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The point spectra for generalized Hausdorff operators

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It is the purpose of this paper to show that the point spectra of a large class of generalized Hausdorff matrices is empty. The generalized Hausdorff matrices under consideration were defined independently by ENDL [3] and JAKIMOVSKI [6]. Each matrix $H^{(\alpha)}$ is a lower triangular matrix with nonzero entries

$$h_{nk}^{(\alpha)} = \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k, \tag{1}$$

where $\{\mu_k\}$ is a real or complex sequence, and Δ is the forward difference operator defined by $\Delta\mu_k=\mu_k-\mu_{k+1}$, $\Delta^{n+1}\mu_k=\Delta(\Delta^n\mu_k)$. Let c denote the space of convergent sequences. The bounded linear operators on c and l^p, $1\leqq p\leqq\infty$, will be denoted by $B(c)$ and $B(l^p)$, respectively. Although (1) is defined for any real α which is not a negative integer, in this paper α is restricted to be nonnegative.

Let $1<p<\infty$, $H^{(\alpha)}\in B(l^p)$. The author showed in [8] that the point spectrum of $H^{(\alpha)*}$, the adjoint of $H^{(\alpha)}$, contains an open set. Let $C^{(\alpha)}$ denote the generalized Hausdorff matrix generated by $\mu_n=(n+\alpha+1)^{-1}$, q the conjugate index of p. It was also shown in [8] that the spectrum of $I-2C^{(\alpha)}/q$ is the closed unit disc. For $p=2$, every $H^{(\alpha)}\in B(l^p)\cap B(c)$ is an analytic function of $C^{(\alpha)}$, so the spectral mapping theorem can be used to obtain the spectrum. GHOSH, RHOADES and TRUTT [5] showed that each $H^{(\alpha)}\in B(l^2)$, for integer α, is subnormal. In [8] the author showed that each $C^{(\alpha)}$ is hyponormal.

In order to establish the point spectra results it will first be necessary to extend some results of FUCHS [4]. Define

$$S = S(a_1, a_2, \ldots) = \{\varphi_k(x)\} = (e^{-cx}x^{a_k}\colon\ c>0;\ k\geqq 1;\ a_1<a_2<\ldots\}. \tag{2}$$

The set S is closed in $L^2(0,\infty)$ if, for each $h\in L^2(0,\infty)$ and for each $\varepsilon>0$, there

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exists a finite linear combination $\Phi(x)$ of the functions φ_k such that

$$\int_0^\infty (h(x)-\Phi(x))^2\,dx < \varepsilon.$$

The set S is said to be complete in $L^2(0,\infty)$ if, for each $h\in L^2(0,\infty)$,

$$\int_0^\infty h(x)\varphi_k(x)\,dx = 0$$

for all $k\geqq 1$ implies $h(x)=0$ a.e. It is well known that the concepts of closed and complete are equivalent.

Theorem 1. *Let* $\{s_n\}\subset C$ *satisfy* $s_n=o(n^{M+\alpha})$, $M>0$, α *a nonnegative real number. Define* $\{t_n\}$ *by*

$$t_n = \sum_{i=0}^{n}\binom{n+\alpha}{n-i}(-1)^i s_i. \tag{3}$$

Then $t_n=0$ *for* $n=a_1, a_2, \ldots$ *implies* $s_n=\Gamma(n+\alpha+1)P(n)/n!$, P *a polynomial of degree less than* M *if and only if* $S=\{e^{-x/2}x^{a_n}\colon n=0,1,2,\ldots\}$ *is closed in* $L^2(0,\infty)$.

Suppose that $s_n=O(n^{M+\alpha})$, $t_n=0$ for $n=a_1, a_2, \ldots$ implies

$$s_n=\Gamma(n+\alpha+1)P(n)/n!,$$

where the degree of P is less than M.

We may write (3) in the form

$$t_n = \sum_{i=0}^{n}\binom{n+\alpha}{n-k}\frac{(-1)^k\Gamma(i+\alpha+1)P(i)}{i!} =$$

$$=\frac{\Gamma(n+\alpha+1)}{n!}\sum_{i=0}^{n}\binom{n}{i}(-1)^iP(i)=\frac{\Gamma(n+\alpha+1)}{n!}\Delta^nP(0).$$

Since the degree of P is less than n, $t_n=0$ for each $n\geqq[M]+1$, and the set S is closed.

