (Section 2). Proofs are given in Section 4.

1. Definitions. A *clone* is a composition closed set of functions (on a fixed universe A) containing all projections (cf. [12]). For two clones $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{B}\supseteqq\mathbf{A}$ we say that $\mathbf{A}$ is a *subclone* of $\mathbf{B}$, while $\mathbf{B}$ is an *extension* of $\mathbf{A}$. If $\mathbf{A}\neq\mathbf{B}$ and $\mathbf{A}$ is not a *trivial clone* (i.e. consisting of projections only), then $\mathbf{A}$ is said to be a *proper subclone* of $\mathbf{B}$. If m is the least integer such that there is an essentially m-ary function in $\mathbf{B}-\mathbf{A}$, then B is called an *m-ary extension* of $\mathbf{A}$.

For any set $\mathbf{F}$ of functions, $P_n(\mathbf{F})$ denotes the set of essentially n-ary functions in $\mathbf{F}$, and $p_n(\mathbf{F})$ is the cardinality of $P_n(\mathbf{F})$. Moreover, we denote $S(\mathbf{F})=\{n\colon p_n(\mathbf{F})>0\}$.

A function $f\colon A^n\to A$ is *idempotent* if it satisfies $f(x,\ldots,x)=x$ identically. If, in addition, it satisfies

$$f(f(x_1^1,x_2^1,\ldots,x_n^1),f(x_1^2,x_2^2,\ldots,x_n^2),\ldots,f(x_1^n,x_2^n,\ldots,x_n^n))=f(x_1^1,x_2^2,\ldots,x_n^n),$$

then it is called *diagonal*. If every function in a clone is idempotent (diagonal), then the clone itself is called *idempotent (diagonal)*.

Other, undefined concepts are standard and can be found in corresponding papers given in our references. Throughout the paper we make use of the fact that clones can be identified with sets of polynomials of universal algebras (cf. [11]).

Received May 26, 1986.

2. Examples. The algebraic property of idempotency is of special interest in studying clones, because for any clone **A**, the idempotent functions in **A**, as it is easy to check, form a composition closed set (the full idempotent subclone of **A**). Consequently, studying minimal clones leads to studying certain idempotent clones (see [2], [12]). Also, p_n-sequences (and free spectra) of idempotent clones are completely different in nature from those of nonidempotent clones (see [4], [7], [8]). Diagonal clones are rather exceptional among idempotent clones and are fully described. Properties of diagonal clones mentioned below are derived from [9] and [6].

(2.1) *Diagonal clones.* A clone **D** generated by a single essentially r-ary diagonal function $d(x_1, ..., x_r)$ is called an *r-dimensional diagonal clone (algebra).* $S(\mathbf{D})=\{2, 3, ..., r\}$ and $p_n(\mathbf{D})$ is finite for all n. A diagonal clone **D** is *finitely generated* iff it is as above. Otherwise, $S(\mathbf{D})=\{2, 3, ...\}$ and $p_n(\mathbf{D})\geqq\aleph_0$ for all $n\geqq 2$. For any diagonal clone **D**, $P_n(\mathbf{D})$ with $n\geqq 2$, if not empty, is a generating set for **D**. Finally, if a clone **A** is generated by diagonal functions only, and has no nondiagonal binary functions, then it is a diagonal clone; it is finitely generated if and only if $p_2(\mathbf{A})$ is finite. Also, the structure of m-ary extensions of r-dimensional diagonal clones with $m>r+1$ is described (see [14]).

(2.2) *Boolean reducts.* For the full idempotent subclone **I** of the clone (of polynomials) of any Boolean group $\langle G, +\rangle$ we have $S(\mathbf{I})=\{3, 5, 7, ...\}$ and $p_n(\mathbf{I})=1$ for odd $n\geqq 3$ (see [8], p. 234). In this paper such clones are called simply *Boolean reducts*. The structure of m-ary extensions of Boolean reducts with $m\geqq 5$ is described in [14].

