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Convergence of Hermite—Fejér interpolation at zeros of generalized Jacobi polynomials

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1. Introduction

The aim of this paper is to find necessary and sufficient conditions for uniform convergence of Hermite—Fejér interpolating processes based at the zeros of generalized Jacobi polynomials. As a by-product of our investigation we also give an answer to a question raised by P. TURÁN [35, Problem XXVII, p. 47] (cf. [36, Sections 2.3.1 and 3.6, pp. 337—338]). If f is a bounded function and w is a nonnegative integrable weight function on the real line, and $x_{1n}(w) > x_{2n}(w) > \ldots > x_{nn}(w)$ are the zeros of the orthonormal polynomials $p_n(w)$ corresponding to w, then the associated Hermite—Fejér interpolating polynomial $H_n(w,f)$ is defined to be the unique polynomial of degree at most $2n-1$ which satisfies

$$H_n(w, f, x_{kn}(w)) = f(x_{kn}(w)) \quad \text{and} \quad H_n'(w, f, x_{kn}(w)) = 0, \quad k = 1, 2, \ldots, n.$$

Ever since the work of L. Fejér, G. Grünwald and G. Szegő there has been a great deal of research performed in conjunction with convergence properties of these polynomials in terms of the weight function w, the point system $\{x_{kn}\}$ and the function f. In particular, when $\{x_{kn}(w^{(a,b)})\}$ are the zeros of the Jacobi polynomials $p_n^{(a,b)}$ which are orthonormal with respect to the Jacobi weight $w^{(a,b)}$ defined by

$$w^{(a,b)}(x) = \begin{cases} (1-x)^a(1+x)^b & \text{for} \quad x \in [-1, 1] \\ 0 & \text{for} \quad x \notin (-1, 1), \end{cases}$$

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$a>-1$, $b>-1$, one has a complete description of the conditions assuring uniform convergence of the corresponding Hermite—Fejér polynomials $H_n(w^{(a,b)},f)$. Namely, roughly speaking, for negative parameters a and b $\lim H_n(w^{(a,b)},f)=f$ uniformly for all continuous functions f, whereas for nonnegative a and b $\lim H_n(w^{(a,b)},f)=f$ takes place uniformly only under additional conditions on f. An accurate synthesis of the results we are interested in is given by the following five statements.

Proposition 1.1. *Let* $a>-1$, $b>-1$ *and* $0<\varepsilon<1$. *Then*

$$\lim_{n\to\infty} \max_{-\varepsilon\le x\le\varepsilon} |f(x)-H_n(w^{(a,b)},f,x)| = 0$$

for every function f continuous in $[-1,1]$.

Proposition 1.2. *Let* $b>-1$ *and* $-1<\varepsilon<1$. *Then*

$$\sup_{n\ge1} \max_{\varepsilon\le x\le1} |H_n(w^{(a,b)},f,x)| <\infty$$

for every function f bounded in $[-1,1]$ *if and only if* $-1<a\le0$.

Proposition 1.3. *Let* $b>-1$ *and* $-1<\varepsilon<1$. *Then*

$$\lim_{n\to\infty} \max_{\varepsilon\le x\le1} |f(x)-H_n(w^{(a,b)},f,x)| = 0$$

for every function f continuous in $[-1,1]$ *if and only if* $-1<a<0$.

The above three theorems are condensed from [4, Vol. II, pp. 9—48, 285—317, 361—417, 502—512, 527—562, 767—801], [28, p. 138], [32, Theorem 14.6, pp. 340—344] and [33, Vol. 1, pp. 335—362].

By Markov's theorem on the derivatives of algebraic polynomials (cf. [16, § VI. 6, p. 141]) if $\{Q_n\}$ ($\deg Q_n=n$) is a uniformly convergent sequence of algebraic polynomials in an interval, then $Q_n^{(r)}$ is $O(n^{2r})$ in the same interval for $r=1,2,\dots$. In view of this observation the following result whose special case of Legendre zeros ($a=b=0$) was also treated by A. Schönhage [27] and J. Szabados [29] is especially satisfying.

Proposition 1.4 [38, Theorem 2.1, p. 84]. *Let* $-1<\varepsilon<1$ *and let f be continuous in* $[-1,1]$. *Let* $a\in[s-1,s)$ *for a fixed positive integer s, and let* $b>-1$. *Then*

$$\lim_{n\to\infty} \max_{\varepsilon\le x\le1} |f(x)-H_n(w^{(a,b)},f,x)| = 0$$

holds if and only if

$$\lim_{n\to\infty} H_n(w^{(a,b)},f,1) = f(1)$$

and (if $a \geqq 1$)

$$\lim_{n\to\infty} n^{-2r}[H_n^{(r)}(w^{(a,b)}, f, x)]|_{x=1} = 0$$

for $r=1, 2, \ldots, s-1$.

The following result of J. Szabados is the culmination of research by several authors including L. FEJÉR [4, Vol. II, pp. 22 and 40], E. EGERVÁRI and P. TURÁN [3], A. SCHÖNHAGE [27] and G. FREUD [9].

Proposition 1.5 [29, Theorems 1 and 3, pp. 470 and 457]. *Let* $b>-1$ *and* $-1<\varepsilon<1$. *Let f be continuous in* $[-1, 1]$. *Then*

$$\lim_{n\to\infty} H_n(w^{(0,b)}, f, 1) = (1+b)2^{-b-1} \int_{-1}^{1} f(t) w^{(0,b)}(t)\, dt,$$

and

$$\lim_{n\to\infty} \max_{\varepsilon \leqq x \leqq 1} |f(x) - H_n(w^{(0,b)}, f, x)| = 0$$

holds if and only if

$$f(1) = (1+b)2^{-b-1} \int_{-1}^{1} f(t) w^{(0,b)}(t)\, dt.$$

In what follows the function w is a generalized Jacobi weight if it can be represented as

$$w = g w^{(a,b)} \quad \text{where} \quad g(>0) \in C^1 \quad \text{and} \quad g' \in \text{Lip}\, 1 \quad \text{on} \quad [-1, 1]$$

for some $a>-1$ and $b>-1$. Because of J. Korous' theorem yielding bounds for the corresponding generalized Jacobi polynomials $p_n(w)$ (cf. [32, Theorem 7.1.3, p. 162]) one expects a close relationship between Jacobi and generalized Jacobi polynomials, in particular, between associated approximation procedures. This is indeed the case as shown in the research conducted by V. M. Badkov, A. Máté, V. Totik and us (cf. [1], [2], [11]—[15], [18]—[22] and [24]).

In [24] we dealt with characterizing weighted mean convergence properties of Hermite—Fejér interpolating sequences associated with generalized Jacobi polynomials and we proved the following

Proposition 1.6 [24, Theorem 5, p. 55]. *Let* $0<p<\infty$, *and let w be a generalized Jacobi weight. Let u be an unrelated Jacobi weight function. Then*

$$\lim_{n\to\infty} H_n(w, f) = f \quad in \quad L_p(u) \quad in \quad [-1, 1]$$

for every function f continuous in $[-1, 1]$ *if and only if* $w^{-1} \in L_p(u)$ *in the interval* $[-1, 1]$.

2. Main results

As announced in [23], we can generalize and/or extend the previous six propositions as follows.

Theorem 2.1. *Let w be a generalized Jacobi weight, and let $0<\varepsilon<1$. Then*

$$\lim_{n\to\infty} \max_{-\varepsilon\leqq x\leqq\varepsilon} |f(x)-H_n(w,f,x)| = 0$$

for every function f continuous in $[-1,1]$.

Theorem 2.2. *Let w be a generalized Jacobi weight. Then for every fixed non-negative integer m there exists a polynomial Π such that R defined by $R(x)=(1-x)^m\Pi(x)$ satisfies*

$$\liminf_{n\to\infty} n^{-2a}|R(1)-H_n(w,R,1)| \geqq 1.$$

Theorem 2.3. *Let w be a generalized Jacobi weight, and let $-1<\varepsilon<1$. Then*

$$\sup_{n\geqq 1} \max_{\varepsilon\leqq x\leqq 1} |H_n(w,f,x)| <\infty$$

for every function f bounded in $[-1,1]$ if and only if $w(1)\neq 0$.