To prove the converse we may assume, without loss of generality, that $\{s_n\}$ is real and that $|s_n|\leqq 1$ for $n<2M+2+s$, $|s_n|\leqq\left[\begin{matrix}n+\alpha\\ n-M\end{matrix}\right]$ for $n\geqq 2M+2+s$, $s=[\alpha]+1$, replacing s_n by some scalar multiple γs_n, if necessary.

Lemma 1 [1, p. 77]. *Let a_{nk}, b_n be real numbers, with* $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$. *Then the system of equations*

$$\sum_{k=0}^{\infty} a_{nk} x_k = b_n \quad (n = 0, 1, 2, \ldots)$$

has a solution satisfying $|x_n| \leqq 1$ *if and only if*

$$\Big|\sum_k \lambda_k b_k\Big| \leqq \sum_{n=0}^{\infty} \Big|\sum_k \lambda_k a_{kn}\Big|$$

for every finite set of real multipliers λ_k.

Lemma 2. *Let* $\{a_n: n=0, 1, 2, \ldots\}$ *be an increasing sequence of natural numbers,* $\{t_n\}$ *as in* (3). *Then* $t_n=0$ *for* $n=a_1, a_2, \ldots$ *implies* $t_{a_0}=0$ *if and only if*

$$\text{(4)} \qquad l \cdot \mathrm{bd} \cdot \left\{ \sum_{h=0}^{2M+1+s} \left| \sum_{k=0}^{N} \lambda_k \binom{a_k+\alpha}{a_k-h} \right| + \sum_{h \geqq 2M+2+s} \binom{h+\alpha}{h-M} \left| \sum_{k=0}^{N} \lambda_k \binom{a_k-\alpha}{a_k-h} \right| \right\} = 0$$

where $s=[\alpha]+1$, $\lambda_0=1$ *and the* λ_k *for* $k>0$ *run through all sets of real numbers for* $N=1, 2, \ldots$.

Proof of Lemma 2. Consider $t_n=0$ for $n=a_1, a_2, \ldots, t_{a_0}=\gamma>0$ as a system of equations for the unknowns x_n, where $x_n=s_n$ for $n<2M+2+s$, $x_n=\begin{bmatrix} n+\alpha \\ n-M \end{bmatrix}^{-1} s_n$ for $n \geqq 2M+2+s$. From Lemma 1 this system has a solution for $|x_n| \leqq 1$ if and only if the left side of (4) is $\geqq \gamma$. Therefore (4) implies that $\gamma=0$.

Conversely, if $\gamma=0$, then (4) is nonnegative for every choice of the λ_n. But the choice $\lambda_k=0$ for $k>0$ gives the lower bound.

To complete the proof of Theorem 1, we shall show that the condition that S be closed is equivalent to (4). Let the set S in (2) be closed and $a_0 \geqq 2M+2+s$. We shall show that (4) is satisfied.

$$\text{(5)} \qquad \sum_{h \geqq 2M+2+s} \binom{h+\alpha}{h-M} \left| \sum_{k=0}^{N} \lambda_k \binom{a_k+\alpha}{a_k-h} \right| \leqq \sum_{h \geqq 2M+2s+2} \binom{h+s}{h-M} \sum_{k=0}^{N} |\lambda_k| \binom{a_k+s}{a_k-h} \leqq$$

$$\leqq \left\{ \sum_h \binom{h+s}{h-M}^2 \binom{h+s}{2M+2s+2}^{-1} \right\}^{1/2} \left\{ \sum_h \binom{h+s}{2M+2s+2} \left(\sum_{k=0}^{N} |\lambda_k| \binom{a_k+s}{a_k-h} \right)^2 \right\}^{1/2} \leqq$$

$$\leqq A \left\{ \sum_h \binom{h+s}{2M+2s+2} \sum_{j,k=0}^{N} |\lambda_j \lambda_k| \binom{a_j+s}{a_j-h} \binom{a_k+s}{a_k-h} \right\}^{1/2},$$

since the first sum is $O(\Sigma h^{-2})$.