(2.3) *Counter-examples.* Let C be the union of two infinite disjoint sets A and B and two further elements a and b. For any $n\geqq 1$ we define two functions on C: $f_n(x_1, ..., x_n)=a$ if $x_1, ..., x_n\in A$ and are pairwise distinct, and $f_n=b$ otherwise. Similarly, $g_n(x_1, ..., x_n)=a$ if $x_1, ..., x_n\in B$ and are pairwise distinct, and $g_n=b$ otherwise. It is easy to check that any set of functions f_i, g_i containing the constant b is a clone. Thus, for any set of positive integers S, there exist a clone **B** and a subclone **A** of **B** such that $S(\mathbf{B}-\mathbf{A})=S$. For these clones $P_n(\mathbf{B})\cup\{b\}$ is always a subclone. Also, examples of clones without constants and having the same properties can be given using constructions applied in [3].

3. Results. Our main result concerns the difference $\mathbf{B}-\mathbf{A}$ of an idempotent clone **B** and its subclone **A**. In the general case, by example (2.3), the set $S(\mathbf{B}-\mathbf{A})$ can be arbitrary. If **B** is assumed to be idempotent, the situation is very different:

Theorem 1. *Let* **B** *be an idempotent clone and* **A** *its proper subclone. Then one of the following conditions holds:*

(i) $S(\mathbf{B}-\mathbf{A})=\{m, m+1, \ldots\}$ *for some* $m \geqq 2$,
(ii) $S(\mathbf{B}-\mathbf{A})=\{2, 3, \ldots, r\}$ *for some* $r \geqq 2$,
(iii) $S(\mathbf{B}-\mathbf{A})=\{2, 3, \ldots, r\} \cup \{m, m+1, \ldots\}$ *for some* $r \geqq 2$ *and* $m > r+1$,
(iv) $S(\mathbf{B}-\mathbf{A})=\{3, 5, 7, \ldots\} \cup \{m, m+1, \ldots\}$ *for some even* $m > 5$.
Moreover, conditions (ii)—(iv) *determine the structure of the clone* **B**. *Namely,*
1. *if* (ii) *holds, then* **B** *is an r-dimensional diagonal clone,*
2. *if* (iii) *holds, then* **B** *is an m-ary extension of an r-dimensional diagonal clone,*
3. *if* (iv) *holds, then* **B** *is an m-ary (or* $(m-1)$*-ary) extension of a Boolean reduct.*

Corollary. *The difference* $\mathbf{B}-\mathbf{A}$ *of an idempotent clone* **B** *and its proper subclone* **A** *is always infinite, unless* **B** *is a finitely generated diagonal clone.*

Theorem 1 is actually a classification of the differences $\mathbf{B}-\mathbf{A}$, analogous to that of [14]. It is of some interest that from such a theorem one can derive a result concerning composition of functions:

Theorem 2. *If* **B** *is an idempotent clone which can be generated by (at most) k-ary functions, then for any* $n \geqq k$, *the set* $P_n(\mathbf{B})$ *of essentially n-ary functions in* **B**, *if not empty, is a generating set for* **B**.

In addition to the examples in (2.3), many others can be constructed showing that our theorem fails to hold for nonidempotent clones.

4. Proofs. At first, we give the proof of Theorem 1 which is based on several lemmas. We use techniques and constructions worked out in [5] and [8]. Throughout, **B** is assumed to be an idempotent clone, and **A** its subclone. The numbers in question are always integers. For every $k \geqq 2$ we consider the following property of the clone **B**:

(+) for every function $f(x_1, \ldots, x_n) \in P_n(\mathbf{B})$ with $n \geqq k$ there exists a function $\bar{f}(x_1, \ldots, x_{n+1}) \in P_{n+1}(\mathbf{B})$ in **B** such that $\bar{f}(x_1, \ldots, x_n, x_i)=f(x_1, \ldots, x_n)$ for some i.

Lemma 1. *If* **B** *satisfies* (+) *for some* k, *then for every* $m \geqq k$, $m \in S(\mathbf{B}-\mathbf{A})$ *implies* $\{m, m+1, \ldots\} \subseteq S(\mathbf{B}-\mathbf{A})$.

