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Some uniform weak-star ergodic theorems

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0. Introduction. Let $\mathfrak{B}$ be a Banach space and let G be a bounded semigroup of adjoint operators in $\mathfrak{B}^*$. We have proved the following result in [3]:

Suppose $\mathfrak{B}$ is weakly complete and G is commutative and separable. If for every $t\in\mathfrak{B}^*$, the w^*-closed convex hull of the orbit $Gt=\{gt: g\in G\}$ contains exactly one G-invariant element t^G, then the mapping $t\to t^G$: $\mathfrak{B}^*\to\mathfrak{B}^*$ is a w^*-continuous linear projection P such that $gP=Pg=P$ $(g\in G)$.

(The term "separable" here means that G contains a countable subset G_0 which is dense in G if G is considered in the topology of pointwise w^*-convergence on $\mathfrak{B}^*$.)

According to [4], the above result also holds if instead of the commutativity of G, we only assume its amenability.

In the present paper we are going to prove analogues of the above result for the uniformly closed convex hull of the orbit Gt. The particular case where $\mathfrak{B}^*$ is a W^*-algebra M and G is a group of *-automorphisms of M may be of some interest.

1. Results. Let $\mathfrak{B}$ be a Banach space with dual $\mathfrak{B}^*$ and let G be a bounded semigroup of w^*-continuous linear operators in $\mathfrak{B}^*$. In other words, $\sup\{\|g\|: g\in G\}<\infty$ and for every $g\in G$, there is a unique bounded linear operator g_* acting in $\mathfrak{B}$, such that $(g_*)^*=g$. Let us consider the following two properties of the pair $\mathfrak{B}$, G:

(N) For every $t\in\mathfrak{B}^*$, the norm-closed convex hull of the orbit $Gt=\{gt: g\in G\}$ contains at least one G-invariant element.

(N_1) For every $t\in\mathfrak{B}^*$, the norm-closed convex hull of the orbit Gt contains exactly one G-invariant element, say t^G.

Theorem 1. *Suppose $\mathfrak{B}$ is weakly complete and G is amenable and countable. Then condition (N) implies condition (N_1). If condition (N_1) (or (N)) is satisfied, then the mapping $t\to t^G$ $(t\in\mathfrak{B}^*)$ is a w^*-continuous linear projection P such that $gP=Pg=P$ $(g\in G)$.*

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Theorem 2. *Suppose $\mathfrak{B}$ is weakly complete and G is commutative and separable. Then condition* (N) *implies condition* (N_1). *If condition* (N) (*or* (N_1)) *is satisfied, then the mapping $t \to t^G$ ($t \in \mathfrak{B}^*$) is a w^*-continuous linear projection P such that $gP = Pg = P$ ($g \in G$).*

Proposition. *Let $\mathfrak{B}^* = M$, a von Neumann algebra and let G be a countable amenable group of *-automorphisms of M or let $\mathfrak{B}^* = M$, a von Neumann algebra in a separable Hilbert space and let G be a commutative group of *-automorphisms of M. Assume that condition* (N) *is satisfied. Then condition* (N_1) *is also satisfied and M is G-finite.* (For this notion, cf. [2].)

2. Proofs.

Proof of Theorem 1. For every $\varphi \in \mathfrak{B}$ and $t \in \mathfrak{B}^*$, let us define the element $f_{\varphi,t} \in l^\infty(G)$ by the equality $f_{\varphi,t}(g) = \varphi(g(t))$ ($g \in G$). Assume that (N) holds and consider a given $t \in \mathfrak{B}^*$. Then there is a sequence $\{v_n\}_{n=1}^\infty$ of elements of the convex hull conv G of G, for which $v_n(t)$ converges in norm to a G-invariant element of $\mathfrak{B}^*$, say t'. Let a positive number ε be given. Using the notation $\|G\| = \sup\{\|g\| : g \in G\}$, we can find a positive integer n_0 such that $\|v_n(t) - t'\| < \varepsilon/\|G\|$ if $n \geqq n_0$. Then $\|gv_n(t) - t'\| = \|g(v_n(t) - t')\| \leqq \|g\|\,\|v_n(t) - t'\| < \|G\|(\varepsilon/\|G\|) = \varepsilon$ uniformly in $g \in G$ for $n \geqq n_0$. Consequently, for a given $\varphi \in \mathfrak{B}$ we have $|f_{\varphi,t}(gv_n) - \varphi(t')| = |\varphi(gv_n(t)) - \varphi(t')| = |\varphi(gv_n(t) - t')| \leqq \|\varphi\|\,\|gv_n(t) - t'\| < \|\varphi\|\varepsilon$ for all $g \in G$ if $n \geqq n_0$. Since $\varepsilon > 0$ was arbitrary, we have proved that the constant function on G which is equal to $\varphi(t')$ can be uniformly approximated by convex combinations of the right translates of the element $f_{\varphi,t}$ of $l^\infty(G)$.