(6) $$\sum_h \binom{h+s}{2s+2M+2}\binom{a_j+s}{a_j-h}\binom{a_k+s}{a_k-h} =$$

$$= \frac{1}{(2s+2M+2)!}\sum_{h=2M+2+s}^{a_k} \frac{(a_j+s)!}{(h-s-2M-2)!(a_j-h)!}\binom{a_k+s}{a_k-h} =$$

$$= \binom{a_j+s}{2s+2M+2}\sum_{i=0}^{a_k-2M-2-s}\binom{a_j-2M-2-s}{i}\binom{a_k+s}{a_k-2M-2-s-i}.$$

For b, c positive noninteger real numbers,

$$(1+t)^b(1+t)^c = \left(\sum_j \binom{b}{j} t^j\right)\left(\sum_j \binom{c}{j} t^j\right) = \sum_n \left(\sum_{j=0}^{n}\binom{b}{j}\binom{c}{n-j}\right) t^n.$$

Since also $(1+t)^{b+c} = \sum_j \begin{bmatrix} b+c \\ j \end{bmatrix} t^j$,

(7) $$\sum_{j=0}^{n}\binom{b}{j}\binom{c}{n-j} = \binom{b+c}{n}.$$

Substituting (7) in (6),

$$\sum_h \binom{h+s}{2s+2M+2}\binom{a_j+s}{a_j-h}\binom{a_k+s}{a_k-h} =$$

$$= \frac{(a_j+a_k-2M-2)!}{(2s+2M+2)!(a_j-2M-2-s)!(a_k-2M-2-s)!} =$$

$$= \frac{1}{(2M+2s+2)!(a_j-2M-2-s)!(a_k-2M-2-s)!}\int_0^\infty e^{-x}x^{a_j+a_k-2M-2}\,dx,$$

and (5) can be written

$$\sum_{h\geqq 2M+2+s}\binom{h+s}{2M+2s+2}\left|\sum_{k=0}^{N}\lambda_k\binom{a_k+\alpha}{a_k-h}\right| \leqq A\left\{\int_0^\infty e^{-x}x^{2M+2+2s}Q^2(x)\,dx\right\}^{1/2},$$

where A is independent of N and the λ_k's and

$$Q(x) = \sum_{k=0}^{N}\frac{|\lambda_k|x^{a_k-2M-2-s}}{(a_k-2M-2-s)!} \quad (\lambda_0 = 1).$$

For $h<2M+2+s$,

$$\left|\sum_{k=0}^{N}\lambda_k\binom{a_k+\alpha}{a_k-h}\right|\leqq\sum_{k=0}^{N}|\lambda_k|\binom{a_k+s}{a_k-h}=$$

$$=\frac{1}{(s+h)!}\int_0^\infty\frac{e^{-x}x^{h+s}}{(2M-h+1+s)!}\int_0^x(x-y)^{2M-h+1+s}Q(y)\,dy\,dx=$$

$$=\int_0^\infty Q(y)\,dy\int_y^\infty e^{-x}x^{h+s}(x-y)^{2M-h+1+s}\,dx=$$

$$=\int_0^\infty Q(y)\,dy\int_0^\infty e^{-y-z}(y+z)^{h+s}z^{2M-h+1+s}\,dz<$$

$$<\int_0^\infty e^{-y}Q(y)\,dy\int_0^\infty e^{-z}(y+z)^{h+s+2M-h+1+s}\,dz=$$

$$=\int_0^\infty e^{-y}Q(y)\,dy\int_0^\infty e^{-z}(y+z)^{2M+1+2s}\,dz<$$

$$<2^{2M+1+2s}\int_0^\infty e^{-y}Q(y)\,dy\int_0^\infty e^{-z}(y^{2M+1+2s}+z^{2M+1+2s})\,dz<$$

$$<B\int_0^\infty e^{-y}Q(y)(1+y^{2M+1+2s})\,dy<$$

$$<B\Big(\int_0^\infty e^{-y}(1+y^{2M+1+2s})^2\,dy\Big)^{1/2}\Big(\int_0^\infty e^{-y}Q^2(y)\,dy\Big)^{1/2}=C\Big(\int_0^\infty e^{-y}Q^2(y)\,dy\Big)^{1/2}.$$

It remains to show that

$$\int_0^\infty e^{-x}Q^2(x)(1+x^{2M+2s+2})\,dx<\varepsilon. \tag{8}$$

Using Lemma 1 and Theorem 4 of [4], the system

$$\{e^{-x/2}(1+x^{2M+2+2s})^{1/2}x^{a_k-2M-2-s}\}\quad(k\geqq 1) \tag{9}$$

is closed since S is closed. Therefore

$$\frac{e^{-x/2}(1+x^{2M+2+2s})^{1/2}x^{a_0-2M-2-s}}{(a_0-2M-2)!}$$

can be approximated arbitrarily close by finite linear combinations of functions from (9). This proves (8).