Proof. $m \in S(\mathbf{B}-\mathbf{A})$ means that there is an essentially m-ary function $f(x_1, \ldots, x_m)$ in $\mathbf{B}-\mathbf{A}$. Then, $\bar{f}(x_1, \ldots, x_{m+1}) \notin \mathbf{A}$, since (by substitution $x_{m+1}=x_i$) it generates $f(x_1, \ldots, x_m)$. It follows that $m+1 \in S(\mathbf{B}-\mathbf{A})$. Now the result follows easily by induction.

Lemma 2. *Let* $g(x, y)=x \cdot y$ *be a binary function in* **B**, *not diagonal. If* $f=$ $=f(x_1, \ldots, x_n) \in P_n(\mathbf{B})$ $(n \geqq 2)$, *then for some* i, $\bar{f}=f(x_1, \ldots, x_i \cdot x_{n+1}, \ldots, x_n) \in P_{n+1}(\mathbf{B})$.

Proof. At first, note that $\bar{f}$ obviously depends on each of the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, since substituting $x_{n+1}=x_i$ in $\bar{f}$ we get f depending on these variables.

Now, suppose that $f(x\cdot y, x_2, ..., x_n)$ does not depend on y. Then by substituting $y=x$, $f(x\cdot y, x_2, ..., x_n)=f(x, x_2, ..., x_n)$. Consequently, $f((x\cdot y)\cdot z, x_2, ..., ..., x_n)=f(x, x_2, ..., x_n)$. Similarly, if $f(x\cdot y, x_2, ..., x_n)$ does not depend on x, then $f((x\cdot y)\cdot z, x_2, ..., x_n)=f(z, x_2, ..., x_n)$. By analogous arguments for all indices i, and in view of the idempotency of f, we infer that, if $f(x_1, ..., x_i\cdot x_{n+1}, ..., x_n)$ is not essentially $(n+1)$-ary for any i, then $(x\cdot y)\cdot z=f((x\cdot y)\cdot z, ..., (x\cdot y)\cdot z)$ does not depend on y. Consequently, $(x\cdot y)\cdot z=x\cdot z$. Similarly, $x\cdot(y\cdot z)=x\cdot z$. This means that $x\cdot y$ is diagonal, a contradiction.

Lemma 3. *If there is a binary nondiagonal function in* **B**, *then condition* (i) *of Theorem* 1 *holds.*

Proof. By Lemma 2, **B** satisfies condition $(+)$ for $k=2$. Now, if m is the least integer such that $m\in S(\mathbf{B}-\mathbf{A})$, then in view of Lemma 1, $S(\mathbf{B}-\mathbf{A})=\{m, m+1, ...\}$, as required.

Lemma 4. *If* **B** *is a diagonal clone, then* $S(\mathbf{B}-\mathbf{A})=\{2, ..., r\}$ *for some* $r\geqq 2$ *whenever* **B** *is finitely generated, and* $S(\mathbf{B}-\mathbf{A})=\{2, 3, ...\}$ *otherwise.*

Proof. If **B** is finitely generated, then the result is by (2.1). Suppose that **B** is not finitely generated and $m\notin S(\mathbf{B}-\mathbf{A})$. Then $P_m(\mathbf{B})=P_m(\mathbf{A})$. However, as $P_m(\mathbf{B})$ generates the clone **B** (cf. (2.1)), $P_m(\mathbf{A})$ also does, and so $\mathbf{A}=\mathbf{B}$, a contradiction.

Lemma 5. *If* **B** *is an m-ary extension of a diagonal clone* **D** *for some* $m\geqq 2$, *then* **D** *is contained in the clone generated by* $P_m(\mathbf{B})$.

Proof. If there is a diagonal function in $P_m(\mathbf{B})$, then by (2.1) $P_m(\mathbf{D})$ generates **D**, and since $P_m(\mathbf{B})\supset P_m(\mathbf{B})$, the result follows. In the opposite case, **D** is an r-dimensional diagonal clone for some $r<m$ $(r\geqq 2)$. In this case we apply the method of diagonal decomposition of the clone **B** with respect to **D** (see [8], p. 244).

