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On additive functions taking values from a compact group

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1. Let G be a metrically compact Abelian topological group, T be the one-dimensional torus. A function $\varphi\colon \mathbf{N}\to G$ will be called additive if $\varphi(mn)=\varphi(m)+\varphi(n)$ holds for every coprime pairs m, n of natural numbers, while if $\varphi(mn)=\varphi(m)+\varphi(n)$ holds for each couple of $m, n\in\mathbf{N}$ then we say that it is completely additive. Let $\mathscr{A}_G$, $\mathscr{A}_G^*$ be the class of additive, and the class of completely additive functions, respectively.

Let $\{x_\nu\}_{\nu=1}^{\infty}$ be an infinite sequence in G. We shall say that it is of property D, if for any convergent subsequence x_{ν_n} the shifted subsequence x_{ν_n+1} has a limit, too. We say that it is of property Δ if $x_{\nu+1}-x_\nu\to 0$ $(\nu\to\infty)$.

Let $\mathscr{A}_G(D)$, $\mathscr{A}_G(\Delta)$ be the set of those $\varphi\in\mathscr{A}_G$ for which the sequence $\{x_n=\varphi(n)\}$ is a property D, Δ, respectively. The classes $\mathscr{A}_G^*(D)$, $\mathscr{A}_G^*(\Delta)$ are defined as follows:

$$\mathscr{A}_G^*(D)=\mathscr{A}_G(D)\cap\mathscr{A}_G^*,\quad \mathscr{A}_G^*(\Delta)=\mathscr{A}_G(\Delta)\cap\mathscr{A}_G^*.$$

It is obvious that $\mathscr{A}_G(\Delta)\subseteqq\mathscr{A}_G(D)$, $\mathscr{A}_G^*(\Delta)\subseteqq\mathscr{A}_G^*(D)$. In [1] we proved that $\mathscr{A}_G^*(\Delta)=\mathscr{A}_G^*(D)$. Recently E. WIRSING [4] proved that $\varphi\in\mathscr{A}_T(D)$ if and only if

$$\varphi(n)\equiv\tau\log n \pmod 1\quad (n\in\mathbf{N}) \tag{1.1}$$

for a $\tau\in\mathbf{R}$. By using Wirsing's theorem we proved in [2] the following assertion.

If $\varphi\in\mathscr{A}_G^*(\Delta)$ $\left(=\mathscr{A}_G^*(D)\right)$ then there exists a continuous homomorphism $\psi\colon \mathbf{R}_x\to G$, $\mathbf{R}_x$ denotes the multiplicative group of the positive reals, such that φ is a restriction of ψ on the set $\mathbf{N}$, i.e. $\varphi(n)=\psi(n)$ for all $n\in\mathbf{N}$. The converse assertion is obvious. If $\psi\colon \mathbf{R}_x\to G$ is a continuous homomorphism, then $\varphi(n):=\psi(n)\in\mathscr{A}_G^*(\Delta)\subseteqq\mathscr{A}_G^*(D)$.

We should like to extend our results for the class $\mathscr{A}_G(D)$. This was done in [3] for $G=T$. Our aim in this paper is to characterize the class $\mathscr{A}_G(\Delta)$ for a general metrically compact Abelian group G.

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Let $\mathbf{N}_1$, $\mathbf{N}_0$ be the set of the odd and the even natural numbers, respectively. For a $\varphi\in\mathcal{A}_G$ let $S(\mathbf{N}_j)$ be the set of limit points of $\{\varphi(n)\mid n\in\mathbf{N}_j\}$ $(j=1, 0)$, and let $S(\mathbf{N})$ be the set of limit points of $\{\varphi(n)\mid n\in\mathbf{N}\}$.

Theorem 1. *Let* $\varphi\in\mathcal{A}_G(D)$. *Then* $S(\mathbf{N}_1)$ *is a compact subgroup of* G, $S(\mathbf{N}_0)=\gamma+S(\mathbf{N}_1)$ *with a suitable* $\gamma\in G$. *There exists a continuous homomorphism* $\psi\colon \mathbf{R}_x\to G$ *such that* $\varphi(n)=\psi(n)$, $n\in\mathbf{N}_1$. *The function* $u(n):=\varphi(n)-\psi(n)$ *is zero for* $n\in\mathbf{N}_1$, *and* $u(2)=u(2^\alpha)$ $(\alpha=1, 2, \ldots)$. *If* $u(2)\in S(\mathbf{N}_1)$, *then* $2u(2)=0$.

