Acta Sci. Math., 52 (1988), 437—441

Compact and Fredholm composite multiplication operators

R. K. SINGH and N. S. DHARMADHIKARI

1. Introduction. Let X be a nonempty set and $V(X)$ be a vector space of complex valued functions on X under the pointwise operations of addition and scalar multiplication. Let T be a mapping of X into X such that $f \circ T$ is in $V(X)$ whenever f is in $V(X)$. Define the composition transformation C_T on $V(X)$ as $C_T f = f \circ T$ for every f in $V(X)$. If $V(X)$ has a Banach space structure and C_T is bounded, then C_T is called the composition operator on $V(X)$ induced by T. Let $\theta \colon X \to \mathbf{C}$ be a function such that M_θ, defined as $M_\theta f = \theta \cdot f$ for every f in $V(X)$ is a bounded linear operator on $V(X)$. Then the product $M_\theta C_T$ which becomes a bounded operator on $V(X)$ is called a composite multiplication operator.

The study of composite multiplication operators becomes significant and interesting due to the fact that the class of composite multiplication operators includes composition operators, multiplication operators, weighted composition operators. LAMBERT and QUINN [4] initiated the study of weighted composition process on L^1-space, having resemblence with composite multiplication operators. HADWIN, NORDGREN, RADJAVI and ROSENTHAL [2] proved that there exists on operator belonging to the class of composite multiplication operators, which does not satisfy Lomonosov's hypothesis [5] pertaining to the wellknown invariant subspace problem in operator theory.

In this paper the necessary and sufficient conditions for $M_\theta C_T \in B(L^2(\lambda))$ to be a compact operator and a Fredholm operator are obtained in case $V(X)$ is an L^2-space of a sigma-finite measure space.

By $\mathscr{B}(\mathfrak{H})$, we mean the Banach algebra of all bounded operators on a Hilbert space $\mathfrak{H}$. If $(X, \mathscr{S}, \lambda)$ is a measure space and $T \colon X \to X$ is a measurable transformation such that $C_T \in \mathscr{B}(L^2(\lambda))$, then the measure λT^{-1}, defined as $\lambda T^{-1}(E) = \lambda(T^{-1}(E))$ for every E in $\mathscr{S}$, is absolutely continuous with respect to the measure λ [7]. Let f_0 denote the Radon—Nikodym derivative of λT^{-1} with respect to λ. If $C_T \in \mathscr{B}(L^2(\lambda))$, then $C_T^* C_T = M_{f_0}$ [7]. The symbols Ker A and Ran A denote the

Received July 5, 1985 and in revised form September 26, 1986.

kernel and the range of the operator $A\in\mathscr{B}(\mathfrak{H})$ and Z_δ^θ denotes the closed subspace of $L^2(\lambda)$ consisting of all those functions which vanish outside $X_\delta^\theta=\{x\in X\,|\,|\theta(x)|>\delta\}$. By Z_θ, we mean the set $\{x\in X\,|\,\theta(x)=0\}$ and Z'_θ is the complement of Z_θ. In this paper we consider $(X, \mathscr{S}, \lambda)$ to be a σ-finite measure space.

2. Some basic results. In this section we present some essential results which are often used in the presentation of this paper.

Theorem 2.1. *Let* $C_T\in\mathscr{B}(L^2(\lambda))$. *Then* C_T *has dense range if and only if* $C_TC_T^*=M_{f_0\circ T}$.

Proof. Suppose that C_T has dense range. Then for every f in $L^2(\lambda)$ we have a sequence $\{f_n\}$ with $f=\lim_n C_Tf_n$ and we get

$$C_TC_T^*f = \lim_n C_TC_T^*C_Tf_n = \lim_n C_TM_{f_0}f_n = \lim_n C_T(f_0\cdot f_n) =$$

$$= \lim_n (f_0\circ T)(f_n\circ T) = \lim_n M_{f_0\circ T}C_Tf_n = M_{f_0\circ T}C_Tf.$$

Hence $C_TC_T^*=M_{f_0\circ T}$.