Theorem 2.4. *Let $w=gw^{(a,b)}$ be a generalized Jacobi weight function, and let $-1<\varepsilon<1$. Then*

$$\lim_{n\to\infty} \max_{\varepsilon\leqq x\leqq 1} |f(x)-H_n(w,f,x)| = 0$$

for every function f continuous in $[-1,1]$ if and only if $w(1)=\infty$.

Theorem 2.5. *Let $w=gw^{(a,b)}$ be a generalized Jacobi weight function, and let $-1<\varepsilon<1$. Let f be continuous in $[-1,1]$. Let $a\in[s-1,s)$ for a fixed positive integer s, and let $b>-1$. Then*

$$\lim_{n\to\infty} \max_{\varepsilon\leqq x\leqq 1} |f(x)-H_n(w,f,x)| = 0$$

holds if and only if

$$\lim_{n\to\infty} H_n(w,f,1) = f(1)$$

and (if $a\geqq 1$)

$$\lim_{n\to\infty} n^{-2r}[H_n^{(r)}(w,f,x)]|_{x=1} = 0$$

for $r=1,2,\ldots,s-1$.

Theorem 2.6. *Let $w=gw^{(0,b)}$ be a generalized Jacobi weight function, and let $-1<\varepsilon<1$. Let f be continuous in $[-1,1]$. Then*

$$\lim_{n\to\infty} H_n(w,f,1) = (2w(1))^{-1}\int_{-1}^{1} f(t)\,d[w(t)(1+t)].$$

Hence,

$$\lim_{n\to\infty} \max_{\varepsilon \le x \le 1} |f(x)-H_n(w,f,x)| = 0$$

holds if and only if

$$f(1) = (2w(1))^{-1} \int_{-1}^{1} f(t)\, d[w(t)(1+t)].$$

Needless to say that analogous results can be proved in the interval $[-1, \varepsilon]$ as well, and therefore one can formulate results that are concerned with convergence in the entire interval $[-1, 1]$.

3. Notations

As a rule of thumb, all positive constants whose value is irrelevant and which are independent of the variables in consideration are denoted by "K". Each time "K" is used it may (or may not) take a different value. The symbol "$\sim$" is used to indicate that if A and B are two expressions depending on some variables then $A\sim B \Leftrightarrow |AB^{-1}| \le K$ and $|A^{-1}B| \le K$. We use **N** and **R** to denote the set of positive integers and real numbers, respectively.

Given a weight function w, the leading coefficient of the corresponding orthonormal polynomial $p_n(w)$ is denoted by $\gamma_n(w)$. $K_n(w)$ is the associated reproducing kernel function, that is

$$K_n(w, x, t) = \sum_{k=0}^{n-1} p_n(w, x) p_n(w, t). \tag{3.1}$$

In terms of the Christoffel—Darboux formula (cf. [32, Theorem 3.2.2, p. 43]), $K_n(w)$ can be expressed as

$$K_n(w, x, t) = (\gamma_{n-1}(w)/\gamma_n(w))[p_n(w, x)p_{n-1}(w, t)-p_{n-1}(w, x)p_n(w, t)]/(x-t). \tag{3.2}$$

The Christoffel function $\lambda_n(w)$ is defined by

$$\lambda_n(w, x) = K_n^{-1}(w, x, x). \tag{3.3}$$

The Cotes numbers $\lambda_{kn}(w)$ in the Gauss—Jacobi quadrature formula are given by

$$\lambda_{kn}(w) = \lambda_{kn}(w, x_{kn}(w)). \tag{3.4}$$

The fundamental polynomials of Lagrange interpolation $\ell_{kn}(w)$ associated with the zeros of $p_n(w)$ are defined by

$$\ell_{kn}(w, x) = p_n(w, x)/[p_n'(w, x_{kn}(w))(x-x_{kn}(w))]. \tag{3.5}$$

Another useful expression for $\ell_{kn}(w)$ is the following

$$\ell_{kn}(w, x) = (\gamma_{n-1}(w)/\gamma_n(w))\lambda_{kn}(w)p_{n-1}(w, x_{kn}(w))p_n(w, x)/(x-x_{kn}(w)) \tag{3.6}$$

(cf. [32, formula (3.4.7), p. 48]).

The usual way of expressing the Hermite—Fejér interpolating polynomial $H_n(w, f)$ is in terms of the fundamental polynomials $\ell_{kn}(w)$, and it is given by

$$H_n(w, f, x) = \sum_{k=1}^{n} f(x_{kn}(w))v_{kn}(w, x)\ell_{kn}^2(w, x) \tag{3.7}$$

where $v_{kn}(w)$ is defined by

$$v_{kn}(w, x) = 1 - p_n''(w, x_{kn}(w))[p_n'(w, x_{kn}(w))]^{-1}(x-x_{kn}(w)) \tag{3.8}$$

(cf. [32, p. 330—331]). For special orthogonal polynomial systems due to available differential equations $p_n''(w, x_{kn}(w))[p_n'(w, x_{kn}(w))]^{-1}$ can be expressed explicitly in terms of the weight function and the zeros of the orthogonal polynomials, the above expression is convenient when investigating Hermite—Fejér interpolation. However, for general weight functions it is difficult (if not impossible) to handle the derivatives of orthogonal polynomials, and thus this formula is of limited value. On the other hand, G. Freud's formula

$$v_{kn}(w, x) = 1 + \lambda_n'(w, x_{kn}(w))\lambda_{kn}(w)^{-1}(x-x_{kn}(w)) \tag{3.9}$$

(cf. [5, p. 113]) involves the Christoffel functions and their derivatives which are much more suitable when the weight function is not one of the classical ones (cf. [5, 8, 24]). If P is a polynomial of degree at most $2n-1$ then in view of the Hermite interpolation formula (cf. [32, pp. 330—331]) we can write

$$P(x) = H_n(w, P, x) + \mathcal{H}_n(w, P', x) \tag{3.10}$$

where

$$\mathcal{H}_n(w, f, x) = \sum_{k=1}^{n} f(x_{kn}(w))(x-x_{kn}(w))\ell_{kn}^2(w, x). \tag{3.11}$$

4. Technicalities

Here, in addition to formulating some useful and known properties of generalized Jacobi polynomials which run parallel to those of Jacobi polynomials, we will also prove a few propositions of technical nature that will subsequently be applied to demonstrate our principal results. In what follows w is a generalized Jacobi weight.

If $x_{kn}(w)=\cos(\theta_{kn}(w))$ where $x_{0n}(w)=1$, $x_{n+1,n}(w)=-1$ and $0\leqq\theta_{kn}(w)\leqq\pi$ then

$$\theta_{k+1,n}(w)-\theta_{kn}(w)\sim 1/n \tag{4.1}$$

uniformly for $0\leqq k\leqq n$ and $n\in\mathbf{N}$ (cf. [18, Theorem 3, p. 367]).

Using Korous' theorem (cf. [32, Theorem 7.1.3, p. 162]), similarly to Jacobi polynomials, the generalized Jacobi polynomials can be estimated in terms of the weight function as follows:

$$|p_n(w,x)|\leqq K\begin{cases}[w(x)(1-x^2)^{1/2}]^{-1/2} & \text{for } x\in[-1+n^{-2},1-n^{-2}]\\ n^{1/2}[w(1-n^{-2})]^{-1/2} & \text{for } x\in[1-n^{-2},1]\\ n^{1/2}[w(-1+n^{-2})]^{-1/2} & \text{for } x\in[-1,-1+n^{-2}],\end{cases} \tag{4.2}$$

uniformly for $n\in\mathbf{N}$ (cf. [32, Theorem 7.32.2, p. 169] or [1, p. 226]),

$$|p_n(w,x)|\sim\begin{cases}n|x-x_{mn}(w)|[w(x)(1-x^2)^{3/2}]^{-1/2} & \text{for } 2x\in[-1+x_{nn}(w),1+x_{1n}(w)]\\ n^{1/2}[w(1-n^{-2})]^{-1/2} & \text{for } 2x\in[1+x_{1n}(w),2]\\ n^{1/2}[w(-1+n^{-2})]^{-1/2} & \text{for } 2x\in[-2,-1+x_{nn}(w)],\end{cases} \tag{4.3}$$

uniformly for $n\in\mathbf{N}$ where m is the index of the zero $x_{kn}(w)$ which is (one of the) closest to x (cf. [19, Theorem 9.33, p. 171]), and

$$|p_{n-1}(w,x_{kn}(w))|\sim w(x_{kn}(w))^{-1/2}(1-x_{kn}(w)^2)^{1/4} \tag{4.4}$$

uniformly for $1\leqq k\leqq n$ and $n\in\mathbf{N}$ (cf. [19, Theorem 9.31, p. 170]).