We shall now show that, if (4) is true for every $a_0 \geqq 2M+2+s$, then S is complete. If (4) is satisfied then, for suitable values of λ_k,

$$\sum_{h\geqq M}\binom{h+\alpha}{h-M}\left|\sum_{k=0}^{N}\lambda_k\binom{a_k+\alpha}{a_k-h}\right| < \varepsilon.$$

It then follows that

$$\sum_{h\geqq M}\binom{h+\alpha}{h-M}\left(\sum_{k=0}^{N}\lambda_k\binom{a_k+\alpha}{a_k-h}\right)^2 < \varepsilon^2. \tag{10}$$

But

$$\sum_{h\geqq M}\binom{h+\alpha}{h-M}\left(\sum_{k=0}^{N}\lambda_k\binom{a_k+\alpha}{a_k-h}\right)^2 = \sum_{j,k=0}^{N}\lambda_j\lambda_k\sum_{h=M}^{a_k}\binom{h+\alpha}{h-M}\binom{a_j+\alpha}{a_j-h}\binom{a_k+\alpha}{a_k-h} =$$

$$= \sum_{j,k=0}^{N}\frac{\lambda_j\lambda_k}{\Gamma(\alpha+M+1)}\sum_{h=M}^{a_k}\frac{\Gamma(a_j+\alpha+1)}{(h-M)!(a_j-h)!}\binom{a_k+\alpha}{a_k-h} =$$

$$= \sum_{j,k=0}^{N}\lambda_j\lambda_k\binom{a_j+\alpha}{M+\alpha}\sum_{i=0}^{a_k-M}\binom{a_j-M}{i}\binom{a_k+\alpha}{a_k-M-i} =$$

$$= \sum_{j,k=0}^{N}\lambda_j\lambda_k\binom{a_j+\alpha}{M+\alpha}\binom{a_j+a_k+\alpha-M}{a_k-M} =$$

$$= \sum_{j,k=0}^{N}\lambda_j\lambda_k\frac{\Gamma(a_j+a_k+\alpha-M+1)}{\Gamma(M+\alpha+1)(a_j-M)!(a_k-M)!} = \frac{1}{\Gamma(M+\alpha+1)}\int_0^\infty e^{-x}R^2(x)\,dx,$$

where

$$R(x) = \sum_{k=0}^{N}\frac{\lambda_k x^{a_k+\alpha/2-M/2}}{(a_k-M)!}.$$

Therefore

$$\frac{1}{\Gamma(M+\alpha+1)}\int_0^\infty e^{-x}R^2(x)\,dx < \varepsilon^2,$$

which implies that

$$e^{-x/2}x^{n-M/2+\alpha/2}, \quad n = 2M+2+s, 2M+3+s, \ldots, \tag{11}$$

can be mean square approximated by linear combinations of the functions $e^{-x/2}x^{a_k-M/2+\alpha/2}$, $k\geqq 1$. From [4, Theorem 5] the set (11) is closed. Thus also is $\{e^{-x/2}x^{a_k-M/2+\alpha/2}\}$. From Lemma 1 of [4] with $p(x)=x^{M/2-\alpha/2}$, S is closed.

Suppose $t_n=0$ for $n=a_1, a_2, \ldots,$ and S is closed. Then one can use condition (4) and mathematical induction to force $t_n=0$ for all $n\geqq a_0$.

Now suppose that $s_n=o(n^{M+\alpha})$, $\{t_n\}$ satisfies (3) with $t_n=0$ for $n\geqq 2M+ +s+2$. Note that (3) is the nth term of a diagonal matrix t satisfying $t=\delta^{(\alpha)}s$, where s is the diagonal matrix with entries s_n and $\delta_{nk}^{(\alpha)}=(-1)^k\begin{bmatrix}n+\alpha\\n-k\end{bmatrix}$. Since $\delta^{(\alpha)}$ is its own inverse, and multiplication is associative, $\delta^{(\alpha)}t=s$; i.e.

$$s_n=\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}t_k=\sum_{k=0}^{2M+s+1}(-1)^k\binom{n+\alpha}{n-k}t_k=$$

$$=\sum_{k=0}^{\varrho}(-1)^k\binom{n+\alpha}{n-k}t_k=\frac{\Gamma(n+\alpha+1)}{n!}\sum_{k=0}^{\varrho}\frac{(-1)^k n!\,t_k}{(n-k)!\,\Gamma(k+\alpha+1)},$$

where ϱ is the largest integer for which $t_k\neq 0$. Therefore $s_n=\Gamma(n+\alpha+1)P(n)/n!$, where P is a polynomial in n of degree ϱ. Since $s_n=o(n^{M+\alpha})$, $\alpha+\varrho<M+\alpha$, and the degree of P is less than M.