Let $f(x_1, ..., x_m)$ be a nondiagonal essentially m-ary function in **B**, which exists by assumption. Then

$$f(x_1, ..., x_m) = \langle f^1(x_1, ..., x_m), ..., f^r(x_1, ..., x_m)\rangle$$

and each f^i is either essentially m-ary or equal to $h_i(x_k)$. Indeed, if e.g. $f^1(x_1, ..., x_m)$ depended on exactly k variables with $1<k<m$, say $f^1(x_1, ..., x_m)=g(x_1, ..., x_k)$, then we would have $\langle g(x_1, ..., x_k), h_2(x_1), ..., h_r(x_1)\rangle\in P_k(\mathbf{B})$. This function is nondiagonal (by properties of diagonal decomposition) contradicting the assumptions in our lemma. Moreover, at least one f^i must be essentially m-ary, since otherwise f would be a diagonal function.

So, suppose that e.g. $f^1(x_1, ..., x_m)$ is essentially m-ary.

$$f(x_1, ..., x_{m-1}, x_{m-1}) = h_1(x_k) \quad \text{for some} \quad k \ (1\leqq k\leqq m-1)$$

(otherwise we can infer a contradiction, as above). Put

$$g(x_1, \ldots, x_m) = \langle f^1(x_1, \ldots, x_m),\ h_2(x_{i_2}), \ldots, h_r(x_{i_r})\rangle$$

where $i_2, \ldots, i_r$ are pairwise distinct and $k \notin \{i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, m\}$. Then $g \in P_m(\mathbf{B})$ and $g(x_1, \ldots, x_{m-1}, x_{m-1}) = \langle h_1(x_k), h_2(x_{i_2}), \ldots, h_r(x_{i_r})\rangle$ is an essentially r-ary diagonal function, and by (2.1), it generates the clone $\mathbf{D}$. This completes the proof.

Lemma 6. *Suppose that $P_2(\mathbf{B})$ is finite, nonempty, and consists of diagonal functions only. If $\mathbf{B}$ is not a diagonal clone, then either $S(\mathbf{B}-\mathbf{A}) = \{m, m+1, \ldots\}$ for some $m \geqq 2$, or $S(\mathbf{B}-\mathbf{A}) = \{2, \ldots, r\} \cup \{m, m+1, \ldots\}$ for some $r \geqq 2$ and $m > r+1$ In the latter case $\mathbf{B}$ is an m-ary extension of an r-dimensional diagonal clone.*

Proof. Denote by $\mathbf{D}$ the clone generated by $P_2(\mathbf{B})$. By (2.1) it is an r-dimensional diagonal clone for some $r \geqq 2$, and consits of all diagonal functions in $\mathbf{B}$. In other words, since by assumption $\mathbf{B} \neq \mathbf{D}$, $\mathbf{B}$ is an m-ary extension of $\mathbf{D}$ for some $m \geqq 2$, just as the second part of the lemma states. Moreover, $\mathbf{B}$ satisfies $(+)$ for $k=m$. Indeed, it is enough to set $\bar{f} = fL$, where L is a mapping defined in [8], Section 5.4.

Now, if s is the least integer such that $s \in S(\mathbf{B}-\mathbf{A})$ and $s \geqq m$, then in view of Lemma 1, $S(\mathbf{B}-\mathbf{A}) = \{s, s+1, \ldots\}$, as required. It remaines to consider the case when $s < m$, which means that there exists a diagonal essentially s-ary function in $\mathbf{B}-\mathbf{A}$ $(s \geqq 2)$. By means of (2.1), it follows that the full diagonal subclone of $\mathbf{A}$ is contained properly in $\mathbf{D}$, and consequently, $\{2, \ldots, r\} \subseteq S(\mathbf{B}-\mathbf{A})$. On the other hand, by Lemma 5, $m \in S(\mathbf{B}-\mathbf{A})$, and since $\mathbf{B}$ satisfies $(+)$ for $k=m$, $\{m, m+1, \ldots\} \subseteq$ $\subseteq S(\mathbf{B}-\mathbf{A})$.

Now, if $r \geqq m-1$, then $S(\mathbf{B}-\mathbf{A}) = \{2, 3, \ldots\}$, while if $r < m-1$, then as $\mathbf{B}$ is an m-ary extension of $\mathbf{D}$, $S(\mathbf{B}-\mathbf{A}) = \{2, \ldots, r\} \cup \{m, m+1, \ldots\}$. This completes the proof.