The derivatives of generalized Jacobi polynomials at ± 1 satisfy

$$|[p_n^{-1}(w,\pm 1)]^{(\ell)}|\leqq Kn^{2\ell}|p_n^{-1}(w,\pm 1)| \tag{4.5}$$

uniformly for $n\in\mathbf{N}$ (cf. [20, formula (23), p. 674]). Writing $p_n^{-2}=(p_n^{-1})(p_n^{-1})$, and using (4.5) and the product differentiation rule (Leibnitz's formula) we obtain

$$|[p_n^{-2}(w,\pm 1)]^{(\ell)}|\leqq Kn^{2\ell}|p_n^{-2}(w,\pm 1)| \tag{4.6}$$

uniformly for $n\in\mathbf{N}$.

For the Christoffel functions and Cotes numbers we have the following estimates

$$\lambda_n(w,x)\sim\begin{cases}n^{-1}w(x)(1-x^2)^{1/2} & \text{for } x\in[-1+n^{-2},1-n^{-2}]\\ n^{-2}w(1-n^{-2}) & \text{for } x\in[1-n^{-2},1]\\ n^{-2}w(-1+n^{-2}) & \text{for } x\in[-1,-1+n^{-2}],\end{cases} \tag{4.7}$$

uniformly for $n\in\mathbf{N}$ (cf. [17, p. 336]) and

$$\lambda_{kn}(w)\sim n^{-1}w(x_{kn}(w))(1-x_{kn}(w)^2)^{1/2} \tag{4.8}$$

uniformly for $1\leqq k\leqq n$ and $n\in\mathbf{N}$ (this follows immediately from estimates (4.1) and (4.7) (cf. 3.4)). The derivatives of the Christoffel functions satisfy

$$(4.9)\qquad |\lambda_n'(w,x)| \leqq K\begin{cases} n^{-1}w(x)(1-x^2)^{-1/2} & \text{for}\quad x\in[-1+n^{-2},1-n^{-2}] \\ w(1-n^{-2}) & \text{for}\quad x\in[1-n^{-2},1] \\ w(-1+n^{-2}) & \text{for}\quad x\in[-1,-1+n^{-2}], \end{cases}$$

uniformly for $n\in\mathbf{N}$ (cf. [24, formula (23), p. 36]) and

$$(4.10)\qquad \lambda_n'(w,x_{kn}(w)) \leqq Kn^{-1}w(x_{kn}(w))(1-x_{kn}(w)^2)^{-1/2}$$

uniformly for $1\leqq k\leqq n$ and $n\in\mathbf{N}$ (cf. [24, formula (24), p. 36]).

A weight function w is said to belong to Szegő's class ($w\in S$) if it is supported in $[-1,1]$ and $\log w(\cos\theta)\in L_1$ in $[0,\pi]$. For instance, all generalized Jacobi weights are in Szegő's class. According to the Szegő Theory (cf. [32, Theorem 12.7.1, p. 309]) the leading coefficients $\gamma_n(w)$ of the orthogonal polynomials $p_n(w)$ satisfy

$$(4.11)\qquad 0<\lim_{n\to\infty} 2^{-n}\gamma_n(w) = \pi^{-1/2}\exp\left\{(2\pi)^{-1}\int_0^\pi \log w(\cos\theta)\,d\theta\right\}<\infty$$

whenever $w\in S$.

The following proposition is a simple but unexpected generalization of (4.2) and (4.4).

Lemma 4.1. *Let $w_1=g_1w^{(a,b)}$ and $w_2=g_2w^{(a,b)}$ be two (not necessarily different) generalized Jacobi weights corresponding to the same parameters $a>-1$ and $b>-1$. Then for every fixed integer ℓ we have*

$$(4.12)\qquad |p_{n+\ell}(w_1,x_{kn}(w_2))| \leqq Kw(x_{kn}(w_2))^{-1/2}(1-x_{kn}(w_2)^2)^{1/4}$$

uniformly for $1\leqq k\leqq n$ and $n\in\mathbf{N}$.

Proof. By Korous' theorem (cf. [32, Theorem 7.1.3, p. 162]) we have

$$|p_{n+\ell}(w_1,x)| \leqq K[|p_{n+\ell}(w_2,x)|+|p_{n+\ell-1}(w_2,x)|]$$

for $x\in[-1,1]$. Being orthogonal polynomials, the generalized Jacobi polynomials satisfy the three-term recurrence

$$xp_n(w,x) = a_{n+1}(w)p_{n+1}(w,x)+b_n(w)p_n(w,x)+a_n(w)p_{n-1}(w,x),$$

and since $w>0$ almost everywhere in $[-1,1]$, we have $\lim a_n(w)=1/2$ and $\lim b_n(w)=0$ (cf. [25], [26], [12, p. 68], [22, Sections 4.5 and 4.13] and (4.11)). Hence by repeated application of the recurrence formula we obtain

$$|p_{n+j}(w_2,x)| \leqq K[|p_{n-1}(w_2,x)|+|p_n(w_2,x)|],\quad x\in[-1,1],$$

for all fixed j. Now inequality (4.12) follows from (4.4) applied with $w=w_2$.

The next step is to compare Christoffel functions of generalized Jacobi weights.

Lemma 4.2. *Let $w_1=g_1w^{(a,b)}$ and $w_2=g_2w^{(a,b)}$ be two generalized Jacobi weights corresponding to the same parameters $a>-1$ and $b>-1$. Then*

$$\left|g_1(x_{kn}(w_1))\lambda_{kn}(w_1)^{-1}-g_2(x_{kn}(w_1))\lambda_{kn}(w_2)^{-1}\right| \leqq \leqq Kw_1(x_{kn}(w_1))^{-1}[1-x_{kn}(w_1)^2]^{1/2} \tag{4.13}$$

holds uniformly for $1\leqq k\leqq n$ and $n\in\mathbf{N}$.

Proof. Let $w_1=gw_2$. Then the identity

$$g(x)\lambda_n^{-1}(w_1,x)-\lambda_n^{-1}(w_2,x)=\int_{\mathbf{R}}K_n(w_1,x,t)K_n(w_2,x,t)[g(x)-g(t)]w_2(t)\,dt$$

is a straightforward consequence of orthogonality relations. Since we have $g'\in\text{Lip }1$, we can write $g(x)-g(t)=g'(x)(x-t)+O([x-t]^2)$. Hence the previous formula becomes

$$g(x)\lambda_n^{-1}(w_1,x)-\lambda_n^{-1}(w_2,x)= =g'(x)\int_{\mathbf{R}}K_n(w_1,x,t)K_n(w_2,x,t)(x-t)w_2(t)\,dt+ +O(1)\int_{\mathbf{R}}|K_n(w_1,x,t)K_n(w_2,x,t)|(x-t)^2w_2(t)\,dt. \tag{4.14}$$

In view of (3.2) the first integral here can explicitly be evaluated in terms of the orthogonal polynomials involved and their leading coefficients. We have

$$\int_{\mathbf{R}}K_n(w_1,x,t)K_n(w_2,x,t)(x-t)w_2(t)\,dt=(\gamma_{n-1}(w_1)/\gamma_n(w_2))p_{n-1}(w_1,x)p_n(w_2,x). \tag{4.15}$$

