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Generalized Toeplitz kernels and dilations of intertwining operators. II. The continuous case

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I. Matricial Toeplitz kernels and intertwining operators

This paper continues a study about the relation between generalized Toeplitz kernels and the problem of the dilation of the commutant of contractive semigroups, started in [2], where only discrete semigroups were considered. In Section II we shall extend that study to general groups. In Section III the group of the real numbers is considered and the basic results of this paper on dilation theory — theorems (III.11) and (III.13) — are obtained; the last includes a continuous version of the theorem on the dilation of the commutant due to Sz.-Nagy and Foiaş.

In this section we start with preliminary results concerning the relation between intertwining operators, unitary representations of groups, and positive definite matricial functions.

We fix a (topological) group Γ with neutral element e and consider $\mathscr{L}(H)$-valued kernels on Γ, i.e. functions $K\colon \Gamma\times\Gamma\to\mathscr{L}(H)$, where $\mathscr{L}(H)$ is the set of bounded operators on a Hilbert space H. Such a kernel is said to be positive definite, p.d., if

$$\sum_{s,t\in\Gamma}\langle K(s,t)h(s),\, h(t)\rangle_H \geqq 0,$$

for every function $h\colon \Gamma\to H$ whose support $\{t\in\Gamma\colon h(t)\neq 0\}$ is a finite set.

If K is such that $K(st, su)=K(t, u)$ holds for all $s, t, u\in\Gamma$, then K is determined by the function G on Γ given by $G(s)=K(s, e)$; conversely, if a function G on Γ is given, setting $K(s, t)=G(t^{-1}s)$ we get a kernel with the above property; in that case we say that K or — informally speaking — G are Toeplitz kernels. When $H=H_1\oplus H_2$ is the direct sum of two Hilbert spaces, H_1 and H_2, then G is given by a matrix $(G_{jk})_{j,k=1}^2$ where $G_{jk}(s)\in\mathscr{L}(H_j, H_k)$ for all $s\in\Gamma$, and we say that G is a matricial Toeplitz kernel.

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A positive definite matricial Toeplitz kernel can be viewed as a relation between two unitary representations of the given group, in the following sense.

Proposition 1. *For $j=1,2$ let H_j be a Hilbert space and $G_j\colon \Gamma\to\mathscr{L}(H_j)$ a positive definite Toeplitz kernel on the group Γ, such that G_j equals the identity on the neutral element of Γ; let U_j be the minimal unitary dilation of G_j to a Hilbert space F_j. Let $\mathscr{R}(U_1, U_2)$ be the set of intertwining operators between U_1 and U_2, considered as a (closed) subspace of $\mathscr{L}(F_1, F_2)$. Then the relation*

$$g(s) = P^{F_2}_{H_2}WU_1(s)|_{H_1}, \quad i.e. \quad \langle WU_1(s)h_1, h_2\rangle_{F_2} = \langle g(s)h_1, h_2\rangle_{H_2},$$

for all $s\in\Gamma$, $h_1\in H_1$, $h_2\in H_2$, gives a bijection $W\leftrightarrow g$ between the unit ball of $\mathscr{R}(U_1, U_2)$ and the set of functions $f\colon \Gamma\to\mathscr{L}(H_1, H_2)$ such that $G=(G_{jk})^2_{j,k=1}$, $G_{11}=G_1$, $G_{12}=g$, $G_{21}=\tilde{g}$, $G_{22}=G_2$ is a positive definite matricial Toeplitz kernel. If moreover Γ is a topological group and G_1, G_2 are continuous in the weak topology of operators, then all such functions g will be continuous in the strong topology.

Notation. When H is a closed subspace of a Hilbert space F, i^F_H denotes the inclusion of H in F and P^F_H the orthogonal projection of F onto H. If g is a function on Γ, we set $\tilde{g}(s)=g^*(s^{-1})$. If $\{S_t\colon t\in M\}$ is a family of subspaces of F, $\bigvee_{t\in M} S_t$ denotes the minimal closed subspace of F that contains S_t for all $t\in M$.

Proof of Proposition 1. For $j=1,2$, $U_j=\{U_j(s)\colon s\in\Gamma\}\subset\mathscr{L}(F_j)$ is such that $G_j(s)=P^{F_j}_{H_j}U_j(s)|_{H_j}$ holds for all $s\in\Gamma$ and the minimality condition $F_j=\bigvee_{s\in\Gamma} U_j(s)H_j$ is also true; that is the content of Naimark's dilation theorem (see [9]). Let G be as in the above statement; set, for all $s, t\in\Gamma$, $h_1\in H_1$, $h_2\in H_2$,

$$B\big(U_1(s)h_1, U_2(t)h_2\big) := \langle G_{12}(t^{-1}s)h_1, h_2\rangle_{H_2}.$$

Taking in account that the elements $U_j(s)h_j$ span the space F_j, it is easy to verify that G is p.d. if and only if B defines a bounded sesquilinear form on $F_1\times F_2$ of norm $\|B\|\leqq 1$. In that case there exists $W\in\mathscr{L}(F_1, F_2)$ such that $\|W\|=\|B\|\leqq 1$ and $\langle Wf_1, f_2\rangle_{F_2}=B(f_1, f_2)$ hold; moreover, from the equalities

$$\langle WU_1(s)[U_1(u)h_1], U_2(t)h_2\rangle_{F_2} = \langle G_{12}(t^{-1}su)h_1, h_2\rangle_{H_2} =$$
$$= \langle U_2(s)WU_1(u)h_1, U_2(t)h_2\rangle_{F_2}$$

and the minimality condition it follows that $WU_1(s)=U_2(s)W$ is true for all $s\in\Gamma$. Hence, W is a contraction belonging to $R(U_1, U_2)$.

By setting $g(s)=P^{F_2}_{H_2}WU_1(s)|_{H_1}$ for all $s\in\Gamma$, the converse also follows.

We now apply the preceding result to the dilation of the commutant of two semigroups of isometries.

