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Absolute summability of double orthogonal series

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Dedicated to Professor B. Sz.-Nagy on his 75th birthday

1. Introduction: Summability of numerical series

We consider a quadruply infinite matrix

$$T = \{t_{ik}^{mn}\colon\ i, k, m, n = 0, 1, \ldots\}$$

of real numbers such that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} |t_{ik}^{mn}| < \infty \qquad (m, n = 0, 1, \ldots). \tag{1.1}$$

Condition (1.1) is trivially satisfied if the matrix T is such that for each m and n there exists an integer $\varkappa_{mn}$ with the property that $t_{ik}^{mn}=0$ whenever $\max(i, k)>\varkappa_{mn}$. In this case T is called *generalized triangular*. In particular, T is called *triangular* if for each m and n we have $t_{ik}^{mn}=0$ whenever at least one of the relations $i>m$ and $k>n$ is satisfied.

With every double series

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} u_{ik} \tag{1.2}$$

of real numbers, we associate a double sequence $\{\sigma_{mn}\}$ given by

$$\sigma_{mn} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} t_{ik}^{mn} u_{ik} \qquad (m, n = 0, 1, \ldots), \tag{1.3}$$

provided the double series on the right converges in the sense of Pringsheim. This is the case if (1.1) is satisfied and the terms u_{ik} of series (1.2) are bounded. We note that in this case the series on the right (1.3) is even absolutely convergent.

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If σ_{mn} tends to a finite limit s as $\min(m, n) \to \infty$ we say that series (1.2) is *T-summable* to the sum s. The σ_{mn} are called the *T-means* of (1.2).

We introduce the following notation:

$$\Delta_{mn} = \sigma_{mn} - \sigma_{m-1,n} - \sigma_{m,n-1} + \sigma_{m-1,n-1} \tag{1.4}$$

with the agreement that

$$\sigma_{-1,n} = \sigma_{m,-1} = \sigma_{-1,-1} = 0 \qquad (m, n = 0, 1, \ldots). \tag{1.5}$$

We say that series (1.2) is *absolutely T-summable* (shortly: $|T|$*-summable*) if

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{mn}| < \infty. \tag{1.6}$$

Clearly, $|T|$-summability implies T-summability. In addition, $|T|$-summability also implies that σ_{mn} converges as $n \to \infty$ for each $m=0, 1, \ldots$ and that σ_{mn} converges as $m \to \infty$ for each $n=0, 1, \ldots$.

2. Main results: Summability of orthogonal series

Let $\varphi = \{\varphi_{ik}(x)\colon i, k=0, 1, \ldots\}$ be a real-valued *orthonormal system* (in abbreviation: ONS) defined on a positive measure space $(X, \mathcal{F}, \mu)$. We consider the double *orthogonal series*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik} \varphi_{ik}(x), \tag{2.1}$$

where $\{a_{ik}\colon i, k=0, 1, \ldots\}$ is a double sequence of real numbers such that

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^2 < \infty. \tag{2.2}$$

The T-means of series (2.1) are defined according to (1.3):

$$\sigma_{mn}(x) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} t_{ik}^{mn} a_{ik} \varphi_{ik}(x) \qquad (m, n = 0, 1, \ldots).$$

If conditions (1.1) and (2.2) are satisfied, then $\sigma_{mn}(x)$ is well defined μ-a.s. for each m and n. In fact, it follows from (2.2), via B. Levi's theorem, that

$$\lim_{\max(i,k) \to \infty} a_{ik} \varphi_{ik}(x) = 0 \qquad \mu\text{-a.s.},$$

and, a foriori, the terms $a_{ik} \varphi_{ik}(x)$ are bounded μ-a.s.

We introduce the following notation:

$$\tau_{ik}^{mn} = t_{ik}^{mn} - t_{ik}^{m-1,n} - t_{ik}^{m,n-1} + t_{ik}^{m-1,n-1} \tag{2.3}$$

with the agreement that

(2.4) $$t_{ik}^{-1,n} = t_{ik}^{m,-1} = t_{ik}^{-1,-1} = 0 \qquad (i, k, m, n = 0, 1, \ldots).$$

Theorem 1. *If conditions* (1.1), (2.2) *are satisfied and*

(2.5) $$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \{\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} [\tau_{ik}^{mn}]^2 a_{ik}^2\}^{1/2} < \infty,$$

then series (2.1) *is* $|T|$*-summable* μ*-a.e. on* X.

The surprising fact is that condition (2.5), under a mild assumption on T, is not only sufficient but also necessary for the μ-a.e. $|T|$-summability of series (2.1) if all ONS φ are taken into consideration.