Using Schwarz's inequality, $w_2\leqq Kw_1$, (3.2) and again orthogonality relations, we can estimate the second integral as follows:

$$\left[\int_{\mathbf{R}}|K_n(w_1,x,t)K_n(w_2,x,t)|(x-t)^2w_2(t)\,dt\right]^2\leqq \leqq K\int_{\mathbf{R}}K_n^2(w_1,x,t)(x-t)^2w_1(t)\,dt\int_{\mathbf{R}}K_n^2(w_2,x,t)(x-t)^2w_2(t)\,dt= =K[\gamma_{n-1}(w_1)/\gamma_n(w_1)]^2[p_{n-1}^2(w_1,x)+p_n^2(w_1,x)]\times \times[\gamma_{n-1}(w_2)/\gamma_n(w_2)]^2[p_{n-1}^2(w_2,x)+p_n^2(w_2,x)]. \tag{4.16}$$

Since generalized Jacobi weights w are in Szegő's class, we can use (4.11) to estimate the ratios of the leading coefficients of generalized Jacobi polynomials. Using this observation and inserting (4.15) and (4.16) into (4.14), we obtain

$$|g(x)\lambda_n^{-1}(w_1,x)-\lambda_n^{-1}(w_2,x)|\leqq \leqq K[|p_{n-1}(w_1,x)|+|p_n(w_1,x)|][|p_{n-1}(w_2,x)|+|p_n(w_2,x)|].$$

Applying this inequality with $x=x_{kn}(w_1)$ (cf. (3.4)) and using Lemma 1 (cf. (4.12)), Lemma 4.2 follows immediately.

Our next goal is to estimate $v_{kn}(w, x)$ (cf. (3.7)—(3.9)) via improving (4.10) regarding the derivatives of the Christoffel functions. For Jacobi polynomials we have

$$v_{kn}(w^{(a,b)}, x) = 1-[1-x_{kn}^2]^{-1}[a-b+(a+b+2)x_{kn}](x-x_{kn}) \tag{4.17}$$

$\left(x_{kn}=x_{kn}(w^{(a,b)})\right)$ (cf. [32, formula (14.5.2), p. 339]). In what follows we will show that the right-hand side of (4.17) is the principal contribution to $v_{kn}(w, x)$ as well.

Lemma 4.3. *Let* $w=gw^{(a,b)}$ *be a generalized Jacobi weight. Then*

$$|\lambda_n'(w, x_{kn}(w))\lambda_{kn}(w)^{-1}+[1-x_{kn}(w)^2]^{-1}[a-b+(a+b+2)x_{kn}(w)]| \leqq K \tag{4.18}$$

uniformly for $1\leqq k\leqq n$ *and* $n\in\mathbf{N}$.

Proof The crux of the matter is the inequality

$$\begin{aligned}|g(x)K_n'(w, x, x)-K_n'(w^{(a,b)}, x, x)| &\leqq\\ \leqq K[|p_{n-2}(w^{(a,b)}, x)|+|p_{n-1}(w^{(a,b)}, x)|+|p_n(w^{(a,b)}, x)|]&\times\\ \times[|p_{n-1}'(w^{(a,b)}, x)|+|p_n'(w^{(a,b)}, x)|] \quad &(x\in[-1, 1]),\end{aligned}$$

$n\in\mathbf{N}$, which is a special case of a general inequality proved in [24, Lemma 1, p. 31]. Setting here $x=x_{kn}(w)$ we can apply Lemma 4.1 to estimate $p_{n+\ell}(w^{(a,b)}, x_{kn}(w))$. Moreover, since

$$p_n'(w^{(a,b)}, x) = \text{const}\, p_{n-1}(w^{(a+1,b+1)}, x)$$

where the constant is of precise order n (cf. [32, formula (4.21.7), p. 63]), we can use (4.1) and (4.2) to estimate $p_{n+\ell}'(w^{(a,b)}, x_{kn}(w))$. We obtain

$$\begin{aligned}\left|g(x_{kn}(w))K_n'(w, x_{kn}(w), x_{kn}(w))-K_n'(w^{(a,b)}, x_{kn}(w), x_{kn}(w))\right| &\leqq\\ \leqq Knw^{-1}(x_{kn}(w))(1-x_{kn}(w)^2)^{-1/2}&\end{aligned} \tag{4.19}$$

for $1\leqq k\leqq n$ and $n\in\mathbf{N}$. Now the point is that $K_n'(w^{(a,b)}, x_{kn}(w), x_{kn}(w))$ can be evaluated. By (3.2)

$$K_n'(w) = \left(\gamma_{n-1}(w)/\gamma_n(w)\right)[p_n''(w)p_{n-1}(w)-p_n(w)p_{n-1}''(w)],$$

and since the Jacobi polynomials satisfy the differential equation

$$(1-x^2)Y'' = -n(n+a+b+1)Y+[a-b+(a+b+2)x]Y'$$

(cf. [32, Theorem 4.2.1, p. 60]) we obtain

$$\begin{aligned}K_n'(w^{(a,b)}, x, x) = (1-x^2)^{-1}\Big\{&[a-b+(a+b+2)x]K_n(w^{(a,b)}, x, x)-\\ &-\left(\gamma_{n-1}(w^{(a,b)})/\gamma_n(w^{(a,b)})\right)(2n+a+b)p_{n-1}(w^{(a,b)}, x)p_n(w^{(a,b)}, x)\Big\}\end{aligned}$$

(which, as a matter of fact, immediately yields formula (4.17)). We have $\gamma_{n-1}(w^{(a,b)})/\gamma_n(w^{(a,b)})\to 1/2$, $n\to\infty$ (cf. (4.11)). Therefore, substituting $x=x_{kn}(w)$ here and applying Lemma 4.1, we get

$$
\begin{aligned}
(4.20)\qquad &\big|K_n'(w^{(a,b)}, x_{kn}(w), x_{kn}(w))-[1-x_{kn}(w)^2]^{-1}\times \\
&\times[a-b+(a+b+2)x_{kn}(w)]K_n(w^{(a,b)}, x_{kn}(w), x_{kn}(w))\big| \leqq \\
&\leqq Knw(x_{kn}(w))^{-1}[1-x_{kn}(w)^2]^{-1/2}
\end{aligned}
$$

for $1\leqq k\leqq n$ and $n\in\mathbf{N}$. Inequalities (4.19) and (4.20) enable us to conclude

$$
\begin{aligned}
&\big|g(x_{kn}(w))K_n'(w, x_{kn}(w), x_{kn}(w))-[1-x_{kn}(w)^2]^{-1}\times \\
&\times[a-b+(a+b+2)x_{kn}(w)]K_n(w^{(a,b)}, x_{kn}(w), x_{kn}(w))\big| \leqq \\
&\leqq Knw^{-1}(x_{kn}(w))(1-x_{kn}(w)^2)^{-1/2}
\end{aligned}
$$

for $1\leqq k\leqq n$ and $n\in\mathbf{N}$. Now we apply Lemma 4.2 with weights $w_1=w$ and $w_2=w^{(a,b)}$. We obtain

$$
\begin{aligned}
&\big|K_n'(w, x_{kn}(w), x_{kn}(w))-[1-x_{kn}(w)^2]^{-1}\times \\
&\times[a-b+(a+b+2)x_{kn}(w)]K_n(w, x_{kn}(w), x_{kn}(w))\big| \leqq \\
&\leqq Knw^{-1}(x_{kn}(w))(1-x_{kn}(w)^2)^{-1/2}
\end{aligned}
$$

for $1\leqq k\leqq n$ and $n\in\mathbf{N}$. Since $K_n=\lambda_n^{-1}$ so that $K_n'/K_n=-\lambda_n'/\lambda_n$, and since the right-hand side here is precisely of order $\lambda_{kn}(w)^{-1}$ (cf. (4.8)), the latter inequality is equivalent to (4.18) what we had to prove.