Conversely, let $C_TC_T^*=M_{f_0\circ T}$. Then since $f_0\circ T\neq 0$ [11], we can conclude from Lemma 1.2 of [9] that $M_{f_0\circ T}$ is an injection. Hence C_T^* is an injection. So the fact that $\{0\}=\operatorname{Ker} C_T^*=(\overline{\operatorname{Ran} C_T})^\perp$ proves that C_T has dense range. Hence the proof is complete.

Theorem 2.2. *Let* $M_\theta C_T\in\mathscr{B}(L^2(\lambda))$. *Then* $M_\theta C_T=0$ *if and only if* θ *vanishes on* $T^{-1}(E)$ *almost everywhere whenever* $\lambda(E)<\infty$.

Proof. In case θ vanishes on $T^{-1}(E)$ a.e. whenever $\lambda(E)<\infty$, we get $M_\theta=0$. Hence $M_\theta C_T=0$. For the converse suppose $M_\theta C_T=0$. Since X is σ-finite measure space, we can write $X=\bigcup_{i=1}^\infty E_i$, where $\{E_i\}$ is the sequence of disjoint sets such that $\lambda(E_i)<\infty$ for each i, $1\leqq i<\infty$. Now $M_\theta C_T\chi_{E_i}=0$, i.e. $M_\theta\chi_{T^{-1}(E_i)}=0$. Hence

$$\theta=0 \quad\text{on}\quad T^{-1}(E_i) \quad\text{for each}\quad i,\ 1\leqq i<\infty.$$

3. Compact composite multiplication operators. Let us recall that an operator $A\in\mathscr{B}(\mathfrak{H})$ is compact if $\{Af\colon f\in\mathfrak{H}$ and $\|f\|<1\}$ is a precompact subset of $\mathfrak{H}$. A measure λ is called atomic if every element E of $\mathscr{S}$ with $\lambda(E)\neq 0$ contains an atom. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathfrak{H})$ is transitive if $\mathscr{A}$ is weakly closed, contains the identity operator and $\operatorname{Lat}\mathscr{A}=\{0, \mathfrak{H}\}$ where $\operatorname{Lat}\mathscr{A}=\bigcap_{A\in\mathscr{A}}\operatorname{Lat} A$.

Theorem 3.1. *Suppose* $C_T\in\mathscr{B}(L^2(\lambda))$ *has dense range. Then* $M_\theta C_T\in\mathscr{B}(L^2(\lambda))$ *is compact if and only if* $Z_\delta^{|\theta|^2f_0\circ T}$ *is finite dimensional for every* $\delta>0$.

Proof. The operator $M_\theta C_T$ is compact if and only if $(M_\theta C_T)(M_\theta C_T)^*$ is compact. So by using the Theorem 2.1, the operator $M_\theta C_T$ becomes compact if and only if $M_{|\theta|^2 f_0\circ T}$ is compact. Hence by the Lemma 1.1 of [10], $M_\theta C_T$ is compact if and only if $Z_\delta^{|\theta|^2 f_0\circ T}$ is finite dimensional for every $\delta>0$.

Corollary 3.2. *Let $T\colon N\to N$ be an injection. Then $M_\theta C_T\in\mathcal{B}(l^2(N))$ is compact if and only if $Z_\delta^{|\theta|^2}$ is finite dimensional for every $\delta>0$.*

Proof. Since T is an injection, C_T has dense range [8] and $f_0\circ T=1$. Hence the proof is immediate.

The main theorem on compact composite multiplication operator on $l^2(N)$ is given below.

Theorem 3.3. *Let $M_\theta C_T\in\mathcal{B}(l^2(N))$. Then $M_\theta C_T$ is compact if and only if $\{\theta(n)\}\to 0$ as $n\to\infty$.*

Proof. Suppose $M_\theta C_T$ is compact. Let $\{e^{(n)}\}$ be the sequence defined by $e^{(n)}(m)=\delta_{nm}$, the Kronecker delta. Since $e^{(n)}\to 0$ weakly and $(M_\theta C_T)^*$ is compact we have

$$\|(M_\theta C_T)^* e^{(n)}\| = |\theta(n)|\,\|C_T^* e^{(n)}\| \to 0.$$

Since $\|C_T^* e^{(n)}\|=\|e^{T(n)}\|=1$, we get $\{\theta(n)\}\to 0$ as $n\to\infty$.