Conversely, let $\psi\colon \mathbf{R}_x\to G$ *be a continuous homomorphism. Let* $\beta\in G$ *an element for which* $\beta\in\psi(G)$ *implies that* $2\beta=0$. *Let* $u\in\mathcal{A}_G$ *be defined by the relation*

$$u(2^\alpha)=\beta \quad (\alpha=1, 2, \ldots), \quad u(n)=0 \quad \textit{for all} \quad n\in\mathbf{N}_1.$$

Then $\varphi=u+\psi\colon \mathbf{N}\to G$ *belongs to* $\mathcal{A}_G(\Delta)$.

2. To prove our theorem we need some auxiliary results that can be proved by a method that was used by E. Wirsing [4] and in our earlier papers [1], [2].

Lemma 1. *If* $\varphi\in\mathcal{A}_G$ *and*

$$\varphi(m+2)-\varphi(m)\to 0 \quad (m\to\infty,\ m\in\mathbf{N}_1) \tag{2.1}$$

then $\varphi(nm)=\varphi(m)+\varphi(n)$ *for each* $m, n\in\mathbf{N}_1$.

Proof. We need to prove only that

$$\varphi(p^\alpha)-\varphi(p^{\alpha-1})-\varphi(p)=0 \quad (\alpha=1, 2, \ldots) \tag{2.2}$$

for each odd prime p. From (2.1) we get that

$$E_m:=\varphi(p^\alpha m)-\varphi(p^\alpha m-2p)\to 0, \quad F_m:=\varphi(p^{\alpha-1}m)-\varphi(p^{\alpha-1}m-2)\to 0,$$

as $m\in\mathbf{N}_1$, $m\to\infty$. Since for $(m(m+2), 2p)=1$ the relation

$$E_m=\varphi(p^\alpha)-\varphi(p^{\alpha-1})-\varphi(p)+F_m$$

holds, therefore (2.2) is true.

Without any important modification of the proof of Wirsing's theorem one can get

Lemma 2. *If the conditions of Lemma* 1 *are satisfied,* $G=T$, *then* $\varphi(n)\equiv\tau\log n \pmod 1$ *for all* $n\in\mathbf{N}_1$, $\tau\in\mathbf{R}$.

Upon this result, in the same way as in [2] one can prove easily the next

Lemma 3. *Assume that the conditions of Lemma* 1 *hold. Then there exists a continuous homomorphism* $\psi\colon \mathbf{R}_x \to G$ *such that* $\varphi(n)=\psi(n)$ *for each* $n\in \mathbf{N}_1$.

In the next section we shall prove that $\varphi\in\mathcal{A}_G(\Delta)$ implies (2.1).

3. Let us assume that $\varphi\in\mathcal{A}_G(\Delta)$. Let S denote the set of limit points of $\{\varphi(n)\mid n\in \mathrm{N}\}$, i.e. $g\in S$ if there exists $n_1<n_2<\ldots<n_\nu\in \mathrm{N}$, for which $\varphi(n_\nu)\to g$. Let $\varphi(n_\nu+1)\to g'$. In [1] we proved that g' is determined by g. So the correspondence $F\colon g\to g'$ is a function. Furthermore, it is obvious that $F(S)=S$. Let $p(n)$ and $P(n)$ denote the smallest and the largest prime factor of $n\in \mathrm{N}$.

Let k be an arbitrary integer,

$$\mathrm{R} = \{R_1 < R_2 < \ldots\} \tag{3.1}$$

be a sequence of natural numbers. We shall say that R belongs to $\mathcal{P}_k$ if for every $d\in\mathrm{N}$, d divides $R_\nu-k$ for every large ν, i.e. if $\nu>\nu_0(\mathrm{R}, d)$. Let $\tilde{\mathcal{P}}_k\subseteq\mathcal{P}_k$ be the set of those $\mathrm{R}\in\mathcal{P}_k$ for which the limit $\lim\limits_{n\to\infty}\varphi(R_n)$ exists. For an arbitrary sequence R let