To be more specific, let $(X, \mathcal{F}, \mu)$ be the familiar unit square

$$U = \{x = (x_1, x_2)\colon\ 0 \leqq x_j \leqq 1 \quad \text{for} \quad j = 1, 2\}$$

with the Borel measurable subsets as $\mathcal{F}$ and with the planar Lebesgue measure as μ. We remind that the ordinary one-dimensional *Rademacher system* $\{r_i(x_1)\}$ is defined as follows

$$r_i(x_1) = \operatorname{sign} \sin (2^i \pi x_1) \qquad (i = 0, 1, \ldots;\ 0 \leqq x_1 \leqq 1)$$

(see, e.g. [1, p. 51] or [15, p. 212]).

Theorem 2. *Assume that conditions* (1.1), (2.2), *are satisfied and*

(2.6) $$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\tau_{ik}^{mn}| < \infty \qquad (i, k = 0, 1, \ldots).$$

If condition (2.5) *is not satisfied, then the two-dimensional Rademacher series*

(2.7) $$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik} r_i(x_1) r_k(x_2)$$

is not $|T|$*-summable a.e. on* U.

Putting Theorems 1 and 2 together, we obtain the following

Corollary 1. *Assume that conditions* (1.1), (2.2), *and* (2.6) *are satisfied. Then series* (2.1) *is* $|T|$*-summable a.e. for every double ONS* φ *defined on* U *if and only if condition* (2.5) *is satisfied.*

The corresponding results for single ONS defined on the unit interval $I = \{x_1\colon 0 \leqq x_1 \leqq 1\}$ were proved by LEINDLER and TANDORI [8].

As an application, we will conclude a number of results on $|C, \alpha, \beta|$-summability of double orthogonal series for $\alpha > -1$ and $\beta > -1$. As is known, (C, α, β)-summability is defined by means of the triangular matrix $T=\{t_{ik}^{mn}\}$:

$$(2.8)\qquad t_{ik}^{mn}=\begin{cases}\dfrac{A_{m-i}^{\alpha}}{A_m^{\alpha}}\dfrac{A_{n-k}^{\beta}}{A_n^{\beta}}, & \text{for } i=0,1,\dots,m;\ k=0,1,\dots,n;\ m,n=0,1,\dots;\\ 0, & \text{otherwise.}\end{cases}$$

Here

$$A_m^{\alpha}=\binom{\alpha+m}{m}=\frac{(\alpha+1)(\alpha+2)\dots(\alpha+m)}{m!}\qquad (m=0,1,\dots;\ \alpha>-1)$$

is the binomial coefficient.

3. Proofs of Theorems 1 and 2

Similarly to (1.4) and (1.5), we set

$$(3.1)\qquad \Delta_{mn}(x)=\sigma_{mn}(x)-\sigma_{m-1,n}(x)-\sigma_{m,n-1}(x)+\sigma_{m-1,n-1}(x)$$

with the agreement that

$$\sigma_{-1,n}(x)=\sigma_{m,-1}(x)=\sigma_{-1,-1}(x)=0\quad (m,n=0,1,\dots)$$

for every x in X.

Proof of Theorem 1. By Minkowski's inequality, orthogonality, and (2.5), we get in turn that

$$\Big\{\int_X\Big[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\Delta_{mn}(x)|\Big]^2 d\mu(x)\Big\}^{1/2}\leqq\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\int_X\Delta_{mn}^2(x)\,d\mu(x)\Big\}^{1/2}=$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}[\tau_{ik}^{mn}]^2a_{ik}^2\Big\}^{1/2}<\infty.$$

This means that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}|\Delta_{mn}(x)|\in L^2(X,\mathcal{F},\mu),$$

and, in particular, series (2.1) is $|T|$-summable μ-a.e.

The proof of Theorem 1 is complete.

In the proof of Theorem 2 we need the following auxiliary result proved in [9].

Theorem A. *Given any measurable set E $(\subset U)$ of positive measure, then there exist an integer n_0 and a constant $C_1>0$ such that for every finite sum*

$$P(x_1,x_2)=\sum_{i=m}^{M}\sum_{k=n}^{N}a_{ik}r_i(x_1)\,r_k(x_2)$$

with $\max(m, n) \geqq n_0$, $M \geqq m \geqq 0$ *and* $N \geqq n \geqq 0$ *we have*

$$\iint_E |P(x_1, x_2)|\, dx_1\, dx_2 \geqq C_1 \Big\{ \sum_{i=m}^{M} \sum_{k=n}^{N} a_{ik}^2 \Big\}^{1/2}.$$

We note that this is an extension of a result due to ORLICZ [10] from the one-dimensional Rademacher system to the two-dimensional one.