Freud's formula (3.9) and Lemma 4.3 immediately yield

Lemma 4.4. *Let $w=gw^{(a,b)}$ be a generalized Jacobi weight. Then*

$$
\begin{aligned}
&|v_{kn}(w, x)-1-[1-x_{kn}(w)^2]^{-1}[a-b+(a+b+2)x_{kn}(w)](x-x_{kn}(w))| \leqq \\
&\leqq K|x-x_{kn}(w)|
\end{aligned}
$$

uniformly for $x\in[-1, 1]$, $1\leqq k\leqq n$ and $n\in\mathbf{N}$.

The following three purely technical lemmas deal with Lebesgue function type estimates.

Lemma 4.5. *Let $w=gw^{(a,b)}$ be a generalized Jacobi weight, and let $c\in\mathbf{R}$. Then the asymptotics*

$$
(4.21)\qquad \sum_{k=1}^{n}[1-x_{kn}(w)]^{-c}[x-x_{kn}(w)]^2\ell_{kn}^2(w, x)\sim
$$

$$
\sim p_n^2(w, x)\begin{cases} n^{-1} & \text{for } a-c+2>0, \\ n^{-1}\log n & \text{for } a-c+2=0, \\ n^{2(c-a-5/2)} & \text{for } a-c+2<0, \end{cases}
$$

holds uniformly for $n\in\mathbf{N}$ *and* $x\in\mathbf{R}$. *In addition, analogous estimates hold when* $[1-x_{kn}(w)]^{-c}$ *is replaced by* $[1+x_{kn}(w)]^{-c}$ *in the left-hand side of* (4.21).

Proof. By (3.6) we have

$$[1-x_{kn}(w)]^{-c}[x-x_{kn}(w)]^2\ell_{kn}^2(w,x)\,p_n^{-2}(w,x) =$$
$$=(\gamma_{n-1}(w)/\gamma_n(w))^2[1-x_{kn}(w)]^{-c}\lambda_{kn}(w)^2\,p_{n-1}^2(w,x_{kn}(w)).$$

Since $\lim[\gamma_{n-1}(w)/\gamma_n(w)]=1/2$ (cf. (4.11)), we can use (4.4) and (4.8) to obtain

$$[1-x_{kn}(w)]^{-c}[x-x_{kn}(w)]^2\ell_{kn}^2(w,x)\,p_n^{-2}(w,x)\sim$$
$$\sim n^{-2}[1-x_{kn}(w)]^{-c+a+3/2}[1+x_{kn}(w)]^{b+3/2}$$

for $n\in\mathbf{N}$, and then (4.21) follows from (4.1) via routine estimates.

Lemma 4.6. *Let* $w=gw^{(a,b)}$ *be a generalized Jacobi weight function, and let* $0<\varepsilon<1$. *Then*

$$\sup_{n\geq 1}\max_{-\varepsilon\leq x\leq\varepsilon}\sum_{k=1}^{n}[1-x_{kn}(w)]^{-c}|v_{kn}(w,x)|\ell_{kn}^2(w,x)<\infty \tag{4.22}$$

for $c\leq a+3/2$ *and*

$$\lim_{n\to\infty}\max_{-\varepsilon\leq x\leq\varepsilon}\sum_{k=1}^{n}[1-x_{kn}(w)]^{-c}|x-x_{kn}(w)|\ell_{kn}^2(w,x)=0 \tag{4.23}$$

for $c<a+5/2$.

Proof. First let $c=0$. For $c=0$ formula (4.23) was proved in [24, Lemma 4, (36), p. 40]. The proof of (4.22) with $c=0$ is based on

$$\sup_{n\geq 1}\max_{-\varepsilon\leq x\leq\varepsilon}\sum_{k=1}^{n}\ell_{kn}^2(w,x)<\infty \tag{4.24}$$

which was verified in [24, Lemma 4, (35), p. 40]. We write

(4.25)

$$\sum_{k=1}^{n}|v_{kn}(w,x)|\ell_{kn}^2(w,x)=\sum_{2|x_{kn}|<1+\varepsilon}|v_{kn}(w,x)|\ell_{kn}^2(w,x)+\sum_{2|x_{kn}|\geq 1+\varepsilon}|v_{kn}(w,x)|\ell_{kn}^2(w,x).$$

By Freud's formula (3.9) and by Lemma 4.3 (cf. (4.18)), $|v_{kn}(w,x)|\leq K$ for $2|x_{kn}|<1+\varepsilon$ and $-\varepsilon\leq x\leq\varepsilon$. Hence (4.24) can be used to estimate the first sum on the right-hand side of (4.25). For $2|x_{kn}|\geq 1+\varepsilon$ and $-\varepsilon\leq x\leq\varepsilon$ we can apply again (3.9) and (4.18) to obtain $|v_{kn}(w,x)|\leq K[1-x_{kn}(w)^2]^{-1}$. Now, in view of (4.4) and (4.8), $\lambda_{kn}(w)p_{n-1}^2(w,x_{kn}(w))[1-x_{kn}(w)^2]^{-1}\sim n^{-1}$. Therefore, the Gauss—Jacobi quadrature formula (cf. [32, Theorem 3.4.1, p. 47]), (3.6) and (4.11) yield

$$\sum_{2|x_{kn}|\geq 1+\varepsilon}|v_{kn}(w,x)|\ell_{kn}^2(w,x)\leq Kn^{-1}p_n^2(w,x)\sum_{2|x_{kn}|\geq 1+\varepsilon}\lambda_{kn}(w)\leq$$
$$\leq Kn^{-1}p_n^2(w,x)\sum_{k=1}^{n}\lambda_{kn}(w)=Kn^{-1}p_n^2(w,x)\int_{\mathbf{R}}w$$

for $-\varepsilon\leqq x\leqq\varepsilon$ and $n\in\mathbf{N}$. By (4.3) the generalized Jacobi polynomials are uniformly bounded for $-\varepsilon\leqq x\leqq\varepsilon$. Therefore, the second sum on the right-hand side of (4.25) converges to 0 as $n\to\infty$ uniformly for $-\varepsilon\leqq x\leqq\varepsilon$. Consequently (4.22) and (4.23) hold for $c=0$ which naturally implies their validity for all $c<0$ as well. The extension of (4.22) and (4.23) to all permissable values of c is done via Lemma 4.5 as follows. We write

$$\sum_{k=1}^{n}[1-x_{kn}(w)]^{-c}|v_{kn}(w,x)|\ell_{kn}^2(w,x)=\sum_{2|x_{kn}|<1+\varepsilon}[1-x_{kn}(w)]^{-c}|v_{kn}(w,x)|\ell_{kn}^2(w,x)+$$
$$+\sum_{2|x_{kn}|\geqq 1+\varepsilon}[1-x_{kn}(w)]^{-c}|v_{kn}(w,x)|\ell_{kn}^2(w,x).$$

To prove (4.22), we can estimate the first sum on the right-hand side here by (4.22) applied with $c=0$, whereas for the second sum Lemma 4.5 can be used in the following way. First, we can assume that $c>a+1$. Second, we do not need to concern ourselves with $p_n^2(w,x)$ since by (4.2) it is uniformly bounded in the interval $[-\varepsilon,\varepsilon]$. Thirdly, we note as before that $|v_{kn}(w,x)|\leqq K[1-x_{kn}(w)^2]^{-1}$ (cf. (3.9) and (4.18)). Thus applying (4.21) with $c+1$ instead of c, inequality (4.22) follows. Formula (4.23) can be proved in a similar way from (4.21) applied with c and then from (4.23) applied with $c=0$.

Lemma 4.7. *Let $w=gw^{(a,b)}$ be a generalized Jacobi weight function, and let $-1<\varepsilon<1$. Then for every nonnegative c we have*

$$\text{(4.26)}\qquad \sup_{n\geqq 1}\max_{\varepsilon\leqq x\leqq 1}(1-x)^c\sum_{k=1}^{n}[1-x_{kn}(w)]^{-c}\ell_{kn}^2(w,x)<\infty$$

if $c-5/2<a<c$,

$$\text{(4.27)}\qquad \sup_{n\geqq 1}\max_{\varepsilon\leqq x\leqq 1}(1-x)^c\sum_{k=1}^{n}[1-x_{kn}(w)]^{-c-1}|x-x_{kn}(w)|\ell_{kn}^2(w,x)<\infty$$

if $c-3/2\leqq a<c$, and

$$\text{(4.28)}\qquad \lim_{n\to\infty}\max_{\varepsilon\leqq x\leqq 1}(1-x)^c\sum_{k=1}^{n}[1-x_{kn}(w)]^{-c}|x-x_{kn}(w)|\ell_{kn}^2(w,x)=0$$

if $c-5/2<a<c$.