$$a(\mathrm{R}) = \lim_{\nu\to\infty}\varphi(R_\nu)$$

if the limit exists. Furthermore, if R is an infinite subsequence of natural numbers increasing monotonically and k is an integer then $\mathrm{R}+k$ denotes the sequence of the positive elements of $R_\nu+k$ written in increasing order. It is obvious that $\mathrm{R}+k\in\mathcal{P}_k$ if and only if $\mathrm{R}\in\mathcal{P}_0$. Furthermore, if $l<k$, $\mathrm{R}\in\tilde{\mathcal{P}}_l$, then $\mathrm{R}+(k-l)\in\tilde{\mathcal{P}}_k$. If $l>k$, then $\mathrm{R}\in\tilde{\mathcal{P}}_l$ implies only that $\mathrm{R}+(k-l)\in\mathcal{P}_k$. In this case we can assert only that there exists a suitable subsequence of $\mathrm{R}+(k-l)$ that belongs to $\tilde{\mathcal{P}}_k$.

Let

$$K_k := \{a(\mathrm{R})\mid \mathrm{R}\in\tilde{\mathcal{P}}_k\}. \tag{3.2}$$

It is obvious that

$$F[K_k] = K_{k+1} \tag{3.3}$$

for every integer k, and that

$$\bigcup_{k=-\infty}^{\infty} K_k \subseteq S. \tag{3.4}$$

Let now $g_1\in K_k$, $g_2\in K_l$, where $k\in\{1,-1\}$. Then there exist $\mathrm{R}\in\tilde{\mathcal{P}}_k$, $\mathrm{S}\in\tilde{\mathcal{P}}_l$ such that $a(\mathrm{R})=g_1$, $a(\mathrm{S})=g_2$. Since $k\in\{1,-1\}$, therefore $p(R_\nu)\to\infty$ $(\nu\to\infty)$. Let now the sequence $Q_\nu=R_{j_\nu}\cdot S_\nu$ be defined as follows: $j_0=0$, $j_\nu>j_{\nu-1}$ such that $p(R_{j_\nu})>P(S_\nu)$. Then $(R_{j_\nu}, S_\nu)=1$, and so $\varphi(Q_\nu)=\varphi(R_{j_\nu})+\varphi(S_\nu)\to g_1+g_2$. But

$Q_\nu \equiv kl \pmod d$ for every $d\in\mathbf{N}$ whenever $\nu>\nu_0(d)$, so $\{Q_\nu\}\in\tilde{\mathscr{P}}_{kl}$, i.e. $g_1+g_2\in K_{kl}$ So we proved

Lemma 4. *For every integer l*

$$K_1+K_l \subseteqq K_l, \tag{3.5}$$

$$K_{-1}+K_l \subseteqq K_{-l}. \tag{3.6}$$

(3.5) gives that $K_1+K_1\subseteqq K_1$, i.e. that K_1 is a semigroup in G. It is clear that K_1 is closed. The closedness of K_1 implies that K_1 is a compact semigroup in G, and so by [5] (9.16) it must be a group.

Lemma 5. *Let $k\in\mathbf{N}$. Then*

$$K_k = K_1+\varphi(k), \quad K_{-k} = K_{-1}+\varphi(k). \tag{3.7}$$

Proof. Let $\tau\in K_k$, $\mathsf{R}\in\tilde{\mathscr{P}}_k$, $a(\mathsf{R})=\tau$. Let $S_\nu:=R_{j_\nu}-k$ be a subsequence of $\mathsf{R}-k$ for which $\mathsf{S}\in\tilde{\mathscr{P}}_0$. Then R_{j_ν} can be written as

$$R_{j_\nu} = k[A_\nu+1], \quad S_\nu = kA_\nu.$$

The sequence $\{A_\nu\}\in\mathscr{P}_0$, therefore $(A_\nu+1, k)=1$ for every large ν, so $\varphi(A_\nu+1)=\varphi(R_{j_\nu})-\varphi(k)$, consequently

$$\varphi(A_\nu+1) \to \tau-\varphi(k)\in K_1.$$

So we proved that $K_k-\varphi(k)\subseteqq K_1$.

Let now $\varrho\in K_1$, $\mathsf{R}\in\tilde{\mathscr{P}}_1$ so that $a(\mathsf{R})=\varrho$. Then the sequence $S_\nu=kR_\nu$ belongs to $\tilde{\mathscr{P}}_k$, $(k, R_\nu)=1$ if ν is large, $\lim\varphi(S_\nu)=\varphi(k)+\lim\varphi(R_\nu)=\varphi(k)+\varrho\in K_k$. This implies that $K_1+\varphi(k)\subseteqq K_k$.

The proof of the second relation of (3.7) is the same, and so we omit it.