Lemma 7. *If $P_2(\mathbf{B})$ is infinite and consists of diagonal functions only, then condition* (i) *in Theorem* 1 *holds.*

Proof. In view of Lemma 1 it is enough to show that $\mathbf{B}$ satisfies condition $(+)$ for $k=2$. Applying again the method of diagonal decomposition [8], we construct a suitable mapping.

Let $f(x_1, \ldots, x_m) \in P_m(\mathbf{B})$, $m \geqq 2$. By virtue of (2.1) (for every m) there exists an essentially $(m+1)$-ary diagonal function in $\mathbf{B}$. This function generates an $(m+1)$-dimensional diagonal clone, a subclone of $\mathbf{B}$. We decompose $\mathbf{B}$ just with respect to this diagonal clone. Thus, we have

$$f(x_1, \ldots, x_m) = \langle f^1, f^2, \ldots, f^{m+1}\rangle.$$

Since each f^i depends on at least one variable, there are a variable x_k and indicies i_1, i_2 such that both f^{i_1} and f^{i_2} depend on x_k. Replacing in f^{i_1} the variable x_k by

x_{m+1}, we obtain an essentially $(m+1)$-ary function $\bar{f}$ which yields f, with the substitution $x_{m+1}=x_k$ as required.

Lemma 8. *If $p_2(\mathbf{B})=0$ and $\mathbf{B}$ is not a q-ary extension of a Boolean reduct for any $q\geqq 4$, then $S(\mathbf{B}-\mathbf{A})=\{m, m+1, \ldots\}$ for some $m\geqq 3$.*

Proof. It is enough to show that $\mathbf{B}$ satisfies condition (+) for $k=3$. For the proof one should consider three cases corresponding to Sections 3, 4 and 6 of [5] (note that in Section 7, actually, 4-ary extensions of Boolean reducts are considered), and observe that the constructions given there satisfy the requirements of our condition (+) for $k=3$.

Lemma 9. *If $\mathbf{B}$ is an m-ary extension of a Boolean reduct $\mathbf{I}$ with $m\geqq 4$, then either $S(\mathbf{B}-\mathbf{A})=\{q, q+1, \ldots\}$ for some $q\geqq m$ or $S(\mathbf{B}-\mathbf{A})=\{3, 5, 7, \ldots\}\cup \cup\{m, m+1, \ldots\}$.*

Proof. For $f\in P_n(\mathbf{B})$ with $n\geqq m-1$ put $\bar{f}=fL_1$, where L_1 is the mapping defined in [8], Section 5.2. It follows (by properties of L_1) that $\mathbf{B}$ satisfies condition (+) for $k=m-1$. (In [7] it is assumed that $m\geqq 5$, but the construction works also for $m=4$, since the conditions (i), (ii) in [8], p. 242, hold for $m=4$ as well (cf. [5], p. 111)).

Now, if q is the least integer such that $q\in S(\mathbf{B}-\mathbf{A})$ and $q\geqq m$, then by Lemma 1, $S(\mathbf{B}-\mathbf{A})=\{q, q+1, \ldots\}$.

In turn, $q<m$ means that one of the essentially n-ary functions of $\mathbf{I}$ with $3\leqq n<m$ is in $\mathbf{B}-\mathbf{A}$, and as each Boolean reduct function generates $\mathbf{I}$, we have $\{3, 5, 7, \ldots\}\subseteq \subseteq S(\mathbf{B}-\mathbf{A})$. Since $\mathbf{B}$ satisfies (+) for $k=m-1$ (applying this condition to Boolean reduct functions), we get $S(\mathbf{B}-\mathbf{A})=\{3, 5, 7, \ldots\}\cup\{m, m+1, \ldots\}$ regardless as to whether m is even or odd. The proof is complete.

Lemma 10. *If $p_2(\mathbf{B})=p_3(\mathbf{B})=0$, then $S(\mathbf{B}-\mathbf{A})=\{m, m+1, \ldots\}$ for some $m\geqq 4$.*

Proof. Let s be the least integer ($\geqq 4$) with the property $p_s(\mathbf{B})>0$. Then $\mathbf{B}$ satisfies (+) for $k=s$. To see this, it is enough to put $\bar{f}=fL_1$, where L_1 is as in the previous proof. The result follows by Lemma 1.