Proposition 2. *Let Γ be a group with neutral element e and Γ_1 a subsemigroup of Γ. Set $\Gamma_1^{-1}=\{s\in\Gamma: s^{-1}\in\Gamma_1\}$ and assume that $\Gamma_1\cap\Gamma_1^{-1}=\{e\}$ and $\Gamma_1\cup\Gamma_1^{-1}=\Gamma$ hold. Let $\{V_1(s): s\in\Gamma_1\}$ and $\{V_2(s): s\in\Gamma_1\}$ be two semigroups of isometries in the Hilbert spaces H_1 and H_2, respectively, and Y a contraction intertwining them, so that*

(2a) $$YV_1(s) = V_2(s)Y, \text{ for every } s\in\Gamma_1, \text{ and } \|Y\| = 1 \text{ hold.}$$

Let a matricial Toeplitz kernel G be associated with the commutator Y by:

(2b) $$\begin{gathered} G_{jj}(s) = V_j(s) \text{ if } s\in\Gamma_1, \quad G_{jj}(s) = V_j(s^{-1}) \text{ if } s\in\Gamma_1^{-1}, \quad j = 1,2; \\ G_{12}(s) = V_2(s)Y \text{ if } s\in\Gamma_1, \quad G_{12}(s) = V_2(s^{-1})Y \text{ if } s\in\Gamma_1^{-1}; \quad G_{21} = \tilde{G}_{12}. \end{gathered}$$

Then G is p.d. if and only if the following conditions hold:

(2c) *for $j=1,2$ there exists a unitary representation U_j of Γ in a Hilbert space F_j that contains H_j and satisfies*

$$U_j(s)|_{H_j} = V_j(s) \text{ for } s\in\Gamma_1, \quad F_j = \bigvee_{s\in\Gamma} [U_j(s)H_j];$$

(2d) *there exists $W\in\mathcal{L}(F_1, F_2)$ that verifies*

$$WU_1(s) = U_2(s)W \text{ for } s\in\Gamma; \quad \|W\| = \|Y\|; \quad P^{F_2}_{H_2}W|_{H_1} = Y.$$

Moreover such a W is unique.

Proof. If $Y=0$, $W=0$ is the only solution of (2d), so we may always assume that $Y\neq 0$ and, by homogeneity, $\|Y\|=1$, as in (2a).

If G is p.d. G_{11} and G_{22} have the same property; let U_1 and U_2 be their minimal unitary dilations, respectively. From $P^{F_j}_{H_j}U_j(s)|_{H_j}=G_{jj}(s)$ and (2b) it follows that $U_j(s)$ is an extension of the isometry $V_j(s)$ for every $s\in\Gamma_1$. Thus (2c) is satisfied. Let W be associated with G as in Proposition 1; then $\|W\|\leqq 1$, W intertwines U_1 and U_2, and $\langle Wh_1, h_2\rangle_{F_2}=\langle G_{12}(e)h_1, h_2\rangle_{H_2}=\langle Yh_1, h_2\rangle_{H_2}$ holds for all $h_1\in H_1$, $h_2\in H_2$; thus $P^{F_2}_{H_2}W|_{H_1}=Y$, so $1=\|Y\|\leqq\|W\|\leqq 1$. Consequently (2d) is also satisfied.

Conversely, assume that (2c) and (2d) hold. From Proposition 1 it follows that it is enough to prove that G_{12} is the same function as $g(s)\overset{\text{def}}{=}P^{F_2}_{H_2}WU_1(s)|_{H_1}$; now, if $s\in\Gamma_1$, then $g(s)=P^{F_2}_{H_2}WV_1(s)=YV_1(s)=G_{12}(s)$; if $s\in\Gamma_1^{-1}$, we have for all $h_1\in H_1$, $h_2\in H_2$,

$$\begin{aligned} \langle g(s)h_1, h_2\rangle_{H_2} &= \langle U_2(s)Wh_1, h_2\rangle_{F_2} = \langle P^{F_2}_{H_2}Wh_1, V_2(s^{-1})h_2\rangle_{H_2} = \\ &= \langle V_2^*(s^{-1})Yh_1, h_2\rangle_{H_2} = \langle G_{12}(s)h_1, h_2\rangle_{H_2}. \end{aligned}$$

The simplest example is perhaps $\Gamma=Z$, the set of integers, $\Gamma_1=Z_1:=\{n\in Z: n\geqq 0\}$. In that case the semigroup $V_j(s)$ is determined by the isometry $V_j(1)$, so that we are concerned with the commutator $YV_1=V_2Y$ of two isometries. Then it is easy to prove ([2], Lemma II.3) that G is p.d. so, if U_1 and U_2 are the minimal unitary extensions of V_1 and V_2, there exists W that verifies $WU_1=U_2W$, $\|W\|=\|Y\|$ and $P_{H_2}^{F_2}W|_{H_1}=Y$. Note that in general the last equality cannot be improved so as to get W to be a strict lifting of Y, i.e., such that $P_{H_2}^{F_2}W=YP_{H_1}^{F_1}$. In fact, the last equation implies $YV_1^*=YP_{H_1}^{F_1}U_1^*i_{H_1}^{F_1}=P_{H_2}^{F_2}WU_1^*i_{H_1}^{F_1}=P_{H_2}^{F_2}U_2^*Wi_{H_1}^{F_1}=V_2^*P_{H_2}^{F_2}Wi_{H_1}^{F_1}$, because U_2 extends V_2; thus $YV_1^*=V_2^*Y$. Now, the last equality is not a consequence of $YV_1=V_2Y$ because if $V_1=V_2=Y=V$ is any non-unitary isometry then $YV_1^*=VV^*\neq$ $\neq I=V_2^*Y$, etc.

Let us now go from the discrete to the continuous case. Set $\Gamma=R=\{\text{real numbers}\}$, $\Gamma_1=R_1=\{s\in R\colon s\geqq 0\}$. In order to apply Proposition 2 we assume that (2a) holds and consider G given by (2b). Working as in [9], page 30, we can prove that G is p.d. whenever the semigroups V_1 and V_2 are weakly continuous. Thus:

Corollary 3. *Let* $\{V_1(s)\}$, $\{V_2(s)\}$, $s\geqq 0$, *be two continuous monoparametric semigroups of isometries in the Hilbert spaces* H_1, H_2, *respectively, and let* $Y\in\mathscr{L}(H_1,H_2)$ *be a contraction intertwining them, i.e., such that*

$$YV_1(s)=V_2(s)Y \quad \textit{for} \quad s\geqq 0, \quad \|Y\|\leqq 1.$$

For $j=1,2$ *let* $\{U_j(s)\}$, $s\in R$, *be a minimal extension of* V_j *to a continuous monoparametric group of unitary operators in a Hilbert space* F_j. *Then there exists a unique operator* $W\in\mathscr{L}(F_1,F_2)$ *such that*

$$WU_1(s)=U_2(s)W, \quad \textit{for every} \quad s\in R; \quad Y=P_{H_2}^{F_2}W|_{H_1}; \quad \|Y\|=\|W\|.$$

II. Generalized Toeplitz kernels and dilations of the commutator of two contractions

When we consider a commutator of two contractions instead of isometries the method of the preceding section does not work. In fact, the associated matricial Toeplitz kernel need not be positive definite. (See [2], II.1b.) Nevertheless a suitable extension of such kind of kernels allows a similar approach to the more general situation.