Lemma 6. *If $g\in K_{-2}$, then*

$$F[g]+F^2[g] = F^2[g+F^3[g]]. \tag{3.8}$$

Proof. Let us start from the identity $n(n+3)+2=(n+1)(n+2)$. If $(n,3)=1$, then $(n, n+3)=1$, furthermore $(n+1, n+2)=1$ for every $n\in\mathbf{N}$. Let $\{n_\nu\}\in\tilde{\mathscr{P}}_{-2}$ such that $a(\{n_\nu\})=g\in K_{-2}$. Then $3\nmid n_\nu$, consequently $\varphi(n_\nu(n_\nu+3))=\varphi(n_\nu)+\varphi(n_\nu+3)$, $\varphi((n_\nu+1)(n_\nu+2))=\varphi(n_\nu+1)+\varphi(n_\nu+2)$. Since $\varphi(n_\nu+k)\to F^k[g]$ $(k=0, 1, 2, 3)$, we get (3.8) immediately.

Since $0\in K_1$, there exists $\mathsf{R}\in\tilde{\mathscr{P}}_1$, $a(\mathsf{R})=0$. Let $R_{j_\nu}-3$ be a subsequence of $R_\nu-3$ for which the limit $\lim\limits_\nu \varphi(R_{j_\nu}-3)=\eta$ exists. Since $\{R_{j_\nu}-3\}_\nu\in\tilde{\mathscr{P}}_{-2}$, therefore $\eta\in K_{-2}$, and $F^3[\eta]=0$. Let us apply (3.8) with $g=\eta$. Then we get $F[\eta]=0$. Since $\eta\in K_{-2}$, therefore $F[\eta]\in K_{-1}$, consequently $0\in K_{-1}$. Furthermore, $0=F^3[\eta]=F^2[F[\eta]]=F^2[0]$. So we proved

Lemma 7. *We have*

$$F^2[0]=0, \tag{3.9}$$

$$0\in K_{-1}. \tag{3.10}$$

Lemma 8. *We have*

$$K_{-1}=K_1. \tag{3.11}$$

Proof. Put $l=1$ in (3.6). We get $K_{-1}+K_1\subseteq K_{-1}$. Since $0\in K_{-1}$, we deduce that $K_1\subseteq K_{-1}$. Let now $l=-1$. Then $K_{-1}+K_{-1}\subseteq K_1$. Since $0\in K_{-1}$ we get that $K_{-1}\subseteq K_1$. Consequently (3.11) is true.

Since $F^2[K_l]=K_{l+2}$ holds for every integer l, we get that $K_{2n+1}=K_1$ for every $n\in\mathbf{N}$. From (3.7) we get that $\varphi(2n+1)\in K_1$. Consequently $S(\mathbf{N}_1)\subseteq K_1$. On the other hand, it is obvious that $K_1\subseteq S(\mathbf{N}_1)$. So we know that

$$S(\mathbf{N}_1)=K_1. \tag{3.12}$$

Since $F[K_m]=K_{m+1}$, we get that $K_1=K_{2n}$ $(n\in\mathbf{N})$, i.e. that $\varphi(2n)-\varphi(2)\in K_1$ for all $n\in\mathbf{N}$, and so $\varphi(2^\alpha)-\varphi(2)\in K_1$ $(\alpha=1,2,\dots)$. So we get that

$$S=\begin{cases}K_1\cup\{\varphi(2)+K_1\} & \text{if} \quad \varphi(2)\notin K_1,\\ K_1 & \text{if} \quad \varphi(2)\in K_1.\end{cases}$$

Lemma 9. *The function $F\colon S\to S$ is continuous.*

For the proof of this quite obvious assertion see [1].

Lemma 10. *If $g\in K_1$, then*

$$F[g]=g+F[0]. \tag{3.13}$$

If $h\in K_2$, then

$$F^2[h]=h+C, \tag{3.14}$$

where

$$C=\varphi(4)-2\varphi(2)+F[0]. \tag{3.15}$$

Proof. Let $k\in\mathbf{N}_1$, $\mathsf{M}\in\tilde{\mathscr{P}}1$, $a(\mathsf{M})=-\varphi(k)$. Then $(k,M_\nu)=1$, and so $\varphi(kM_\nu)\to 0$, $\varphi(kM_\nu+k)\to F^k[0]=F[0]$. Furthermore, $(k,M_\nu+1)=1$, therefore $\varphi(kM_\nu+k)=\varphi(k)+\varphi(M_\nu+1)$, $\varphi(M_\nu+1)\to F[-\varphi(k)]$. This implies that

$$F[-\varphi(k)]=-\varphi(k)+F[0]. \tag{3.16}$$

$\{\varphi(k)|k\in\mathbf{N}_1\}$, and so $\{-\varphi(k)|k\in\mathbf{N}_1\}$ is everywhere dense in K_1, F is continuous on K_1, therefore (3.13) is true.