Let m now be a right invariant mean on $l^\infty(G)$. The result above implies that $m(f_{\varphi,t}) = \varphi(t')$. In particular, if t'' is another element of the norm-closed convex hull of Gt, then $m(f_{\varphi,t}) = \varphi(t'')$. Consequently, $\varphi(t') = \varphi(t'')$ for every $\varphi \in \mathfrak{B}$, and thus $t' = t''$. Therefore, since $t \in \mathfrak{B}^*$ was arbitrary, we have proved that (N) implies (N_1) (even without assuming the weak completeness of $\mathfrak{B}$ or the countability of G).

Now let $\{G_n\}_{n=1}^\infty$ be a right-hand summing sequence for G, i.e., let $(1/\text{card}\, G_n)\, \text{card}\,([G_n \cup G_n g] \setminus [G_n \cap G_n g]) \to 0$ as $n \to \infty$. (For the existence of such a sequence, see [1].) We are going to prove that for $\varphi \in \mathfrak{B}$ and $t \in \mathfrak{B}^*$,

$$(*) \qquad (1/\text{card}\, G_n) \sum_{h \in G_n} f_{\varphi,t}(h) \to \varphi(t^G)$$

as $n \to \infty$. To prove this, we fix $\varphi \in \mathfrak{B}$ and $t \in \mathfrak{B}^*$ and assume that for some filter F finer than the filter base $\{\{n : n \geqq k\} : k \in \mathbf{N}\}$, $\lim_F (1/\text{card}\, G_n) \sum_{h \in G_n} f_{\varphi,t}(h)$ exists. Let F_1 be an ultrafilter finer than F. Then for every $f \in l^\infty(G)$ and $g \in G$, the limit $\lim_{F_1} (1/\text{card}\, G_n) \sum_{h \in G_n} f(hg)$ exists (since F_1 is an ultrafilter) and is independent of

$g\in G$ (because of the summing sequence property, since F_1 is finer than $\{\{n: n\geqq k\}: k\in\mathbf{N}\}$). Consequently, $m(f)=\lim_{F_1}(1/\text{card}\, G_n)\sum_{h\in G_n} f(h)$ is a right invariant mean on $l^\infty(G)$. By the beginning of our proof,

$$m(f_{\varphi,t}) = \lim_{F_1}(1/\text{card}\, G_n)\sum_{h\in G_n} f_{\varphi,t}(h) = \varphi(t^G).$$

This means that $\lim_F (1/\text{card}\, G_n)\sum_{h\in G_n} f_{\varphi,t}(h)=\varphi(t^G)$ for every filter F which is finer than the filter base $\{\{n: n\geqq k\}: k\in\mathbf{N}\}$ and for which $\lim_F (1/\text{card}\, G_n)\sum_{h\in G_n} f_{\varphi,t}(h)$ exists. This means that $\lim_{n\to\infty}(1/\text{card}\, G_n)\sum_{h\in G_n} f_{\varphi,t}(h)=\varphi(t^G)$. Since $\varphi\in\mathfrak{B}$ and $t\in\mathfrak{B}^*$ were arbitrary fixed elements, we have proved (*).

Let us write $w_n=(1/\text{card}\, G_n)\sum_{h\in G_n} h$. Then $w_n\in\text{conv}\, G$ and by (*), $w_n t\to t^G$ in the w^*-topology for every $t\in\mathfrak{B}^*$. From this point we proceed in the same way as in the first paragraph of Proof of Theorem 2 in [3]. For the sake of completeness, we repeat that reasoning here.

