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One-dimensional perturbations of singular unitary operators

N. G. MAKAROV

Introduction and results. Let $\mathbf{T}$ denote the unit circle and m be the normalized Lebesgue measure on $\mathbf{T}$. Recall that a closed subset e of $\mathbf{T}$ is said to be a *Carleson set* if

$$\int \log [\operatorname{dist}(\zeta, e)]\, \mathrm{dm}(\zeta) > -\infty.$$

These sets arise as sets of nonuniqueness for functions analytic in the unit disc and smooth up to the boundary, see [1]. Also we introduce the class (C_σ) consisting of all countable unions of Carleson sets.

This class plays a crucial role in the description of point spectrum of almost unitary operators acting on a *separable* Hilbert space. It was proved in [3] that if U is a unitary and K is a trace class operators, then

$$\sigma_p(U+K)\cap\mathbf{T}\in(C_\sigma).$$

In the opposite direction, given $e\in(C_\sigma)$, there is a one-dimensional perturbation of the shift operator $f(z)\mapsto zf(z)$ on $L^2\equiv L^2(m)$ with point spectrum equal to e.

It is not immediatly clear from the proof whether the appearing of an uncountable point spectrum relies on the absolutely continuous properties of the unitary operator. The question seems also natural from the viewpoint of spectral analysis of general noncontractive operators (cf. [4]), and it was stated in [2] p. 120 as a reseach problem. In the present paper we give an answer to this question.

A unitary operator is said to be *singular* if its spectral measure is singular with respect to the Lebesgue measure.

Theorem 1. *Let $e\in(C_\sigma)$. There exist a singular unitary operator U and an operator K of rank one such that $e\subset\sigma_p(U+K)$ and, moreover, each point ζ in e is an eigenvalue of $U+K$ having infinite multiplicity (i.e. for any positive integer n,*

$$\ker(U+K-\zeta I)^{n+1}\neq\ker(U+K-\zeta I)^n).$$

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As an application, we consider a question concerning inner functions. By z we denote the identity mapping of the unit disc and by H^2 the standard Hardy space. Let φ_1 and φ_2 be two nonequal inner functions. On which subsets e of $\mathbf{T}$ can such functions "coincide" in the sense that $(z-\zeta)^{-1}(\varphi_1-\varphi_2)\in H^2$ for all $\zeta\in e$? As it follows from Theorem 2 in [3], e has to be of class (C_σ). On the other hand, e is at most countable if, for instance, $\varphi_1=1$. One possible way to see this is as follows.

Assume, for simplicity, that $\varphi(0)=0$, $\varphi\equiv\varphi_2$. Let P denote the orthogonal projection in H^2 onto $H^2\ominus\varphi H^2$. The point $\zeta\in\mathbf{T}$ is an eigenvalue of the unitary operator

$$f\mapsto Pzf+\langle f,\bar{z}\varphi\rangle 1$$

acting on $H^2\ominus\varphi H^2$ if and only if $(z-\zeta)^{-1}(\varphi-1)\in H^2$. Hence, the set of all such points is at most countable.

By similar reasoning, we shall obtain from Theorem 1 the following result.

Theorem 2. *Let $e\in(C_\sigma)$. There exist two nonequal inner functions φ_1 and φ_2 such that for any $\zeta\in e$ and any integer n, the function $(z-\zeta)^{-n}(\varphi_1-\varphi_2)$ belongs to H^2.*

At the same time, the author does not dispose of any explicit construction of such functions.

The proof of both theorems appeals to some properties of almost unitary operators, and thus this work could be considered as an illustration to the theory presented in [4].

Proof of Theorem 1. Fix $e\in(C_\sigma)$. There exists a bounded analytic function h, $h(0)=-1$, satisfying $(z-\zeta)^{-n}h\in H^2$ for all integer n and $\zeta\in e$. In case e is a Carleson set, for h, one can take an infinitely smooth up to the boundary analytic function which vanishes on e together with all its derivatives. For an arbitrary $e\in(C_\sigma)$, one can consider an appropriate product of smooth functions, see [3] for a detailed proof.