Let Γ_1 be a sub-semigroup of the group Γ. A generalized Toeplitz kernel (GTK) on (Γ,Γ_1) is by definition a set

$$K=\{(K_{jk}),\ j,k=1,2;\ H_1,H_2\}$$

composed of two Hilbert spaces, H_1 and H_2, and four functions

$$K_{11}\colon \Gamma\to\mathscr{L}(H_1);\quad K_{12}\colon \Gamma_1\to\mathscr{L}(H_1,H_2);\quad K_{21}\colon \Gamma_1^{-1}\to\mathscr{L}(H_2,H_1);\quad K_{22}\colon \Gamma\to\mathscr{L}(H_2).$$

We say that K is positive definite when

$$\sum_{j,k=1,2}\sum_{s,t\in\Gamma}\langle K_{jk}(t^{-1}s)h_j(s),\ h_k(t)\rangle_{H_k}\geqq 0$$

holds for every pair of functions of finite support $h_1\colon \Gamma_1\to H_1$, $h_2\colon \Gamma_1^{-1}\to H_2$.

When $\Gamma_1=\Gamma$ we have a matricial Toeplitz kernel.

Before, in [5], the vectorial case was considered, and in [2] the subject was related to the dilation of a commutant of two contractions. Here we shall consider the general relation between GTK and lifting properties.

We start extending Proposition (I.1).

Proposition 1. *For $j=1,2$ let H_j be a Hilbert space and K_j an $\mathscr{L}(H_j)$-valued positive definite Toeplitz kernel on an abelian group Γ, such that K_j equals the identity on the neutral element e of Γ; call U_j the minimal unitary dilation of K_j to a Hilbert space F_j. Let $\Gamma_1\subset\Gamma$ be a semigroup such that $e\in\Gamma_1$ and every $u\in\Gamma$ can be written as $u=t-s$, $t, s\in\Gamma_1$. Set:*

$$E_+ := \bigvee_{s\in\Gamma_1}[U_1(s)H_1]\subset F_1,\quad E_- := \bigvee_{t\in\Gamma_1}[U_2(-t)H_2]\subset F_2.$$

Then the formula

$$\text{(1a)}\qquad k(s) = P^{E_-}_{H_2}YU_1(s)|_{H_1},\quad s\in\Gamma_1$$

gives a bijection between the operators $Y\in\mathscr{L}(E_+, E_-)$ that satisfy

$$\text{(1b)}\qquad YU_1(s)i^{F_1}_{E_+} = P^{F_2}_{E_-}U_2(s)Y,\quad for\quad s\in\Gamma_1,\quad \|Y\| = 1,$$

and the functions k such that $K=\{(K_{jk}), k=1,2; H_1, H_2\}$, given by $K_{11}=K_1$, $K_{12}=k$, $K_{21}=\tilde{k}$, $K_{22}=K_2$, is a p.d. GTK on (Γ,Γ_1). Set

$$L(Y) = \{W\in\mathscr{L}(F_1, F_2)\colon\ W\in R(U_1, U_2),\ \|W\|\leqq 1,\ P^{F_2}_{E_-}W|_{E_+}=Y\}.$$

Then

$$\text{(1c)}\qquad g(s) = P^{F_2}_{H_2}WU_1(s)|_{H_1},\quad s\in\Gamma,$$

gives a bijection between $L(Y)$ and the set of functions $g\colon\Gamma\to\mathscr{L}(H_1, H_2)$ such that $G_{11}=K_{11}$, $G_{12}=g$, $G_{21}=\tilde{g}$, $G_{22}=K_{22}$ defines an element $G=(G_{jk})^2_{j,k=1}$ of the class $\mathscr{G}(K)$ of the p.d. matricial Toeplitz kernels that extend K. In particular, $L(Y)$ is non void if and only if $\mathscr{G}(K)$ is non void.

Proof. Assume first that (1b) holds; then K satisfies the following equations for every h_1, h_2 as in the definition of p.d. GTK:

$$\sum_{j,k=1,2}\sum_{s,t\in\Gamma}\langle K_{jk}(s-t)h_j(s),\ h_k(t)\rangle_{H_k} = \sum_{s,t\in\Gamma}\{\langle U_1(s)h_1(s),\ U_1(t)h_1(t)\rangle_{E_+}+$$
$$+2\,\mathrm{Re}\,\langle YU_1(s)h_1(s),\ U_2(t)h_2(t)\rangle_{E_-}+\langle U_2(s)h_2(s),\ U_2(t)h_2(t)\rangle_{E_-}\} =$$
$$= \Big\|\sum_{s\in\Gamma}U_1(s)h_1(s)\Big\|^2_{E_+}+2\,\mathrm{Re}\Big\langle Y\sum_{s\in\Gamma}U_1(s)h_1(s),\ \sum_{t\in\Gamma}U_2(t)h_2(t)\Big\rangle_{E_-}+\Big\|\sum_{t\in\Gamma}U_2(t)h_2(t)\Big\|^2_{E_-},$$

which is a non-negative real number because $\|Y\|\leqq 1$; thus, K is positive definite.

Conversely, if the last is assumed, set for all h_1, h_2 as above

$$D(h_1, h_2) = \sum \{\langle K_{12}(s-t)h_1(s), h_2(t)\rangle_{H_2}\colon s\in\Gamma_1,\ t\in\Gamma_1^{-1}\}.$$

Then D defines a sesquilinear form on $E_+\times E_-$ such that $\|D\|\leqq 1$. So there exists $Y\in\mathcal{L}(E_+, E_-)$ which satisfies $\|Y\|\leqq 1$ and $\langle Ya, b\rangle_{E_-}=D(a, b)$ for all $(a, b)\in E_+\times\times E_-$. The proof of Naimark's dilation theorem shows that we may assume $U_j(s)h_j(t)=h_j(t-s)$ to be always true. Thus

$$\langle YU_1(u)h_1, h_2\rangle_{E_-} = \sum \{\langle K_{12}(s-t)h_1(s-u), h_2(t)\rangle_{H_2}\colon s\in\Gamma_1,\ t\in\Gamma_1^{-1}\} =$$
$$= \langle Yh_1, U_2(-u)h_2\rangle_{E_-} = \langle P^{F_2}_{E_-}U_2(u)Yh_1, h_2\rangle_{E_-}.$$

From the definitions of E_+ and E_- it follows that (1b) holds. Our first assertion is proved.