The converse is trivial.

Corollary 3.4. *If $\mathcal{A}$ is a transitive algebra of $\mathcal{B}(l^2)$ containing $M_\theta C_T$ such that $\{\theta(n)\}\to 0$ as $n\to\infty$, then $\mathcal{A}=\mathcal{B}(l^2)$.*

Proof. Since $\mathcal{A}$ is a transitive algebra of $\mathcal{B}(l^2)$ and contains the compact operator $M_\theta C_T$, $\mathcal{A}=\mathcal{B}(l^2)$, [6].

Example 3.5. Let $X=N$ and λ be the counting measure. Define $T\colon N\to N$ as $T(n)=\begin{cases} n, & \text{if } n=1 \\ n-1, & \text{if } n\geqq 2\end{cases}$ and define $\theta\colon N\to \mathbf{C}$ as $\theta(n)=1/n^2$. Then $M_\theta C_T\in\mathcal{B}(l^2)$ is compact by an application of the Theorem 3.3.

Theorem 3.6. *Suppose $(X,\mathcal{S},\lambda)$ is a nonatomic measure space and $C_T\in\mathcal{B}(L^2(\lambda))$ has dense range. Then $M_\theta C_T\in\mathcal{B}(L^2(\lambda))$ is compact if and only if $\theta=0$ on $Z'_{f_0\circ T}$.*

Proof. Let $M_\theta C_T$ be compact. Then in view of the Theorem 2.1 $(M_\theta C_T)C_T^*(=M_{\theta\cdot f_0\circ T})$ is compact. Thus $\theta\cdot f_0\circ T=0$ a.e. by a theorem of [10]. If $\theta\neq 0$ on $Z'_{f_0\circ T}$, then $f_0\circ T=0$ on $Z'_{f_0\circ T}$. Hence $f_0\circ T=0$ a.e. This is a contradiction to the fact that $f_0\circ T\neq 0$ a.e. for $C_T\in\mathcal{B}(L^2(\lambda))$ [11]. Hence $\theta=0$ on $Z'_{f_0\circ T}$.

Conversely, if $\theta=0$ on $Z'_{f_0\circ T}$, then $|\theta|^2 f_0\circ T=0$ a.e. Hence the operator

$$M_{|\theta|^2 f_0\circ T}(=(M_\theta C_T)(M_\theta C_T)^*)$$

is compact. This proves that $M_\theta C_T$ is compact.

Theorem 3.7. *Let* $\theta \in L^\infty(\lambda)$ *be such that* $|\theta|=1$ *a.e. and* $M_\theta C_T \in \mathscr{B}(L^2(\lambda))$, *Then* $M_\theta C_T$ *is an injective compact operator only if* X *is an atomic measure space.*

Proof. Since $C_T^* C_T = M_{f_0}$, [7], we get $\operatorname{Ker} M_\theta C_T = \operatorname{Ker} (M_\theta C_T)^*(M_\theta C_T) = \operatorname{Ker} M_{f_0}$. Also the operator $M_\theta C_T$ is compact if and only if $(M_\theta C_T)^*(M_\theta C_T)$ $(=M_{f_0})$ is compact. Since $M_\theta C_T$ is an injective compact operator, we get M_{f_0} to be an injective compact multiplication operator. Then by a result of [10], we conclude that X is an atomic measure space.

Theorem 3.8. *Let* $\theta \in L^\infty(\lambda)$ *be such that* $|\theta|=1$ *a.e. and suppose* $M_\theta C_T \in \mathscr{B}(L^2(\lambda))$. *Then the following are equivalent:*

(i) $M_\theta C_T$ *is compact,*

(ii) C_T *is compact,*

(iii) $Z_\delta^{f_0}$ *is finite dimensional for every* $\delta>0$.

Proof. Obvious.