Proof of Theorem 2. We will prove that if series (2.7) is $|T|$-summable on a subset of U with positive measure, then condition (2.5) necessarily holds.

To realize this goal, then by Egorov's theorem there exist a constant C_2 and a subset E $(\subset U)$ of positive measure such that

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{mn}(x_1, x_2)| \leqq C_2 \quad \text{for} \quad (x_1, x_2) \in E, \tag{3.2}$$

where this time $\Delta_{mn}(x_1, x_2)$ is defined by (3.1) in the case of the two-dimensional Rademacher functions and $x = (x_1, x_2)$.

We are going to apply Theorem A formulated above. To this effect, we must get rid of the functions $r_i(x_1)$, $r_k(x_2)$ in the definition of $\Delta_{mn}(x_1, x_2)$ for which $\max(i, k) < n_0$. Therefore, we set

$$\tilde{a}_{ik} = \begin{cases} a_{ik} & \text{if} \quad \max(i, k) \geqq n_0, \\ 0 & \text{if} \quad \max(i, k) < n_0; \end{cases}$$

and denote by $\tilde{\Delta}_{mn}(x_1, x_2)$ the corresponding difference of the T-means for the "truncated" double series

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_{ik} r_i(x_1)\, r_k(x_2). \tag{3.3}$$

Since $|r_i(x_1) r_k(x_2)| \leqq 1$ for every x_1, x_2, an elementary estimation shows that

$$\left| \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\Delta_{mn}(x_1, x_2)| - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |\tilde{\Delta}_{mn}(x_1, x_2)| \right| \leqq$$

$$\leqq \sum\sum_{\max(m,n) \geqq n_0} \sum_{i=0}^{\min(m, n_0-1)} \sum_{k=0}^{\min(n, n_0-1)} |\tau_{ik}^{mn} a_{ik}| =$$

$$= \sum_{i=0}^{n_0-1} \sum_{k=0}^{n_0-1} |a_{ik}| \Big\{ \sum_{m=i}^{n_0-1} \sum_{n=n_0}^{\infty} + \sum_{m=n_0}^{\infty} \sum_{n=k}^{n_0-1} + \sum_{m=m_0}^{\infty} \sum_{n=n_0}^{\infty} \Big\} |\tau_{ik}^{mn}| \leqq$$

$$\leqq 3 \sum_{i=0}^{n_0-1} \sum_{k=0}^{n_0-1} |a_{ik}| \sum_{m=i}^{\infty} \sum_{n=k}^{\infty} |\tau_{ik}^{mn}| < \infty,$$

the last inequality is due to (2.6). Consequently, the $|T|$-summability of series (2.7) and (3.3) are equivalent for every x_1, x_2.

So, we may assume without loss of generality that $a_{ik}=0$ in (2.7) for $i, k = 0, 1, \ldots, n_0-1$, and use the notations a_{ik} and $\Delta_{mn}(x_1, x_2)$ rather than $\tilde{a}_{ik}$ and $\tilde{\Delta}_{mn}(x_1, x_2)$. On the one hand, by (3.2),

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\iint_E |\Delta_{mn}(x_1, x_2)|\, dx_1 dx_2 \leqq C_2\mu(E), \tag{3.4}$$

μ being the plane Lebesgue measure here. On the other hand, applying Theorem A yields

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\iint_E |\Delta_{mn}(x_1, x_2)|\, dx_1 dx_2 \geqq \tag{3.5}$$

$$\geqq C_1 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}[\tau_{ik}^{mn}]^2 a_{ik}^2\Big\}^{1/2}.$$

Combining inequalities (3.4) and (3.5) results in (2.5) to be proved.

4. Application of Theorem 1: Sufficient conditions for $|C, \alpha, \beta|$-summability of orthogonal series

The next seven theorems will be consequences of Theorem 1. We make the following convention: by 2^{-1} we mean 0 in this paper.

Theorem B. *If* $\alpha>1/2$, $\beta>1/2$, *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.1}$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

This theorem was proved in [9] by the first named author, extending the relevant results of Tandori [14] ($\alpha=1$) and Leindler [5] ($\alpha>1/2$) from single to double orthogonal series. The proving method in [9] is a direct one. Nevertheless, it is instructive to present here how Theorem B can be deduced from Theorem 1. Since the same technique will be used in the proofs of Theorems 3—8 below, we enter into full details.

Proof of Theorem B. We will prove that condition (4.1) implies (2.5), and a fortiori, Theorem 1 implies Theorem B.