Now let $W\in L(Y)$. From (I.1) we know that G is p.d. For any $s\in\Gamma_1$, $x_1\in H_1$, $x_2\in H_2$ we have that

$$\langle G_{12}(s)x_1, x_2\rangle_{H_2} = \langle P^{F_2}_{H_2}WU_1(s)x_1, x_2\rangle_{H_2} = \langle P^{E_-}_{H_2}(P^{F_2}_{E_-}|_{E_+})U_1(s)x_1, x_2\rangle_{H_2} =$$
$$= \langle P^{E_-}_{H_2}YU_1(s)x_1, x_2\rangle_{H_2} = \langle K_{12}(s)x_1, x_2\rangle_{H_2},$$

so $G\in\mathcal{G}(K)$. If we start by assuming this, we know that (1c) defines a contraction $W\in\mathcal{R}(U_1, U_2)$. For all $x_1\in H_1$, $x_2\in H_2$, $s\in\Gamma_1$, $t\in\Gamma_1^{-1}$ we have:

$$\langle P^{F_2}_{E_-}WU_1(s)x_1, U_2(t)x_2\rangle_{E_-} = \langle WU_1(s)x_1, U_2(t)x_2\rangle_{F_2} = \langle WU_1(s-t)x_1, x_2\rangle_{F_2} =$$
$$= \langle P^{F_2}_{H_2}WU_1(s-t)x_1, x_2\rangle_{H_2} = \langle g(s-t)x_1, x_2\rangle_{H_2} = \langle k(s-t)x_1, x_2\rangle_{H_2} =$$
$$= \langle P^{E_-}_{H_2}YU_1(s-t)x_1, x_2\rangle_{H_2} = \langle P^{E_-}_{H_2}U_2(-t)YU_1(s)x_1, x_2\rangle_{H_2} = \langle YU_1(s)x_1, U_2(t)x_2\rangle_{E_-}.$$

Thus $P^{F_2}_{E_-}W|_{E_+} = Y$.

If K is a p.d. GTK, the (possibly void) set $\mathcal{G}(K)$ is naturally related with the set $\mathcal{U}(K)$ of the minimal unitary dilations of K, i.e., of the unitary representations U of Γ on a Hilbert space F such that:

$$H_1, H_2\subset F;\ F = \bigvee_{j=1,2}\{\bigvee_{s\in\Gamma}[U(s)H_j]\};$$

$$K_{jk}(s-t) = P^F_{H_k}(s-t)|_{H_j},\quad \text{for}\quad (s, t)\in\Gamma_j\times\Gamma_k,\quad j, k = 1, 2,\quad \text{with}\quad \Gamma_2 = \Gamma_1^{-1}.$$

In fact, if $U\in\mathcal{U}(K)$, $G_{jk}(s)=P^F_{H_k}U(s)|_{H_j}$, for $s\in\Gamma$, defines an element $G=(G_{jk})^2_{j,k=1}$ of $\mathcal{G}(K)$. Conversely, if $G\in\mathcal{G}(K)$, its minimal unitary dilation U satisfies the conditions required to belong to $\mathcal{U}(K)$ and, by its very definition, is related to G by the last equality. Moreover, this correspondence between U and G is a bijection if we identify in $\mathcal{U}(K)$ the representations that are equivalent under unitary isomorphisms that leave invariant all the elements of H_1 and H_2. Thus, if $\mathcal{G}(K)$ non void for every positive definite generalized Toeplitz kernel K on (Γ, Γ_1), it follows that

Naimark's dilation theorem extends to these kernels. In such a case we could say that (Γ, Γ_1) has *Naimark's property.*

When $\Gamma = Z$, $\Gamma_1 = Z_1$ (1b) reduces to $YV_+ = V_-^* Y$, with V_+, V_- isometries. It is known that $L(Y)$ and $\mathcal{G}(K)$ are both non void and these two facts have been proved independently. Because of (1) each of them can be deduced from the other one. In fact, (Z, Z_1) has Naimark's property [5]. On the other side the lifting of $YV_+ = V_-^* Y$ to a commutator of isometries can be obtained as a particular case of the theorem of Sz.-Nagy and Foiaş. More precisely, this theorem is based on a previous result ([9], Proposition II.2.2) which implies that, if V' is a minimal isometric dilation of V_-^* to $E' \supset E_-$, then there exists $Y' \in \mathcal{L}(E_+, E')$ such that $Y'V_+ = V'Y'$, $Y = P_{E_-}^{E'} Y'$ and $\|Y'\| \leq 1$. Now, it is well known that every commutator of isometries can be lifted to a commutator of their minimal unitary extensions (this has also been proved in the previous section); if U_2^* is a minimal unitary dilation of V_-, U_2 has the same property with respect to V_-^* and V'; it follows that there exists $W \in L(Y)$, so that this set is non void. In particular, this gives another proof of the fact that $\mathcal{G}(K)$ is non void (which is certainly less simple than the original one presented in [5]).

Now we can state the relation between GTK and commutators of semigroups of contractions by means of the following extension of proposition (I.2).

Proposition 2. *Let Γ be an Abelian group with neutral element e and Γ_1 a sub-semigroup of Γ. Set $(-\Gamma_1) = \{s \in \Gamma : -s \in \Gamma_1\}$ and assume that $\Gamma_1 \cap (-\Gamma_1) = \{e\}$ and $\Gamma_1 \cup (-\Gamma_1) = \Gamma$. Let $\{T_1(s) : s \in \Gamma_1\}$ and $\{T_2(s) : s \in \Gamma_1\}$ be two semigroups of contractions in the Hilbert spaces H_1 and H_2, respectively. Let $X \in \mathcal{L}(H_1, H_2)$ be such that:*

(2a) $$XT_1(s) = T_2(s)X, \quad \textit{for} \quad s \in \Gamma_1, \quad \textit{and} \quad \|X\| = 1.$$

Let the GTK $K = \{(K_{jk}), j, k = 1, 2;\ H_1, H_2\}$ associated with the commutator (2a) *be defined by*

(2b) $$K_{jj}(s) = T_j(s) \ \textit{if} \ s \in \Gamma_1, \quad K_{jj}(s) = T_j^*(-s) \ \textit{if} \ s \in (-\Gamma_1), \quad j = 1, 2;$$
$$K_{12}(s) = T_2(s)X \ \textit{for} \ s \in \Gamma_1; \quad K_{21} = \tilde{K}_{12}.$$

Then K is p.d. and $\mathcal{G}(K)$ is non void if and only if the following conditions hold:

(2c) *for $j = 1, 2$ there exists a unitary representation U_j of Γ in a Hilbert space F_j that contains H_j and satisfies*

$$T_j(s) = P_{H_j}^{F_j} U_j(s)|_{H_j}, \quad \textit{for} \quad s \in \Gamma_1,\ F_j = \bigvee_{s \in \Gamma} [U_j(s)H_j];$$

(2d) *there exists $W \in \mathcal{L}(F_1, F_2)$ that satisfies $WU_1(s) = U_2(s)W$ for all*

$$s \in \Gamma,\ \|W\| = \|X\| \ \text{and} \ P_{H_2}^{F_2} W|_{E_1} = XP_{H_1}^{E_1}, \quad \textit{where} \quad E_1 := \bigvee_{s \in \Gamma_1} [U_1(s)H_1].$$

Moreover, if these conditions are satisfied, (1c) *gives as in Proposition* 1 *a bijection between $\mathcal{G}(K)$ and the set of operators W as in* (2d).