Let $\varphi\in\mathfrak{B}$ be given. Then for every $t\in\mathfrak{B}^*$ we have $(w_n^*\varphi-w_m^*\varphi, t)= =(\varphi, (w_n-w_m)t)\to 0$ as $n, m\to\infty$. Therefore, the sequence $\{w_n^*\varphi\}_{n=1}^\infty$ is a weak Cauchy sequence in $\mathfrak{B}$. Since $\mathfrak{B}$ is weakly complete, there is an element $P_*\varphi$ of $\mathfrak{B}$ such that $(w_n^*\varphi, t)\to(P_*\varphi, t)$ for every $t\in\mathfrak{B}^*$ as $n\to\infty$. It is obvious that P_* is a bounded linear operator in $\mathfrak{B}$. Furthermore, letting $n\to\infty$, we obtain that $(\varphi, w_n t)= =(w_n^*\varphi, t)\to(P_*\varphi, t)=(\varphi, (P_*)^* t)$ for $\varphi\in\mathfrak{B}$, $t\in\mathfrak{B}^*$. Consequently, for every $t\in\mathfrak{B}^*$ we have $w_n t\to(P_*)^* t$ $(n\to\infty)$ in the w^*-topology of $\mathfrak{B}^*$ and thus $t^G=(P_*)^* t$ $(t\in\mathfrak{B}^*)$. Since $(P_*)^*$ is obviously w^*-continuous, this completes the proof of Theorem 1.

Remark. The first part of the proof of Theorem 1 shows that if G is a bounded amenable semigroup of linear operators in a Banach space $\mathfrak{C}$ and for every $t\in\mathfrak{C}$, the norm-closed convex hull of the orbit Gt contains at least one G-invariant element, then it contains exactly one G-invariant element. This can be seen in the same way as in the first part of the proof of Theorem 1 if we replace $\mathfrak{B}^*$ by $\mathfrak{C}$ and $\mathfrak{B}$ by $\mathfrak{C}^*$ there.

Proof of Theorem 2. Assume (N). We shall prove that for every $t\in\mathfrak{B}^*$, the w^*-closed convex hull of the orbit Gt contains exactly one G-invariant element. Then Theorem 2 of [3] will imply the statement of Theorem 2 of this paper.

First we prove that for every $t\in\mathfrak{B}^*$, the norm-closed convex hull of Gt contains exactly one G-invariant element. (This follows from the above Remark, but in the commutative case the proof is simpler and we prefer to give an independent proof.) In fact, let t' and t'' be two G-invariant elements in the norm-closed convex hull of Gt and let ε be a positive number. There exist v and w in $\text{conv}\, G$, such that $\|vt-t'\|<\varepsilon$ and $\|wt-t''\|<\varepsilon$. We have $\|t'-t''\|\leqq\|t'-vwt\|+\|vwt-t''\|= =\|w(t'-vt)\|+\|v(wt-t'')\|\leqq\|t'-vt\|+\|wt-t''\|<2\varepsilon$, since $vw=wv$ and $\|v\|\leqq 1$,

$\|w\| \leqq 1$. Since $\varepsilon > 0$ was arbitrary, this proves that $t' = t''$ and thus the norm-closed convex hull of Gt contains exactly one G-invariant element, say t^G.

Now we prove that for every $t \in \mathfrak{B}^*$, the only G-invariant element in the w^*-closed convex hull of Gt is t^G. In fact, let $t \in \mathfrak{B}^*$ and let t_0 be a G-invariant element in the w^*-closure of $[\operatorname{conv} G]t$. Given $\varepsilon > 0$, there is $w \in \operatorname{conv} G$ such that $\|wt - t^G\| < \varepsilon$. Furthermore, there exists a net v_n in $\operatorname{conv} G$, such that $v_n t \to t'$ in the w^*-topology. Then $wv_n t \to wt' = t'$ in the w^*-topology. On the other hand, $\|wv_n t - t^G\| = \|v_n(wt - t^G)\| < \varepsilon$. Consequently, $\|t' - t^G\| \leqq \sup_n \|wv_n t - t^G\| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves that $t' = t^G$ is the only G-invariant element in the w^*-closure of $[\operatorname{conv} G]t$. This completes the proof of Theorem 2.

Proof of Proposition. The Proposition is a special case of Theorems 1 and 2. We only have to note that if M is a von Neumann algebra in a separable Hilbert space and G is a group of *-automorphisms of M, then G is separable, as was pointed out in [3].

Problem. If $\mathfrak{B}$ is weakly complete and separable, does condition (N_1) imply that the mapping $t \to t^G$ is w^*-continuous on $\mathfrak{B}^*$?

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