4. Fredholm composite multiplication operator. Let $\mathscr{C}(\mathfrak{H})$ be the ideal of compact operators in $\mathscr{B}(\mathfrak{H})$ and π be the natural homomorphism from $\mathscr{B}(\mathfrak{H})$ into $\mathscr{B}(\mathfrak{H})/\mathscr{C}(\mathfrak{H})$ which is known as the Calkin algebra. Then an operator $A \in \mathscr{B}(\mathfrak{H})$ is said to be a Fredholm operator if $\pi(A)$ is invertible in $\mathscr{B}(\mathfrak{H})/\mathscr{C}(\mathfrak{H})$.

Atkinson Theorem. [1] *If* $\mathfrak{H}$ *is a Hilbert space, then* $T \in \mathscr{B}(\mathfrak{H})$ *is a Fredholm operator if and only if the range of* T *is closed,* $\dim \ker T$ *is finite and* $\dim \ker T^*$ *is finite.*

Theorem 4.1. *Let* $\theta \in L^\infty(\lambda)$ *be bounded away from zero and* C_T^*, *the adjoint of* $C_T \in \mathscr{B}(L^2(\lambda))$ *be a composition operator. Then* $M_\theta C_T \in \mathscr{B}(L^2(\lambda))$ *is a Fredholm operator if and only if* C_T *is a Fredholm operator.*

Proof. Since $\operatorname{Ker} M_\theta C_T = \operatorname{Ker} C_T$ and $\operatorname{Ker} (M_\theta D_T)^* = \operatorname{Ker} C_T^*$, in the light of Atkinson's theorem it is enough to prove that $M_\theta C_T$ has closed range if and only if C_T has closed range. For this, suppose $M_\theta C_T$ has closed range. Let $f \in \overline{\operatorname{Ran} C_T}$. Then there exists a sequence $\{f_n\}$ in $L^2(\lambda)$ such that $C_T f_n \to f$. Hence $M_\theta C_T f_n \to M_\theta f$. Since $M_\theta C_T$ has closed range, $M_\theta C_T f_n \to M_\theta C_T g$ for some g in $L^2(\lambda)$. Hence $M_\theta f = M_\theta C_T g$. Since M_θ is invertible, $f = C_T g$. This proves that C_T has closed range.
The converse can be proved similarly.

References

[1] R. G. DOUGLAS, *Banach algebra techniques in operator theory*, Academic Press Inc. (New York—London, 1972).

[2] D. W. HADWIN, E. A. NORDGREN, H. RADJAVI and P. ROSENTHAL, An operator not satisfying Lomonosov's hypothesis, *J. Funct. Anal.*, **38** (1980), 410—415.

[3] P. R. HALMOS, *A Hilbert space problem book*, Van Nostrand (Princeton, 1967).

[4] A. LAMBERT and J. QUINN, Invariant measures and weighted composition process on L', *Abstracts Amer. Math. Soc.*, **3** (1982), #792—47—389.

[5] V. LOMONOSOV, Invariant subspaces for operators commuting with compact operators, *Funktsional Anal. i Prilozhen*, **7** (1973), 55—56.

[6] H. RADJAVI and P. ROSENTHAL, *Invariant subspaces*, Springer-Verlag (Berlin, 1973).

[7] R. K. SINGH, Compact and quasinormal composition operators, *Proc. Amer. Math. Soc.*, **45** (1974), 80—82.

[8] R. K. SINGH, Normal and hermitian composition operators, *Proc. Amer. Math. Soc.*, **47** (1975), 348—350.

[9] R. K. SINGH and A. KUMAR, Characterization of invertible, unitary and normal composition operators, *Bull. Austral Math. Soc.*, **19** (1978), 81—95.

[10] R. K. SINGH and A. KUMAR, Compact composition operators, *J. Austral Math. Soc. (series A)*, **28** (1979), 309—314.

[11] R. K. SINGH and T. VELUCHAMY, Non atomic measure spaces and Fredholm composition operators, preprint.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF JAMMU
JAMMU—180 001
J&K STATE, INDIA