To this end, we introduce the notations

$$n_q = \begin{cases} 2^{q-1} & \text{if } q = 1, 2, \ldots, \\ 0 & \text{if } q = 0; \end{cases} \tag{4.2}$$

and

$$\mathscr{A}_{mn} = \Big\{\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}[\tau_{ik}^{mn}]^2 a_{ik}^2\Big\}^{1/2} \quad (m, n = 0, 1, \ldots). \tag{4.3}$$

Thus, the left-hand side of (2.5) can be rewritten as follows

$$(4.4)\qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}[\tau_{ik}^{mn}]^2 a_{ik}^2\Big\}^{1/2} = \mathscr{A}_{00}+\sum_{n=1}^{\infty}\mathscr{A}_{0n}+\sum_{m=1}^{\infty}\mathscr{A}_{m0}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathscr{A}_{mn}.$$

According to this, the proof is divided into four parts.

Part 1. By (2.3), (2.4) and (2.8)

$$(4.5)\qquad \tau_{00}^{00}=1 \quad\text{and}\quad \tau_{ik}^{00}=0 \quad\text{otherwise,}$$

whence

$$(4.6)\qquad \mathscr{A}_{00}=|a_{00}|.$$

Part 2. By definition, for $n=1, 2, \ldots$

$$\tau_{0k}^{0n}=\begin{cases}\dfrac{A_{n-k}^{\beta}}{A_n^{\beta}}-\dfrac{A_{n-k-1}^{\beta}}{A_{n-1}^{\beta}} & \text{if}\quad k=0,1,\ldots,n-1;\\[2ex] \dfrac{1}{A_n^{\beta}} & \text{if}\quad k=n;\end{cases}$$

and $\tau_{ik}^{0n}=0$ if $i>0$ or $k>n$. Using the relevant estimates in [5], we have, for $\beta>-1$,

$$(4.7)\qquad \tau_{ik}^{0n}=\begin{cases}O_\beta\big(kn^{-\beta-1}(n+1-k)^{\beta-1}\big) & \text{if}\quad k=0,1,\ldots,n;\\ 0 & \text{if}\quad i>0 \quad\text{or}\quad k>n \quad (n=1,2,\ldots).\end{cases}$$

By the Cauchy inequality,

$$\sum_{n=1}^{\infty}\mathscr{A}_{0n}=\sum_{n=1}^{\infty}\Big\{\sum_{k=0}^{n}[\tau_{0k}^{0n}]^2a_{0k}^2\Big\}^{1/2}=\sum_{q=0}^{\infty}\sum_{n=n_q+1}^{n_{q+1}}\Big\{\sum_{k=0}^{n}[\tau_{0k}^{0n}]^2a_{0k}^2\Big\}^{1/2}\leqq$$

$$\leqq\sum_{q=0}^{\infty}\Big\{(n_{q+1}-n_q)\sum_{n=n_q+1}^{n_{q+1}}\sum_{k=0}^{n}[\tau_{0k}^{0n}]^2a_{0k}^2\Big\}^{1/2}=$$

$$=O(1)\sum_{q=1}^{\infty}\Big\{(n_{q+1}-n_q)\sum_{n=n_q+1}^{n_{q+1}}\sum_{k=0}^{n-1}k^2n^{-2\beta-2}(n-k)^{2\beta-2}a_{0k}^2\Big\}^{1/2}+$$

$$+O(1)\sum_{q=0}^{\infty}\Big\{(n_{q+1}-n_q)\sum_{n=n_q+1}^{n_{q+1}}n^{-2\beta}a_{0k}^2\Big\}^{1/2}=O(1)\,(\Sigma_1+\Sigma_2),\quad\text{say.}$$

Since

$$(4.8)\qquad n_{q+1}-n_q=n_q \quad (q=1,2,\ldots),$$

it immediately follows from (4.1) that $\Sigma_2<\infty$.

Now we turn to Σ_1. A simple computation gives that

$$\Sigma_1 = \sum_{q=1}^{\infty} \Big\{(n_{q+1}-n_q) \sum_{n=n_q+1}^{n_{q+1}} \sum_{r=0}^{q} \sum_{k=n_r}^{\min(n_{r+1},n)-1} k^2 n^{-2\beta-2}(n-k)^{2\beta-2} a_{0k}^2\Big\}^{1/2} =$$

$$= \sum_{q=1}^{\infty} \Big\{(n_{q+1}-n_q) \sum_{r=0}^{q} \sum_{k=n_r}^{n_{r+1}-1} \sum_{n=\max(n_q,k)+1}^{n_{q+1}} k^2 n^{-2\beta-2}(n-k)^{2\beta-2} a_{0k}^2\Big\}^{1/2} \leqq$$