Proof. We start assuming K p.d. and $\mathcal{G}(K)$ non void; then K_{11} and K_{22} are also p.d. and (2c) follows from Naimark's dilation theorem. For a given $G\in\mathcal{G}(K)$, (I.1) shows that there exists a contraction $W\in R(U_1, U_2)$ such that $G_{12}(s)=$ $=P^{F_2}_{H_2}WU_1(s)|_{H_1}$ holds for all $s\in\Gamma$. Then, for $x\in H_1$, $s\in\Gamma_1$, we have $P^{F_2}_{H_2}WU_1(s)x=$ $=G_{12}(s)x=K_{12}(s)x=XT_1(s)x=XP^{F_1}_{H_1}U_1(s)x=XP^{E_1}_{H_1}U_1(s)x$ and (2d) follows. Assume conversely that (2c) and (2d) are true. First of all, it is easy to see that U_j is a minimal unitary dilation of K_{jj}; thus the last is positive definite and U_j is essentially unique. Let $G=(G_{jk})^2_{j,k=1}$ be the matricial Toeplitz kernel associated with the commutator (2d); then $G_{jj}=K_{jj}$, $j=1, 2$, and $G_{12}(s)=\tilde{G}_{21}(s)=P^{F_2}_{H_2}WU_1(s)|_{H_1}$, for $s\in\Gamma$. We know that G is p.d.; moreover, for $s\in\Gamma_1$, $U_1(s)H_1\subset E_1$ so we have $K_{12}(s)=T_2(s)X=XT_1(s)=XP^{E_1}_{H_1}U_1(s)|_{H_1}=P^{F_2}_{H_2}WU_1(s)|_{H_1}=G_{12}(s)$. Thus $G\in\mathcal{G}(K)$.

In the next section what has been done up to now will be applied to commutators of continuous monoparametric semigroups of contractions. Here, as an example, we shall recall and complete some results of [2]. The following holds.

Let T_1 and T_2 be contractions in Hilbert spaces H_1 and H_2, respectively, and $X\in\mathcal{L}(H_1, H_2)$ such that $XT_1=T_2X$. Let $V_1\in\mathcal{L}(E_1)$, $V_2\in\mathcal{L}(E_2)$ be the minimal isometric dilations and $U_1\in\mathcal{L}(F_1)$, $U_2\in\mathcal{L}(F_2)$ the minimal unitary dilations of T_1, T_2, respectively. The following two problems are considered:

i) find $Y\in\mathcal{L}(E_1, E_2)$ such that $YV_1=V_2Y$, $P^{E_2}_{H_2}Y=XP^{E_1}_{H_1}$, $\|Y\|=\|X\|$;

ii) find $W\in\mathcal{L}(F_1, F_2)$ such that $WU_1=U_2W$, $P^{F_2}_{H_2}W|_{E_1}=XP^{E_1}_{H_1}$, $\|W\|=\|X\|$.

If $X=0$ both problems have only the trivial solution, so it is also assumed that $\|X\|=1$.

Let $K=\{(K_{jk}), j, k=1, 2; H_1, H_2\}$ be the GTK on (Z, Z_1) given by $K_{jj}(n)=T_j^n$ if $n\geq 0$, $K_{jj}(n)=T_j^{*-n}$ if $n\leq 0$, $j=1, 2$; $K_{12}(n)=XT_1^n$ for $n\geq 0$ and $K_{21}=\tilde{K}_{12}$.

Theorem 3.

a) *Both problems have solutions.*

b) *K is positive definite.*

c) *There is a bijection between the sets of solutions of these problems and with the set $\mathcal{G}(K)$ of all the positive definite matricial Toeplitz kernels that extend K.*

d) *This bijection can be obtained as follows: given $G\in\mathcal{G}(K)$, let $F=F_1\vee F_2$ be the space of the minimal dilation of G; then set $W=P^F_{F_2}|_{F_1}$, solution of* (i), *and $Y=P^F_{E_2}|_{E_1}=P^{F_2}_{E_2}W|_{E_1}$, solution of* (ii).

e) *The solution of these problems is unique if and only if one of the following equalities is satisfied:*

$$\{(I-X^*X)^{1/2}H_1\}^-\oplus\{(U_1-T_1)H_1\}^-=\{(I-X^*X)^{1/2}T_1h+(U_1-T_1)h:\ h\in H_1\}^-,$$

$$\{(I-X^*X)^{1/2}H_1\}^-\oplus\{(U_2-T_2)H_2\}^-=\{(I-X^*X)^{1/2}h\oplus(U_2-T_2)Xh:\ h\in H_1\}^-.$$

Proof.

a) Follows from (c) and (b) which imply $\mathscr{G}(K)\neq\emptyset$;

b) was proved in [2], Proposition II.1;

c)—d) the assertions concerning Problem (ii) stem from Proposition 2; those concerning Problem (i), from [2], Theorem II.4;

e) follows from (c) and the theorem on the uniqueness of the lifting [1]; also, because $\mathscr{G}(K)$ has only one element if and only if one of these equalities is satisfied ([2], Theorem II.8).

The proof is done.

Remark. The above theorem includes the following result (see [8]) for T_1, T_2, X, U_1, U_2 as before there exists $W\in\mathscr{L}(F_1, F_2)$ such that $WU_1=U_2W$, $\|X\|=\|W\|$ and $X=P^{F_2}_{H_2}W|_{H_1}$ hold.

III. The continuous case

Our task in this section is to show that the results for the discrete case can be extended to the continuous one also. Specifically, we shall show that Proposition II.2 gives positive results when $\Gamma=R$, the set of real numbers, and $\Gamma_1=R_1=\{s\in R\colon s\geqq 0\}$.

Following our general approach we shall first see that (R, R_1) has Naimark's property; in other words, we shall state the dilation theorem for continuous operator-valued GTK, proofs of which were given in [6] and [7] for the scalar case. That result will then be applied to the commutator of two continuous semigroups of contractions.

Our method will be to relate each GTK on (R, R_1) with another on (Z, Z_1) by means of a systematic use of the results concerning semigroups, their dilations and cogenerators, of Sz.-Nagy and Foiaş ([9], Sections III.8 and III.9).