Let $w=h+\bar{h}$ and the operator L_0 be defined on L^2 by the equality

$$L_0f = zf+\langle f,\bar{z}\rangle w.$$

If $\zeta\in e$ and $n\in\mathbf{N}$, then $w_{\zeta,n}\equiv(z-\zeta)^{-n}w\in L^2$ and

$$\langle w_{\zeta,n},\bar{z}\rangle = \left\langle\frac{zh}{(z-\zeta)^n},1\right\rangle+\left\langle\frac{\bar{z}^{n-1}\bar{h}}{(1-\bar{z}\zeta)^n},1\right\rangle = \begin{cases}-1, & n=1\\ 0, & n\geqq 2.\end{cases}$$

Therefore,

$$(L_0-\zeta I)w_{\zeta,n} = \begin{cases}0, & n=1\\ w_{\zeta,n-1}, & n\geqq 2\end{cases}$$

and ζ is an eigenvalue of infinite multiplicity. Remark that the operator L_0 is invertible, since otherwise the origin would be an eigenvalue of L_0 and hence $\langle w, 1\rangle=-1$; on the other hand, by construction, $\langle w, 1\rangle=-2$.

Let $E=\operatorname{span}\{\ker(L_0-\zeta I)^n\colon \zeta\in e,\ n\in\mathbf{N}\}$. It is a hyperinvariant subspace of L_0. Consider the imbedding $j\colon E\to L^2$ and define the operator L on E by $L=j^*L_0 j$. Obviously, L is invertible and any $\zeta\in e$ is its eigenvalue of infinite multiplicity. Also consider the one-dimensional operator $K=\langle\cdot, a\rangle b$ with

$$a=\frac{j^*\bar z}{\|j^*\bar z\|},\quad b=La-\frac{L^{*-1}a}{\|L^{*-1}a\|}.$$

(Note that $j^*\bar z\neq 0$ because $\langle w_{\zeta,1}, \bar z\rangle=-1$ for $\zeta\in e$ and so $\bar z$ is not orthogonal to E.) Let $U=L-K$. We shall prove that U is a unitary operator and that it is singular.

If $f\in E$, then

$$Uf=L(f-\langle f, a\rangle a)+\langle f, a\rangle\|L^{*-1}a\|^{-1}L^{*-1}a.$$

Observe that the terms on the right are orthogonal. Hence

$$\|Uf\|^2=\|L(f-\langle f, a\rangle a)\|^2+|\langle f, a\rangle|^2=$$
$$=\|f-\langle f, a\rangle a\|^2+|\langle f, a\rangle|^2=\|f\|^2.$$

Since U is a Fredholm operator of index zero, it is unitary.

To prove the singularity of U, it suffices to verify that for all f and g in E,

(1) $\quad \langle (U-r\eta I)^{-1}f-(U-r^{-1}\eta I)^{-1}f, g\rangle\to 0$ as $r\to 1$ for a.e. $\eta\in\mathbf{T}$,

cf. Proposition 6.7 and Remark 6.10 in [4]. Let $\lambda\notin\sigma(L)\cup\mathbf{T}$ and R_λ denote $(L-\lambda I)^{-1}$. Direct calculation gives $\langle R_\lambda b, a\rangle\neq 1$ and

$$(U-\lambda I)^{-1}=R_\lambda+\langle\cdot, R_\lambda^* a\rangle\big(1-\langle R_\lambda b, a\rangle\big)^{-1}R_\lambda b.$$

Consequently, (1) follows from the corresponding fact concerning L. But the latter is obvious since linear combinations of root vectors of L are dense in E and for $f\in\ker(L-\zeta I)^n$, $\langle R_\lambda f, g\rangle$ is a polynomial in $(\lambda-\zeta)^{-1}$.

Proof of Theorem 2. By Theorem 1, given $e\in(C_\sigma)$ there exists an operator L, one-dimensional perturbation of a singular unitary operator, such that any point in e is its eigenvalue of infinite multiplicity. Without loss of generality, we can assume that L is completely nonunitary, i.e. it has no reducing subspaces on which it is unitary. (Otherwise, L is the direct sum of a unitary and a completely nonunitary operators, and we can take the latter instead of L. Obviously, all the required properties would persist.) Such an operator admits a representation

$$L=T+\Omega_* A\Omega^*$$

where T is a completely nonunitary partial isometry with two-dimensional defect subspaces $\mathfrak{D}=\operatorname{im}(I-T^*T)$ and $\mathfrak{D}_*=\operatorname{im}(I-TT^*)$, $\Omega\colon \mathbf{C}^2\to\mathfrak{D}$ and $\Omega_*\colon \mathbf{C}^2\to\mathfrak{D}_*$ are some unitary operators and A is a (2×2)-matrix, cf. [4] §3.