Let $\sigma_p(A)$ denote the point spectrum of an operator A, and write H for $H^{(0)}$.

Theorem 2. (a) *Let* $1<p<\infty$, $H^{(\alpha)}\in B(l^p)\cap B(c)$. *Then* $\sigma_p(H^{(\alpha)})$ *is empty.*

(b) *Let* $H^{(\alpha)}\in B(l)$, $\alpha\geqq 0$. *Then* $B_p(H^{(\alpha)})$ *is empty.*

(c) *Let* $H^{(\alpha)}\in B(c)$. *For* $\alpha>0$, $\sigma_p(H^{(\alpha)})$ *is empty. For* $\alpha=0$, *if* H *is multiplicative, then* $\sigma_p(H)=\{\mu_0\}$.

Proof of (a). Suppose there exists an $x\in l^p$ with $H^{(\alpha)}x=\lambda x$. Then $(H^{(\alpha)}-\lambda I)x=0$. But $H^{(\alpha)}\in B(l^p)\cap B(c)$ implies that $K^{(\alpha)}=H^{(\alpha)}-\lambda I\in B(l^p)\cap B(c)$. Moreover, $K^{(\alpha)}$ is also a generalized Hausdorff matrix. Thus, we are looking for solutions of the system $K^{(\alpha)}x=0$. One may write $K^{(\alpha)}=\delta^{(\alpha)}\mu\delta^{(\alpha)}$, where μ is a diagonal matrix with diagonal entries μ_n and $\delta_{nk}^{(\alpha)}=(-1)^k\binom{n+\alpha}{n-k}$. Since $\delta^{(\alpha)}$ is its own inverse, and each matrix forming $K^{(\alpha)}$ is row finite, the system $K^{(\alpha)}x=0$ is equivalent to $\mu\delta^{(\alpha)}x=0$; i.e.,

$$\text{(12)}\qquad \mu_n\sum_{i=0}^{n}(-1)^i\binom{n+\alpha}{n-i}x_i=0,\quad n=0,1,2,\ldots.$$

Since $H^{(\alpha)}\in B(c)$, so also does $K^{(\alpha)}$, so that μ is a moment sequence. This means that

$$\psi(z)=\int_0^1 t^{z+\alpha}\,d\beta(t)$$

is analytic for $\mathrm{Re}(z)>0$, where β and μ_n satisfy

$$\mu_n=\int_0^1 t^{n+\alpha}\,d\beta(t).$$

From [2], the integer values b_n for which $\psi(b_n)=0$ satisfy the condition $\Sigma_k b_k^{-1}<\infty$. Therefore (12) implies that $t_n=0$ for all values of n except possibly a subset $\{b_n\}$ satisfying $\Sigma_k b_k^{-1}<\infty$. Using Theorem 3 of [4], the set S of integers n for which

$t_n=0$ remains closed. Since $\{x_n\}\subset l^p$, $1<p<\infty$, $x_n=o(n^{1/2+\alpha})$. Applying Theorem 1, $x_n=\Gamma(n+\alpha+1)P(n)/n!$, where $P(x)$ is a polynomial of degree less than $M=1/2$; i.e., P is a constant polynomial. But, unless P is the zero polynomial, $x\notin l^p$, so $H^{(\alpha)}$ has empty point spectrum.

Proof of (b). The author has shown in [7] that $H^{(\alpha)}\in B(l)$ implies $H^{(\alpha)}\in B(c)$. The rest of the proof is the same as that of (a).

Proof of (c). Following the proof of (a), since $\{x_n\}\in c$, $\{x_n\}$ is bounded, hence $x_n=o(n^{1/2+\alpha})$, and again $\sigma_p(H^{(\alpha)})$ is empty, for $\alpha>0$.

For $\alpha=0$, $x_n=o(n^{1/2})$, and the only nonzero sequence satisfying (12) is $e=(1, 1, \ldots)$. With $\alpha=0$, each row sum of H is μ_0. Therefore $\sigma_p(H)=\{\mu_0\}$.

A matrix A is multiplicative if $\lim Ax=t\lim x$ for some scalar t, $x\in c$. In terms of the matrix entries, multiplicativity of A translates into A having all zero column limits. For Hausdorff matrices in $B(c)$ this condition is equivalent to the mass function $\beta(t)$ being continuous from the right at zero, and specifically excludes the compact Hausdorff matrix generated by $\mu_0=1$, $\mu_n=0, n>0$. Theorem 1 does not apply to this matrix since there are too many zeros on the main diagonal, but a direct analysis yields the point spectrum to be $\{0, 1\}$.

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