Let now $h=\varphi(2)-\varphi(k)$, k and M as above. Then $\varphi(M_\nu)\to-\varphi(k)=h$. Since $2^2|(2M_\nu+2)$, $2^3\nmid(2M_\nu+2)$, we have

$$\varphi(2M_\nu+2)=\varphi(4)-\varphi(2)+\varphi(M_\nu+1),$$

and so that

$$F^2[h]=\varphi(4)-\varphi(2)+F[-\varphi(k)].$$

Since $-\varphi(k)\in K_1$, from (3.13) we get that $F[-\varphi(k)]=-\varphi(k)+F[0]$, and so that $F^2[h]=h+C$, $h=\varphi(2)-\varphi(k)$ with the C defined in (3.15).

Since $\{-\varphi(k)|k\in\mathbf{N}_1\}$ is everywhere dense in K_1, therefore $\{\varphi(2)-\varphi(k)|k\in\mathbf{N}_1\}$ is everywhere dense in K_2, F^2 being a continuous function, we get (3.14) immediately.

For a sequence x_n let $\Delta x_n:=x_{n+1}-x_n$, $\Delta^2x_n:=x_{n+2}-x_n$.

Lemma 11. *We have*

$$\lim_{m\in\mathbf{N}_1}\Delta\varphi(m)=F[0], \tag{3.16}$$

$$\lim_{m\in\mathbf{N}_0}\Delta^2\varphi(m)=C, \tag{3.17}$$

$$\lim_{m\in\mathbf{N}_1}\Delta^2\varphi(m)=0. \tag{3.18}$$

Furthermore, $C=0$.

Proof. Assume that (3.16) is not true. Then there exists a subsequence $2n_\nu+1$ of positive integers such that $\varphi(2n_\nu+2)-\varphi(2n_\nu+1)\to\delta$, $\delta\neq F[0]$. Then for a suitable subsequence $2n_{j_\nu}+1$ there exists the limit $\lim\varphi(2n_{j_\nu}+1)=\alpha\in K_1$, and $F[\alpha]=\alpha+\delta$. This contradicts (3.13).

The proof of (3.17) is the same and so we omit it.

Since $\Delta^2\varphi(2n-1)=\Delta^2\varphi(4n-2)+\Delta^2\varphi(4n)$, from (3.17) we get that

$$\Delta^2\varphi(2n-1)\to 2C. \tag{3.19}$$

Observe that

$$\Delta\varphi(2n-1)-\Delta\varphi(2n-1)=\Delta^2\varphi(2n)-\Delta^2\varphi(2n-1).$$

From (3.16), (3.17), (3.19) we get that $0=F[0]-F[0]=C-2C$, and so that $C=0$. This proves (3.18).

4. We have almost finished the proof. We know that $\Delta^2\varphi(2n-1)\to0$. The condition of Lemma 1 is satisfied. Then, by Lemma 3 there exists a continuous homomorphism $\psi\colon\mathbf{R}_x\to G$ such that $\varphi(n)=\psi(n)$ for all $n\in\mathbf{N}_1$. Let $u(n):=\varphi(n)-$

$-\psi(n)$. Then $u\in\mathcal{A}$, $u(n)=0$ for all $n\in N_1$. Since ψ is continuous, therefore $\psi(n+k)-\psi(n)=\psi(1+k/n)\to 0$ as $n\to\infty$ for every fixed k. From (3.16) we get that $u(2n+2)\to F(0)$ as $n\to\infty$, that is $u(2)=u(2^\alpha)=F[0]$ $(\alpha=1, 2, \ldots)$.

If, in addition, $F[0]\in K_1$, then $S=K_1$, and (3.13) can be applied twice. This gives $F^2[g]=F[F[g]]=F[g+F[0]]=g+2F[0]$, that by $F^2[0]=0$ gives that $2F[0]=0$.

By this the first assertion in our Theorem is proved. The converse is obvious.

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