We start with a p.d. GTK on (R, R_1), $K=\{(K_{jk}), j, k=1, 2;\ H_1, H_2\}$, such that the K_{jk} are weakly continuous functions. We keep the notation of the preceding section, in particular that of Propositions II.1 and II.2. Let U_1' and U_2' be the cogenerators of U_1 and U_2, minimal unitary dilations of K_{11} and K_{22}, respectively. It is known that U_1' and U_2' are unitary operators and that the following holds ([9], Theorem III.8.1):

1) $U_j'=\text{strong}\lim\limits_{s\to 0+}\Phi_s[U_j(s)]$, $j=1, 2$, where Φ_s is the holomorphic function in the complex plane minus the point $(1+s)$ given by $\Phi_s(z)=(z-1+s)/(z-1-s)$, for $s\in R_1$.

What follows is based in the next equalities ([9], III.9.6, III.9.10).

2) $F_j \equiv \bigvee_{s\in R} [U_j(s)H_j] = \bigvee_{n\in Z} [U_j'^n H_j], \quad j = 1, 2;$

$E_+ \equiv \bigvee_{s\in R_1} [U_1(s)H_1] = \bigvee_{n\in Z_1} [U_1'^n H_1],$

$E_- \equiv \bigvee_{-s\in R_1} [U_2(s)H_2] = \bigvee_{-n\in Z_1} [U_2'^n H_2].$

As we said, a GTK on (Z, Z_1), $K' = \{(K'_{jk}), j, k=1, 2;\ H_1, H_2\}$ will be associated with K. We start defining K'_{11} and K'_{22} in the natural way:

3) $K'_{jj}(m) := P^{F_j}_{H_j} U_j'^m|_{H_j}, \quad \text{for} \quad m\in Z, \quad j = 1, 2.$

Then, from (1), we get that

3a) $K'_{jj}(m) = \underset{s\to 0^+}{\text{strong limit}}\, P^{F_j}_{H_j} \Phi_s^{|m|} [U_j(s\cdot \operatorname{sign} m)]|_{H_j}, \quad j = 1, 2.$

Let D be the sesquilinear form on $E_+ \times E_-$ determined by

4) $D(U_1(s)h_1, U_2(t)h_2) = \langle K_{12}(s-t)h_1, h_2\rangle_{H_2}, \quad s \geqq 0, \quad t \leqq 0, \quad h_1\in H_1,$
$h_2\in H_2.$

From the very definition we get the identity

4a) $D(U_1(s-t)h_1, h_2) = D(U_1(s)h_1, U_2(t)h_2) = D(h_1, U_2(t-s)h_2).$

We want to prove the corresponding result for the discrete case, that is,

4b) $D(U_1'^{m-n}h_1, h_2) = D(U_1'^m h_1, U_2'^n h_2) = D(h_1, U_2'^{n-m} h_2)$

for all $m\geqq 0$, $n\leqq 0$, $h_1\in H_1$, $h_2\in H_2$. In order to do that we refer to the identity ([9], III.9.9) and to the one we obtain from it by conjugation. They imply, respectively,

4c) $U_1'^m h_1 = \underset{s\to 0^+}{\text{limit}} \sum_{k=0}^{\infty} d_k(s, m) U_1(ks)h_1, \quad m \geqq 0, \quad h_1\in H_1,$

and

4d) $U_2'^n h_2 = \underset{-s\to 0^+}{\text{limit}} \sum_{k=0}^{\infty} d_k(-s, -n) U_2(ks)h_2, \quad n \leqq 0, \quad h_2\in H_2,$

where, for $s\in R_1$ and $m\in Z_1$, $\{d_k(s, m)\}_{k=0}^{\infty}$ are the coefficients of the Taylor series of the function Φ_s^m. Since K is positive definite, D is bounded and consequently the following hold:

4e) $D(U_1'^{m-n}h_1, h_2) = \underset{s\to 0^+}{\text{limit}} \sum_{k=0}^{\infty} d_k(s, m-n) D(U_1(ks)h_1, h_2)$,

$$D(U_1'^m h_1, U_2'^n h_2) = \underset{s\to 0^+}{\text{limit}} \sum_{k,j=0}^{\infty} d_k(s, m)\, \overline{d_j(s, -n)}\, D(U_1(ks)h_1, U_2(-js)h_2) =$$

$$= \underset{s\to 0^+}{\text{limit}} \sum_{v=0}^{\infty} \Big[\sum_{j=0}^{v} d_{v-j}(s, m)\, d_j(s, -n)\Big] D(U_1(vs)h_1, h_2).$$

In order to prove the last, recall (4a), set $v=k+j$ and remark that the $d_k(s, m)$ are real numbers. Then the first equality (4b) stems from

$$d_k(s, m+n) = \sum_{j=0}^{k} d_{k-j}(s, m)\, d_j(s, n), \quad s\in R_1, \quad m, n, k\in Z_1,$$

which is a consequence of $\Phi_s^m \Phi_s^n \equiv \Phi_s^{m+n}$. The second equality (4b) can be proved in the same way.

We now complete the definition of K' by setting

5) $\langle K_{12}'(m)h_1, h_2\rangle_{H_2} = D(U_1'^m h_1, h_2), \quad m\in Z_1, \quad h_1\in H_1, \quad h_2\in H_2; \; K_{21}' = \tilde{K}_{12}'$.

From (3a), (4) and (4e) we get the following direct formulas for K' in terms of K.

6) $K_{jj}'(m) = \underset{b\to 0^+}{\text{strong limit}} \sum_{k=0}^{\infty} d_k(s, |m|) K_{jj}[(\text{sign}\, m)ks], \quad m\in Z, \quad j=1, 2;$

$$K_{12}'(m) = \underset{s\to 0^+}{\text{strong limit}} \sum_{k=0}^{\infty} d_k(s, m) K_{12}(ks), \quad m\in Z_1.$$

We shall see that K' is p.d. Set $Z_2 = -Z_1$ and let $f_j\colon Z_j \to H_j$ be functions with finite support, $j=1, 2$. From definitions (3) and (5), and the identity (4b) it follows that

$$\sum_{j,k=1,2} \sum_{(m,n)\in Z_j\times Z_k} \langle K_{jk}'(m-n) f_j(m), f_k(n)\rangle_{H_k} = \Big\|\sum_{m\in Z_1} U_1'^m f_1(m)\Big\|_{F_1}^2 +$$

$$+ 2\,\mathrm{Re}\, D\Big(\sum_{m\in Z_1} U_1'^m f_1(m), \sum_{n\in Z_2} U_2'^n f_2(n)\Big) + \Big\|\sum_{n\in Z_2} U_2'^n f_2(n)\Big\|_{F_2}^2 \geqq 0,$$

because K p.d. implies $\|D\| \leqq 1$.