Proof. Unfortunately, we were unable to find a nontechnical proof, not even one with partially soft features. On the other hand, the computation yielding (4.26)—(4.28) is totally routine, and thus we can (and must) save the reader from the details. Instead, we provide a few hints and instructions as to the nature of the computations. Thus, let $c\geqq 0$ satisfy the appropriate conditions. First, by Lemma 4.6 we can assume $\varepsilon=1/2$. Second, in view of Lemma 4.5 and inequality (4.2), one needs to consider only those values of k in (4.26)—(4.28) for which $x_{kn}(w)$ is positive. Third, since

$$\ell_{mn}^2(w, x) \leq \lambda_{mn}(w) \sum_{k=1}^{n} \ell_{kn}^2(w, x)\lambda_{kn}(w)^{-1} = \lambda_{mn}(w)\lambda_n^{-1}(w, x)$$

(cf. (3.3) and [7, formula (1.4.7), p. 25]), we have by (4.1), (4.7) and (4.8)

$$\sup_{n\geq 1} \max_{\varepsilon\leq x\leq 1} (1-x)^c[1-x_{mn}(w)]^{-c}\ell_{mn}^2(w, x) < \infty,$$

$$\sup_{n\geq 1} \max_{\varepsilon\leq x\leq 1} (1-x)^c[1-x_{mn}(w)]^{-c-1}|x-x_{mn}(w)|\ell_{mn}^2(w, x) < \infty$$

and

$$\lim_{n\to\infty} \max_{\varepsilon\leq x\leq 1} (1-x)^c[1-x_{mn}(w)]^{-c}|x-x_{mn}(w)|\ell_{mn}^2(w, x) = 0$$

for all nonnegative c. Here (and in what follows) m is the index of one of the zeros $x_{kn}(w)$ which are closest to x. Hence it is sufficient to estimate the sums in (4.26)—(4.28) for which $x\geq 1/2$, $x_{kn}(w)>0$ and $k\neq m$. For such values of x and $x_{kn}(w)$ we can use (4.1) to verify $(1-x)\leq K(m/n)^2$, $(1-x_{kn}(w))\sim(k/n)^2$ and $|x-x_{kn}(w)|\sim\sim|m^2-k^2|n^{-2}$. Moreover, in view of expression (3.6) for the fundamental polynomials $\ell_{kn}(w)$, we also need inequalities for $\gamma_{n-1}(w)/\gamma_n(w)$, $\lambda_{kn}(w)$, $|p_{n-1}(w, x_{kn}(w))|$ and $|p_n(w, x)|$. The required estimates are given by formulas (4.11), (4.8), (4.4) and (4.2), respectively (cf. (4.1) as well). Putting all the pieces together, the proof of the lemma is reduced to showing

$$\sup_{n\geq 1} \max_{1\leq m\leq n} m^{-2a+2c-1} \sum_{\substack{k=1\\k\neq m}}^{n} k^{2a-2c+3}|m^2-k^2|^{-2} < \infty \tag{4.29}$$

if $c-5/2<a<c$,

$$\sup_{n\geq 1} \max_{1\leq m\leq n} m^{-2a+2c-1} \sum_{\substack{k=1\\k\neq m}}^{n} k^{2a-2c+1}|m^2-k^2|^{-1} < \infty \tag{4.30}$$

if $c-3/2\leq a<c$, and

$$\lim_{n\to\infty} \max_{1\leq m\leq n} n^{-2}m^{-2a+2c-1} \sum_{\substack{k=1\\k\neq m}}^{n} k^{2a-2c+3}|m^2-k^2|^{-1} = 0 \tag{4.31}$$

if $c-5/2<a<c$. Estimating sums such as the ones in (4.29)—(4.31) is a routine exercise, and it is easily accomplished via splitting up the range of the index k into four subsets given by the inequalities $1\leq k\leq[m/2]$, $[m/2]<k<m$, $m<k<2m$ and $2m\leq k\leq n$. Or, as an alternative, one can apply [19, Lemma 6.3, p. 109] from which (4.29)—(4.31) follow immediately.

5. Underlying ideas (Part I)

Even though a significant portion of results concerning Hermite—Fejér interpolation is proved via hard analysis, such an approach is not always capable of producing the right result. For instance, if one tries to prove the uniform boundedness of the Hermite—Fejér interpolating polynomials associated with the zeros of Legendre polynomials in $[-1, 1]$ by splitting up the interpolating polynomials and by attempting to prove the uniform boundedness of $\sum |x-x_{kn}|(1-x_{kn}^2)^{-1}\ell_{kn}^2(x)$ and $\sum \ell_{kn}^2(x)$ which comes to one's mind when examining (3.7) and (4.17) with $a=b=0$, then one is destined to fail since the maximums of the latter two expressions are of precise order $\log n$, and thus they are unbounded. In other words, Proposition 1.2 with $a=b=0$ holds for more delicate reasons. These reasons are of the soft variety related to the positivity of the operator sequence $\{H_n(w^{(0,0)})\}$. Since for generalized Jacobi weights of the form $w=gw^{(0,b)}$ both sequences

$$\sum |x-x_{kn}(w)|(1-x_{kn}(w)^2)^{-1}\ell_{kn}^2(w, x)$$

and $\sum \ell_{kn}^2(w, x)$ are also unbounded on $[\varepsilon, 1]$, one is forced again into finding a more sensible and sensitive approach to estimating $\{H_n(w,f)\}$. This is the subject of this section, and we will accomplish it via soft analysis which is based on some quasi-positivity properties of the former sequence.

Theorem 5.1. *Let* $w=gw^{(0,b)}$ *be a generalized Jacobi weight function, and let* $-1<\varepsilon<1$. *Then*

$$\sup_{n\geqq 1}\max_{\varepsilon\leqq x\leqq 1}|H_n(w,f,x)|<\infty$$

for every function f bounded in $[-1, 1]$.

Proof. According to (3.7) we have to prove

$$\sup_{n\geqq 1}\max_{\varepsilon\leqq x\leqq 1}\sum_{k=1}^{n}|v_{kn}(w,x)|\ell_{kn}^2(w,x)<\infty. \tag{5.1}$$

Step 1. Here we will show quasi-positivity of $v_{kn}(w)$ in some sense which helps to reduce the Lebesgue function in (5.1) to an expression which can be subjected to rougher handling without ruining its essential behavior. Our main tool is Lemma 4.4 applied with $a=0$ according to which

$$v_{kn}(w, 1)\geqq(1+b)(1-x_{kn}(w))(1+x_{kn}(w))^{-1}-K(1-x_{kn}(w)). \tag{5.2}$$

Hence there exists $d\in[-1, \varepsilon)$ such that $v_{kn}(w, 1)\geqq 1$ for $-1<x_{kn}(w)\leqq d$. But $v_{kn}(w)$ is a linear function which takes the value 1 at $x_{kn}(w)$. Consequently,

$$v_{kn}(w, x)\geqq 1 \quad \text{for} \quad -1<x_{kn}(w)\leqq d \quad \text{and} \quad x\in[\varepsilon, 1]. \tag{5.3}$$