Let Θ denote the characteristic function of T. Since T is a finite-dimensional perturbation of a singular unitary operator, Θ is an inner function, see [4] §§ 5 and 6. Since T is partially isometric, $\Theta(0)=0$. We shall replace T by its functional model [5]. Thus we shall assume that T acts on $K_\Theta \equiv H^2(\mathbf{C}^2)\ominus \Theta H^2(\mathbf{C}^2)$ by the formula $Tf=Pzf$ where P is the orthogonal projection in $H^2(\mathbf{C}^2)$ onto K_Θ. In this model representation, L is given by

$$Lf = zf-(\Theta-A)x_f, \quad x_f = \langle z\Theta^* f, 1\rangle \in \mathbf{C}^2. \tag{2}$$

Lemma. *Let the operator L be defined on K_Θ by* (2). *If* $\zeta\in\mathbf{T}$, $n\in\mathbf{N}$ *and* $\ker(L-\zeta I)^n \neq \ker(L-\zeta I)^{n-1}$, *then*

$$(z-\zeta)^{-n}\det(\Theta-A)\in H^2. \tag{3}$$

Proof. If $f\neq 0$ is in $\ker(L-\zeta I)$, then $(z-\zeta)f=(\Theta-A)x_1$ for some $x_1\neq 0$ in $\mathbf{C}^2$. Hence, $(z-\zeta)^{-1}(\Theta-A)x_1=f\in H^2(\mathbf{C}^2)$. If $f\in\ker(L-\zeta I)^2\setminus\ker(L-\zeta I)$, then, for some $x_1, x_2\in\mathbf{C}^2$,

$$(z-\zeta)f-(\Theta-A)x_2 = (L-\zeta I)f = (z-\zeta)^{-1}(\Theta-A)x_1, \quad x_1\neq 0,$$

and

$$(\Theta-A)[(z-\zeta)^{-2}x_1+(z-\zeta)^{-1}x_2]\in H^2(\mathbf{C}^2).$$

Proceeding by induction, we obtain

$$(\Theta-A)[(z-\zeta)^{-n}x_1+\ldots+(z-\zeta)^{-1}x_n]\in H^2(\mathbf{C}^2), \quad x_1\neq 0.$$

Let V be an analytic matrix-function such that $V(\Theta-A)=[\det(\Theta-A)]I$. Then

$$(z-\zeta)^{-n}\det(\Theta-A)[x_1+\ldots+(z-\zeta)^{n-1}x_n]\in H^2(\mathbf{C}^2).$$

Because of $x_1\neq 0$, we have (3).

Now we are able to complete the proof of Theorem 2. Let δ_1 denote $\det(\Theta-A)$. By the established lemma, the function $(z-\zeta)^{-n}\delta_1$ belongs to H^2 for any $\zeta\in e_0$ and $n\in\mathbf{N}$. Fix a positive number C greater than $\sup|\delta_1|$. Then $h_1\equiv\delta_1+C$ is an outer function. Let δ denote the inner function $\det\Theta$ and h the function $\det(I-A^*\Theta)+C\delta$. We have $h_1=\delta\bar{h}$. Consider the inner-outer factorization $h=h_i h_0$ of the function h. Since $|h_1|=|h_0|$, we can assume that $h_1=h_0$. Hence $\delta h_i^{-1}=h_1\bar{h}_1^{-1}$ and

$$\delta-h_i = h_i\bar{h}_1^{-1}(\delta_1-\bar{\delta}_1).$$

Therefore, $(z-\zeta)^{-n}(\delta-h_i)$ is in H^2 for all $\zeta\in e$ and $n\in\mathbf{N}$. It remains to observe that $\delta\not\equiv h_i$. Indeed, if it were not so then the last equality would imply that $\delta_1\equiv\text{const}$ $(\neq 0)$ and $(z-\zeta)^{-1}\delta_1\notin H^2$ for any $\zeta\in\mathbf{T}$. The assertion now follows with $\varphi_1=\delta$ and $\varphi_2=h_i$.

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LOMI, FONTANKA 27
191011 LENINGRAD, USSR