Consequently, $\mathscr{U}(K') \approx \mathscr{G}(K') \neq \emptyset$. With each $G'\in\mathscr{G}(K')$ we shall associate a $G\in\mathscr{G}(K)$, getting in that way a bijection and, in particular, proving that $\mathscr{G}(K)$ is non void. In order to do that we refer once more to the relation between matricial Toeplitz kernels and intertwining operators. Given $G' = (G_{jk}')_{j,k=1}^2 \in \mathscr{G}(K')$ let $T_{G'}\in\mathscr{L}(F_1, F_2)$ be the operator determined by

7) $\langle T_{G'} U_1'^m h_1, U_2'^n h_2\rangle_{F_2} = \langle G_{12}'(m-n)h_1, h_2\rangle_{H_2}, \quad m, n\in Z, \quad h_1\in H_1, \quad h_2\in H_2.$

As we know,

7a) $T_{G'}U'_1 = U'_2 T_{G'}, \quad \|T_{G'}\| \leqq 1,$

and it is clear that

7b) $G'_{12}(m) = P^{F_2}_{H_2} T_{G'} U'^m_1|_{H_1} = P^{F_2}_{H_2} U'^m_2 T_{G'}|_{H_1}, \quad \text{for} \quad m \in Z.$

We shall show that $T_{G'}$ also intertwines U_1 and U_2:

7c) $T_{G'} U_1(s) = U_2(s) T_{G'}, \quad \text{for} \quad s \in R.$

It is enough to see it for all $s>0$; in order to do that we refer to a reciprocal ([9], III.9.8) of a formula we have already used; it implies that

8) $$U_j(s)h_j = \operatorname*{limit}_{r\to 1^-} \sum_{k=0}^{\infty} r^k c_k(s) U'^k_j h_j, \quad \text{for} \quad s>0, \quad h_j \in H_j, \quad j=1,2,$$

where $\{c_k(s)\}_{k=0}^{\infty}$ are the Taylor coefficients of the function $e_s(z) = \exp\left(\frac{z+1}{z-1}\right)$, $s>0$. From (8) it follows that $T_{G'} U_1(s) h_1 = \operatorname*{limit}_{r\to 1^-} \sum_{k=0}^{\infty} r^k c_k(s) U'^k_2 T_{G'} h_1 = U_2(s) T_{G'} h_1$, so (7c) is proved.

Let $G_{12}\colon R \to \mathscr{L}(H_1, H_2)$ be given by

9) $\langle G_{12}(s) h_1, h_2 \rangle_{H_2} = \langle T_{G'} U_1(s) h_1, h_2 \rangle_{F_2}, \quad s \in R, \quad h_1 \in H_1, \quad h_2 \in H_2.$

Setting $G_{11} := K_{11}, G_{12}, G_{21} := \tilde{G}_{12}, G_{22} := K_{22}$ we define a p.d. matricial Toeplitz kernel G. It only remains to see that G extends K. Since

$$D(U'^k_1 h_1, h_2) = \langle K'_{12}(k) h_1, h_2 \rangle_{H_2} = \langle G'_{12}(k) h_1, h_2 \rangle_{H_2} = \langle T_{G'} U'^k_1 h_1, h_2 \rangle_{F_2},$$

it follows from (8) that, for $s>0$, we have

$$\langle K_{12}(s) h_1, h_2 \rangle_{H_2} = D\big(U_1(s) h_1, h_2\big) = \big\langle T_{G'} [\operatorname*{limit}_{r\to 1^-} \sum_{k=0}^{\infty} r^k c_k(s) U'^k_1 h_1], h_2 \big\rangle_{F_2} =$$

$$= \langle T_{G'} U_1(s) h_1, h_2 \rangle_{F_2} = \langle G_{12}(s) h_1, h_2 \rangle_{H_2}.$$

Thus, $G \in \mathscr{G}(K)$. Also, it follows from (9), (8) and (7b) that

10) $$G_{12}(s) = \operatorname*{strong\,limit}_{r\to 1^-} \sum_{k=0}^{\infty} r^k c_k(|s|) G'_{12}[k(\operatorname{sign} s)], \quad \text{for} \quad s \in R.$$

Conversely:

10a) $$G'_{12}(m) = \operatorname*{strong\,limit}_{s\to 0^+} \sum_{k=0}^{\infty} d_k(s, |m|) G_{12}[ks(\operatorname{sign} m)], \quad \text{for} \quad m \in Z.$$

So we have proved the following

Theorem 11. *Let* $K=\{(K_{jk}),\ j,k=1,2;\ H_1, H_2\}$ *be a positive definite generalized Toeplitz kernel on* (R, R_1) *such that the functions* K_{jk} *are continuous in the weak topology of operators. Then there exists a p.d. GTK* K' *on* (Z, Z_1), *such that there is a bijection between* $\mathcal{G}(K)$ *and* $\mathcal{G}(K')$.

The correspondence $K\to K'$ *given by this theorem is reversible; the converse of formula* (6) *is the following:*

$$\text{11a)}\quad K_{jj}(s) = \operatorname*{strong\,limit}_{r\to 1^-} \sum_{k=0}^{\infty} r^k c_k(|s|) K'_{jj}[(\operatorname{sign} s)k], \quad s\in R, \quad j=1,2;$$

$$K_{12}(s) = \operatorname*{strong\,limit}_{r\to 1^-} \sum_{k=0}^{\infty} r^k c_k(s) K'_{12}(k), \quad s\in R_1.$$

Theorem (11) allows us to transfer a uniqueness condition from the discrete case ([2], Proposition I.6) to the continuous one.

Corollary 12. *Let* K *be as in theorem* (11). *Then* $\mathcal{G}(K)$ *contains only one element if and only if at least one of the following two conditions is satisfied:*

i) $\{(I-Q^*Q)^{1/2}E\}^- = \{(I-Q^*Q)^{1/2}U_1'E_+\}^-$,

ii) $\{(I-QQ^*)^{1/2}E_-\}^- = \{(I-QQ^*)^{1/2}U_2'E_-\}^-$,

where U_1', U_2' *are the cogenerators of the minimal unitary dilations of* K_{11}, K_{22}, *respectively, and* Q *is the operator from*

$$E_+ = \bigvee_{s\geqq 0} U_1(s)H_1 \quad \textit{to} \quad E_- = \bigvee_{s\leqq 0} U_2(s)H_2,$$

given for $s\geqq 0$, $t\leqq 0$, $h_1\in H_1$, $h_2\in H_2$ *by*

$$\langle QU_1(s)h_1, U_2(t)h_2\rangle_{E_-} = \langle K_{12}(s+t)h_1, h_2\rangle_{H_2}.$$

As an application of what has been done in this section we shall state a theorem on the commutator of two semigroups of contractions which is the continuous version of (II.3).