If $d<x_{kn}(w)<1$ then by (5.2)

$$v_{kn}(w,1)\geqq -K(1-x_{kn}(w)),$$

and by (3.9), (4.8) and (4.10) we have $|v'_{kn}(w,x)|\leqq K(1-x_{kn}(w))^{-1}$. Therefore we obtain

$$v_{kn}(w,x)\geqq -\tilde{K}(1-x_{kn}(w))-$$
$$-\tilde{K}(1-x)(1-x_{kn}(w))^{-1} \quad \text{for} \quad d<x_{kn}(w)<1 \quad \text{and} \quad x\in[\varepsilon,1] \tag{5.4}$$

with an appropriate positive constant $\tilde{K}$. Now by (5.3) and (5.4) we have

$$\sum_{k=1}^{n}|v_{kn}(w,x)|\ell_{kn}^2(w,x)=\sum_{x_{kn}\leqq d}|v_{kn}(w,x)|\ell_{kn}^2(w,x)+\sum_{x_{kn}>d}|v_{kn}(w,x)|\ell_{kn}^2(w,x)\leqq$$
$$\leqq \sum_{x_{kn}\leqq d}v_{kn}(w,x)\ell_{kn}^2(w,x)+\sum_{x_{kn}>d}v_{kn}(w,x)\ell_{kn}^2(w,x)+$$
$$+2\tilde{K}\sum_{x_{kn}>d}(1-x_{kn}(w))\ell_{kn}^2(w,x)+2\tilde{K}(1-x)\sum_{x_{kn}>d}(1-x_{kn}(w))^{-1}\ell_{kn}^2(w,x)=$$
$$=1+2\tilde{K}\sum_{kn^{x}>d}(1-x_{kn}(w))\ell_{kn}^2(w,x)+2\tilde{K}(1-x)\sum_{x_{kn}>d}(1-x_{kn}(w))^{-1}\ell_{kn}^2(w,x)$$

since Hermite—Fejér interpolation preserves the constant function. Using the asymptotics for the Cotes numbers (4.8) we obtain from here

$$\sum_{k=1}^{n}|v_{kn}(w,x)|\ell_{kn}^2(w,x)\leqq 1+Kn^{-1}\sum_{k=1}^{n}(1-x_{kn}(w))^{3/2}\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x)+$$
$$+Kn^{-1}(1-x)\sum_{k=1}^{n}(1-x_{kn}(w))^{-1/2}\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x) \tag{5.5}$$

which is the inequality we were to establish in the first step of the proof.

Step 2. The first sum on the right-hand side of (5.5) can be estimated by applying the same techniques that led to (4.26) in Lemma 4.7. However, we will proceed in a different way which consists of evaluating the sum $\sum(1-x_{kn}(w))\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x)$ in a closed form. We have

$$\sum_{k=1}^{n}(1-x_{kn}(w))\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x)=(1-x)\sum_{k=1}^{n}\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x)+$$
$$+\sum_{k=1}^{n}(x-x_{kn}(w))\lambda_{kn}(w)^{-1}\ell_{kn}^2(w,x).$$

Here the first sum on the right-hand side equals $\lambda_n^{-1}(w,x)$ (cf. [7, formula (1.4.7), p. 25]), whereas the second one can be obtained from (3.6) and the Lagrange interpolation formula. We get

$$\sum_{k=1}^{n}(1-x_{kn}(w))\lambda_{kn}(w)^{-1}\ell_{kn^2}(w,x)=$$
$$=(1-x)\lambda_n^{-1}(w,x)+(\gamma_{n-1}(w)/\gamma_n(w))p_n(w,x)p_{n-1}(w,x).$$

Thus, applying (4.2), (4.7) and (4.11) we obtain

$$\sup_{n\ge 1}\max_{\varepsilon\le x\le 1} n^{-1}\sum_{k=1}^{n}(1-x_{kn}(w))\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)<\infty$$

from which the inequality

$$\sup_{n\ge 1}\max_{\varepsilon\le x\le 1} n^{-1}\sum_{k=1}^{n}(1-x_{kn}(w))^{3/2}\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)<\infty \tag{5.6}$$

follows as well.

Step 3. The uniform boundedness of the second sum on the right-hand side of (5.5) was established in Lemma 4.7 (cf. (4.26)). This can also be shown via replacing computations by some properties of Christoffel functions as follows. By Cauchy's inequality

$$\Big[\sum_{k=1}^{n}(1-x_{kn}(w))^{-1/2}\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)\Big]^{2}\le\sum_{k=1}^{n}\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)\times$$

$$\times\sum_{k=1}^{n}(1-x_{kn}(w))^{-1}\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)=\lambda_{n-1}(w,x)\lambda_{n-1}(\tilde{w},x)$$

(here $\tilde{w}$ is defined by $\tilde{w}(x)=(1-x)w(x)$) where we used two identities involving Christoffel functions (cf. [7, formula (1.4.7), p. 25], [6, Lemma 2, formula (15), p. 251] and [19, Lemma 6.1.4, p. 59]). Since both w and $\tilde{w}$ are generalized Jacobi weights, we can use (4.7) to obtain

$$\sup_{n\ge 1}\max_{\varepsilon\le x\le 1} n^{-1}(1-x)\sum_{k=1}^{n}(1-x_{kn}(w))^{-1/2}\lambda_{kn}(w)^{-1}\ell_{kn}^{2}(w,x)<\infty. \tag{5.7}$$

Inequality (5.1) follows from (5.5)—(5.7), and so does the theorem.

6. Underlying ideas (Part II)

Here we will be concerned about the connection between uniform convergence of Hermite—Fejér interpolation and its behavior at one single point. In other words, we look behind the scenes that govern the phenomenon described in Theorem 5.1.

For s nonnegative integer define the function u_s by $u_s(x)=(1-x)^s$. Then it turns out that under certain circumstances it is more convenient to approximate $f\in C[-1,1]$ by $u_sH_n(w,fu_s^{-1})$ then by $H_n(w,f)$. Since the former vanishes at $x=1$ for $s>0$, it can only approximate such functions f which also vanish at $x=1$. What is against $H_n(w,f)$ is that if $w(1)=0$ then the sequence of the corresponding Lebesgue functions becomes unbounded at 1 and thus $\lim H_n(w,f)=f$ cannot be expected at the point 1 or uniformly in a neighborhood of 1. What $u_s H_n(w, fu_s^{-1})$ does is that it tempers the quick growth of $p_n(w)$ in such a way that and operator $U_s H_n(w) U_s^{-1}$ (where U_s^{-1} is the multiplication operator defined by the formula $U_s(g)=u_s g$) becomes appropriately balanced with the right choice of s.

The real role of $u_s H_n(w, fu_s^{-1})$ is that it is the principal term in the Hermite—Fejér type interpolating polynomial $H_{n,s}(w,f)$ defined by

$$H_{n,s}(w,f,x_{kn}(w)) = f(x_{kn}(w)), \quad H'_{n,s}(w,f,x_{kn}(w)) = 0,$$

$k=1, 2, \ldots, n$, and

$$H_{n,s}^{(j)}(w,f,1) = 0$$

for $j=0, 1, \ldots, s-1$. The closed formula for $H_{n,s}(w,f)$ is given by

$$H_{n,s}(w,f) = u_s H_n(w, fu_s^{-1}) + u_s \mathscr{H}_n(w, f[u_s^{-1}]') \tag{6.1}$$

(cf. (3.7) and (3.11)) which is easy to verify directly (cf. [38, Section 3.2, p. 88]). It was E. Egerváry and P. Turán [3] who first realized how $H_{n,1}(w,f)$ can be used to investigate uniform convergence of $H_n(w,f)$ for the Legendre weight function $w=w^{(0,0)}$. The process $H_{n,s}(w^{(a,b)},f)$ was fully investigated in [38] where it was shown that it can be used to prove necessary and sufficient conditions for uniform convergence of Hermite—Fejér interpolation at zeros of Jacobi polynomials. The reason for the usefulness of $u_s H_n(w, fu_s^{-1})$ and $H_{n,s}(w,f)$ lies in the representation

$$H_n(w,f,x) = H_{n,s}(w,f,x) + p_n^2(w,x) \sum_{k=0}^{s-1} (1/k!)[H_n(w,f,1)p_n^{-2}(w,1)]^{(k)}(x-1)^k \tag{6.2}$$

which provides a direct link between $H_n(w,f)$, $u_s H_n(w, fu_s^{-1})$, $p_n(w)$ and $H_n(w,f,1)$. The verification of (6.2) is again easily done by checking out the interpolation conditions. The following is not only a tool necessary for proving one our main results (cf. Theorem 2.5) but the special case $s=1$ is also a de facto solution of P. Turán's Problem XXVII in his collection of "On some open problems of approximation theory" (cf. [35, p. 47]).