Now, Theorem 1 is a consequence of Lemmas 3, 4, 6, 7, 8, 9, and 10. The Corollary is an immediate consequence of Theorem 1. We prove Theorem 2.

To this end suppose that $P_n(\mathbf{B})$ is nonempty (i.e. $n\in S(B)$), and denote by $\mathbf{A}$ the subclone of $\mathbf{B}$ generated by $P_n(\mathbf{B})$. Then we have $n\notin S(\mathbf{B}-\mathbf{A})$. Now, if $\mathbf{A}=\mathbf{B}$, then the result is true. Hence, suppose that $\mathbf{A}$ is a proper subclone of $\mathbf{B}$ and apply Theorem 1. Observe that in cases (ii)—(iv) of Theorem 1, the second part of the theorem combined with Urbanik's result [14] yields that we always have $S(\mathbf{B}-\mathbf{A})=\emptyset$.

This contradicts the fact that $n\in S(\mathbf{B})$, while $n\notin S(\mathbf{B}-\mathbf{A})$. It follows that under our assumptions case (i) in Theorem 1 holds for some $m>n$. In particular, for every $i\leqq n$, $i\notin S(\mathbf{B}-\mathbf{A})$, i.e. $P_i(\mathbf{B})=P_i(\mathbf{A})$. Since by assumption $k\leqq n$, and $\mathbf{B}$ is generated by k-ary functions, it follows that $\mathbf{A}=\mathbf{B}$, completing the proof of Theorem 2.

References

[1] J. Berman, Free spectra of 3-element algebras, in: *Universal Algebra and Lattice Theory,* (Proc. Conf. Puebla, 1982), Springer-Verlag, Lecture Notes 1004 (1983); pp. 10—53.

[2] B. Csákány, All minimal clones on the three-element set, *Acta Cybernet.* **6** (1983), 227—238.

[3] G. Grätzer, Composition of functions, in: *Proc. Conf. on Universal Algebra at Queens University* (Kingston, Ontario, 1969); pp. 1—106.

[4] G. Grätzer, Universal algebra and lattice theory: a story and three research problems, in: *Universal Algebra and its Links with Logic, Algebra, Combinatorics and Computer Science* (P. Burmeister et al. eds.) Heldermann (Berlin, 1984); pp. 1—13.

[5] G. Grätzer and J. Płonka, On the number of polynomials of an idempotent algebra. II, *Pacific J. Math.,* **14** (1973), 99—113.

[6] A. Kisielewicz, A remark on diagonal algebras, *Algebra Universalis,* **12** (1981), 200—204.

[7] A. Kisielewicz, Characterization of p_n-sequences for nonidempotent algebras, *J. Algebra,* **108** (1987), 102—115.

[8] A. Kisielewicz, The p_n-sequences of idempotent algebras are strictly increasing. *Algebra Universalis,* **13** (1981), 233—250.

[9] J. Płonka, Diagonal algebras, *Fund. Math.,* **58** (1966), 309—321.

[10] R. Quackenbusch, On the composition of idempotent functions, *Algebra Universalis,* **1** (1971), 7—12.

[11] I. Rosenberg, Completeness properties of multiple-valued logic algebras, in: *Computer Science and Multiple-Valued Logic,* (2nd. ed.), North-Holland (Amsterdam, 1984); pp. 144—186.

[12] I. Rosenberg, Minimal clones I: The five types, in: *Lectures in Universal Algebra* (Proc. Conf. (Szeged, 1983), Colloq. Math. Soc. J. Bolyai, vol. 43, North-Holland (Amsterdam, 1986); pp. 405—427.

[13] W. Sierpiński, Sur les functions de plusieurs variables, *Fund. Math.,* **33** (1945), 169—173.

[14] K. Urbanik, On algebraic operations in idempotent algebras, *Colloq. Math.,* **13** (1965), 129—157.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
WYBRZEŻE WYSPIAŃSKIEGO 27
50-370 WROCŁAW, POLAND