Theorem 13. *Let* $\{T_1(s)\colon s\geqq 0\}$, $\{T_2(s)\colon s\geqq 0\}$ *be continuous monoparametric semigroups of contraction an Hilbert spaces* H_1, H_2, *respectively, and* $X\in\mathcal{L}(H_1, H_2)$ *such that* $XT_1(s)=T_2(s)X$ *holds for all* $s\geqq 0$. *Let* $\{V_1(s)\colon s\geqq 0\}\subset\mathcal{L}(E_1)$, $\{V_2(s)\colon s\geqq 0\}\subset\mathcal{L}(E_2)$ be *minimal isometric dilations and* $\{U_1(s)\colon s\in R\}\subset\mathcal{L}(F_1)$, $\{U_2(s)\colon s\in R\}\subset\mathcal{L}(F_2)$ *minimal unitary dilations of the semigroups* T_1, T_2, *respectively. The following problems are considered:*

i) *find* $Y\in\mathcal{L}(E_1, E_2)$ *such that* $YV_1(s)=V_2(s)Y$, *for* $s\geqq 0$, $P^{E_2}_{H_2}Y=XP^{E_1}_{H_1}$ *and* $\|Y\|=\|X\|$;

ii) *find* $W\in\mathcal{L}(F_1, F_2)$ *such that* $WU_1(s)=U_2(s)W$, *for* $s\in R$, $P^{F_2}_{H_2}W|_{E_1}=XP^{E_1}_{H_1}$ *and* $\|W\|=\|X\|$.

Discarding the trivial case, it may be assumed by homogeneity that $\|X\|=1$. *Let* $K=\{(K_{jk}), j, k=1,2; H_1, H_2\}$ *be the GTK on* (R, R_1) *given by*

$$K_{jj}(s)=T_j(s) \quad \text{if} \quad s \geqq 0, \quad K_{jj}(s)=T_j^*(-s) \quad \text{if} \quad s \leqq 0, \quad j=1,2;$$

$K_{12}(s)=T_2(s)X$ *if* $s \geqq 0$, $K_{21}=\tilde{K}_{12}$. *Then:*

a) *Both problems have solutions.*

b) K *is positive definite.*

c) *There exist bijections between the set of solutions of* (i), *the one of* (ii) *and* $\mathcal{G}(K)$, *and these bijections are determined by*

$$\langle G_{12}(s-t)h_1, h_2\rangle_{H_2} = \begin{cases} \langle WU_1(s)h_1, U_2(t)h_2\rangle_{F_2}, & \text{for} \quad s, t \in R, \quad h_1 \in H_1, \quad h_2 \in H_2 \\ \langle YV_1(s)h_1, V_2(t)h_2\rangle_{E_2}, & \text{for} \quad s, t \geqq 0, \quad h_1 \in H_1, \quad h_2 \in H_2. \end{cases}$$

d) *The solution of both problems is unique if and only if at least one of the following equalities is satisfied:*

$$\{(I-X^*X)^{1/2}H_1\}^- \oplus \{(U_1'-T_1')H_1\}^- = \{(I-X^*X)^{1/2}T_1'h+(U_1'-T_1')h\colon h \in H_1\}^-,$$

$$\{(I-X^*X)^{1/2}H_1\}^- \oplus \{(U_2'-T_2')H_2\}^- = \{(I-X^*X)^{1/2}h \oplus (U_2'-T_2')Xh\colon h \in H_1\}^-,$$

where U_1', U_2', T_1', T_2' *are the cogenerators of* U_1, U_2, T_1, T_2, *respectively.*

Proof. *First step:* some properties that we have already used ([9], Sections III.8 and III.9) show that U_j' (V_j') is the minimal unitary (isometric) dilation of T_j', where V_j' denotes the cogenerator of the semigroup V_j; moreover, $XT_1'=T_2'X$ holds; from $WU_1'=U_2'W$ it follows that $WU_1(s)=U_2(s)W$ for all $s \in R$, and from $YV_1'=V_2'Y$, that $YV_1(s)=V_2(s)Y$ for all $s \in R_1$.

Second step: apply Theorem II.3 to T_1', T_2', V_1', V_2', U_1', U_2', calling K' the GTK on (Z, Z_1) that in its statement is called K.

Third step: note that (11a) relates precisely the kernels K and K' we are considering here.

Fourth step: apply Theorem 11 of this section.

Remark. The applications of generalized and matricial Toeplitz kernels to the realization of linear systems and scattering theory are considered in [3] and [4].

Added in proofs. A more conceptual appraach to the concept of section III is given in [10].

References

[1] T. Ando, Z. Ceauşescu and C. Foiaş, On intertwining dilations. II, *Acta Sci. Math.*, **39** (1977), 3—14.

[2] R. Arocena, Generalized Toeplitz kernels and dilations of intertwining operators, *Integral Equations and Operator Theory*, **6** (1983), 759—788.

[3] R. Arocena, A theorem of Naimark, linear systems and scattering operators, *J. Funct. Anal.*, **69** (1986), 281—288.

[4] R. Arocena, Scattering functions, Fourier transforms of measures, realization of linear systems and dilations of operators to Kreĭn spaces, *Seminar on harmonic analysis* 1983—84, Publ. Math. Orsay, 85—2, Univ. Paris XI (1985), pp. 1—55.

[5] R. Arocena and M. Cotlar, Dilations of generalized Toeplitz kernels and some vectorial moment and weighted problems, *Harmonic analysis*, Lecture Notes in Math., Vol. 908, Springer (1982), pp. 169—188.

[6] R. Arocena and M. Cotlar, A generalized Herglotz—Bochner theorem and L^2-weighted inequalities, *Conference on Harmonic Analysis in honor of A. Zygmund* (Chicago, 1981), Wadsworth (1983), pp. 258—269.

[7] R. Arocena and M. Cotlar, Continuous generalized Toeplitz kernels in **R**, *Portugaliae Math.*, **39** (1980), 419—434.

[8] R. G. Douglas, P. S. Muhly and C. Pearcy, Lifting commuting operators, *Michigan Math. J.*, **15** (1968), 385—395.

[9] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert space*, North Holland (Amsterdam, 1970).

[10] R. Bruzual, Local semigroups of contractions and some applications to Fourier representation theorems, *Integral Equations and Operator Theory*, **10** (1987), 780—801.

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