Theorem 6.1. *Let $w=gw^{(a,b)}$ be a generalized Jacobi weight function, and let $-1<\varepsilon<1$. Let f be continuous in $[-1, 1]$ such that $f(1)=0$. Let $a\geq 0$, $b>-1$, and let s be a fixed positive integer such that $a\in[s-1, s)$. Then*

$$\lim_{n\to\infty} \max_{\varepsilon\leq x\leq 1} |f(x)-u_s(x)H_n(w, fu_s^{-1}, x)| = 0 \tag{6.3}$$

and

$$\lim_{n\to\infty} \max_{\varepsilon\leq x\leq 1} |f(x)-H_{n,s}(w,f,x)| = 0. \tag{6.4}$$

Proof. In view of (3.11), (6.1) and Lemma 4.7 we have

$$\lim_{n\to\infty} \max_{\varepsilon\le x\le 1} |H_{n,s}(w,f,x)-u_s(x)H_n(w,fu_s^{-1},x)| = 0$$

so that it is sufficient to prove (6.3).

Step 1. First we prove (6.3) for the special case when the function f is given by $f(x)=1-x$. Then

$$f(x)-u_s(x)H_n(w,fu_s^{-1},x) = u_1(x)\big[1-u_{s-1}(x)H_n(w,(u_{s-1})^{-1},x)\big]$$

so that applying (6.1) and (6.2) with $f\equiv 1$ and $s-1$ we obtain

$$f(x)-u_s(x)H_n(w,fu_s^{-1},x) = u_1(x)\big[1-H_{n,s-1}(w,1,x)+u_{s-1}(x)\mathscr{H}_n(w,[u_{s-1}^{-1}]',x)\big]=$$

$$= u_1(x)p_n^2(w,x)\sum_{k=0}^{s-2}(1/k!)[p_n^{-2}(w,1)]^{(k)}(x-1)^k+u_1(x)\big[u_{s-1}(x)\mathscr{H}_n(w,[u_{s-1}^{-1}]',x)\big].$$

Here the first term on the right-hand side can be estimated using (4.2) and (4.6), while the second one by Lemma 4.7 (cf. (3.11) and (4.28) applied with $c=s$). This proves (6.3) for $f=u_1$.

Step 2. Now let f be continuous and $f(1)=0$. The point is that the sequence of operators from $C[-1,1]$ into $C[\varepsilon,1]$ given by $f\to u_sH_n(w,fu_s^{-1})$ is uniformly bounded by (3.7), (3.9), (4.8), (4.10) and Lemma 4.7 (cf. (4.26) and (4.27) applied with $c=s$). Therefore we can finish the proof in the routine fashion as follows. Given $\delta>0$ there exists a polynomial P such that $P(1)=0$ and $|f(x)-P(x)|\le\delta$ for $x\in[-1,1]$ (cf. [34, Theorem 2, p. 259]). Write $P=u_1Q$. With this polynomial P we have

$$(6.5)\quad f(x)-u_s(x)H_n(w,fu_s^{-1},x) = [f(x)-P(x)]-u_s(x)H_n(w,(f-P)u_s^{-1},x)-$$

$$-u_s(x)H_n(w,[Q-Q(x)][u_{s-1}]^{-1},x)+Q(x)[u_1(x)-u_s(x)H_n(w,u_1u_s^{-1},x)].$$

By (3.7), (3.9), (4.8), (4.10) and Lemma 4.7 (cf. (4.28) applied with $c=s$)

$$\lim_{n\to\infty}\max_{\varepsilon\le x\le 1}\big|u_s(x)H_n(w,[Q-Q(x)][u_{s-1}]^{-1},x)\big| = 0$$

since $|Q(x_{kn})-Q(x)|\le K|x_{kn}-x|$, whereas the last term on the right-hand side was taken care of in the first part of the proof. Therefore letting $n\to\infty$ in (6.5) we obtain

$$\limsup_{n\to\infty}\max_{\varepsilon\le x\le 1}|f(x)-u_s(x)H_n(w,fu_s^{-1},x)| \le K\delta$$

from which (6.3) follows.

7. The proofs

On the basis of the results in Sections 4—6 this can be accomplished virtually in a few lines.

Proof of Theorem 2.1. This follows from Lemma 4.6 applied with $c=0$. The details are as follows. By (3.7) and (4.22) the sequence of Hermite—Fejér interpolating polynomials is a bounded sequence of operators from $C[-1, 1]$ to $C[-\varepsilon, \varepsilon]$. By (3.10), (3.11) and (4.23) it converges for polynomials, that is for a dense set of function in $C[-1, 1]$.

Proof of Theorem 2.2. This was de facto proved in [24, Lemma 5, formula (46), p. 43] where it is given with n^{-2a} replaced by $np_n^{-2}(w, 1)$. However, in view of (4.3), they are of the same order.

Proof of Theorem 2.3. First let $w(1)=\infty$. Then by (3.7), (3.9), (4.2), (4.8), (4.10), Lemma 4.5 and Lemma 4.7 (cf. (4.26) and (4.27) applied with $c=0$) the Hermite—Fejér interpolating polynomials are uniformly bounded in $[\varepsilon, 1]$ (here inequality (4.2) and Lemma 4.5 are needed to estimate the expression $\sum [1+x_{kn}(w)]^{-1}[x-x_{kn}(w)]\ell_{kn}^2(w, x)$). If $0<w(1)<\infty$ then this is given in Theorem 5.1. The necessity of the condition $w(1)\neq 0$ follows from Theorem 2.2.

Proof of Theorem 2.4. If $w(1)=\infty$ then by formulas (3.7), (3.10), (3.11) and Lemma 4.7 (cf. (4.28) applied with $c=0$) the Hermite—Fejér interpolating polynomials $H_n(w, P)$ converge uniformly in $[\varepsilon, 1]$ for every fixed polynomial P. Thus Theorem 2.3 yields convergence for every continuous function. The necessity of the condition $w(1)=\infty$ for uniform convergence in $[\varepsilon, 1]$ follows from Theorem 2.2.

Proof of Theorem 2.5. If $\lim H_n(w,f)=f$ uniformly in a left neighborhood of the point 1 then by Markov's theorem (cf. [16, § VI.6, p. 141]) the r-th derivative of $H_n(w,f)$ is $o(n^{2r})$ in the same interval for every $r=1, 2, \ldots$. On the other hand, if we have information concerning the behavior of $H_n(w,f)$ at 1 then we can use Theorem 6.1 (either of (6.3) and (6.4)) and formulas (6.1) and (6.2). First, we can assume without loss of generality that $f(1)=0$ (cf. (3.7), (3.10) and (3.11)). We need to prove

$$\lim_{n\to\infty} \max_{[\varepsilon,1]} p_n^2(w, x) \sum_{k=0}^{s-1} (1/k!)[H_n(w,f, 1)p_n^{-2}(w, 1)]^{(k)}(x-1)^k = 0.$$

This follows immediately by straightforward application of inequalities (4.2), (4.6) and the conditions $H_n^{(r)}(w,f, 1)=O(n^{2r})$, $r=0, 1, \ldots, s$.

Proof of Theorem 2.6. We use an observation by G. Freud in [9, formula (2), p. 176] according to which since $H_n'(w,f)$ vanishes at the zeros of $p_n(w)$ we have $H_n'(w,f)=p_n(w)Q_{n-2}$ where Q_{n-2} is a polynomial of degree at most $n-2$. Thus by orthogonality

$$\int_{-1}^{1} H_n'(w,f,t)[w(t)(1+t)]\,dt = 0,$$

and integration by parts yields

$$H_n(w,f,1) = (2w(1))^{-1} \int_{-1}^{1} H_n(w,f,t)\,d[w(t)(1+t)]$$

(cf. [9, formula (4), p. 176]). Now we can use Proposition 1.6 applied with $u=w$ to pass to the limit of the integral which together with Theorem 2.5 proves Theorem 2.6.

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