$$\leqq \sum_{q=2}^{\infty} \Big\{(n_{q+1}-n_q) \sum_{r=0}^{q-2} \sum_{k=n_r}^{n_{r+1}-1} \sum_{n=n_q+1}^{n_{q+1}} k^2 n^{-2\beta-2}(n-k)^{2\beta-2} a_{0k}^2\Big\}^{1/2} +$$

$$+ \sum_{q=1}^{\infty} (n_{q+1}-n_q)^{1/2} n_q^{\beta-1} \sum_{r=q-1}^{q} \Big\{\sum_{k=n_r}^{n_{r+1}-1} k^2 a_{0k}^2 \sum_{n=\max(n_q,k)+1}^{n_{q+1}} (n-k)^{2\beta-2}\Big\}^{1/2} =$$

$$= \Sigma_{11}+\Sigma_{12}, \quad \text{say.}$$

It is easy to see that

$$\text{(4.9)} \qquad \sum_{n=\max(n_q,k)+1}^{n_{q+1}} (n-k)^{2\beta-2} = O(n_q^{2\beta-1})$$

$$\text{if} \quad n_{q-1} \leqq k < n_{q+1} \quad \Big(q = 1, 2, \ldots;\ \beta > \frac{1}{2}\Big).$$

Consequently, (4.1) and (4.8) yield $\Sigma_{12} < \infty$.
Now we treat Σ_{11}. It is not hard to check that

$$\text{(4.10)} \qquad (n-k)^{2\beta-2} \leqq 4(n_q-n_{r+1})^{2\beta-2}$$

$$\text{if} \quad n_q < n \leqq n_{q+1};\ n_r \leqq k < n_{r+1};$$

$$r = 0, 1, \ldots, q-2;\ q = 2, 3, \ldots;\ \beta > \frac{1}{2}.$$

Using this inequality together with

$$(u+v+\ldots)^{1/2} \leqq u^{1/2}+v^{1/2}+\ldots \quad (u \geqq 0,\ v \geqq 0, \ldots),$$

we find that

$$\Sigma_{11} = \sum_{q=2}^{\infty} \Big\{(n_{q+1}-n_q) \sum_{n=n_q+1}^{n_{q+1}} \sum_{r=0}^{q-2} \sum_{n=n_r}^{n_{r+1}-1} k^2 n^{-2\beta-2}(n-k)^{2\beta-2} a_{0k}^2\Big\}^{1/2} \leqq$$

$$\leqq \sum_{q=2}^{\infty} (n_{q+1}-n_q)^{1/2} n_q^{-\beta-1} \Big\{(n_{q+1}-n_q) \sum_{r=0}^{q-2} (n_q-n_{r+1})^{2\beta-2} \sum_{k=n_r}^{n_{r+1}-1} k^2 a_{0k}^2\Big\}^{1/2} =$$

$$= O(1) \sum_{q=2}^{\infty} (n_{q+1}-n_q)\, n_q^{-\beta-1} \sum_{r=0}^{q-2} n_r (n_q-n_{r+1})^{\beta-1} \Big\{\sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\Big\}^{1/2} =$$

$$= O(1) \sum_{r=0}^{\infty} n_r \Big\{\sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=r+2}^{\infty} (n_{q+1}-n_q) n_q^{-\beta-1}(n_q-n_{r+1})^{\beta-1} = \Sigma, \quad \text{say.}$$

It is easy to see that

(4.11) $$(n_q - n_{r+1})^{\beta-1} = O(n_q^{\beta-1})$$

$$\text{if} \quad q \geqq r+2;\ r = 0, 1, \ldots;\ \beta > \frac{1}{2}.$$

Using this, (4.1) and (4.8) we can conclude that

(4.12) $$\Sigma = O(1) \sum_{r=0}^{\infty} n_r \Big\{\sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=r+2}^{\infty} n_q^{-1} = O(1) \sum_{r=0}^{\infty} \Big\{\sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\Big\}^{1/2} < \infty.$$

Consequently, $\Sigma_{11} < \infty$, $\Sigma_1 < \infty$, and

(4.13) $$\sum_{n=1}^{\infty} \mathscr{A}_{0n} < \infty.$$

Remark. A careful examination of the method used just above shows that if $\{C_k\colon k=0, 1, \ldots\}$ is a sequence of nonnegative numbers, then

(4.14) $$\sum_{n=1}^{\infty} \Big\{\sum_{k=0}^{n} [\tau_{0k}^{0n}]^2 C_k\Big\}^{1/2} = O_\beta(1) \sum_{r=0}^{\infty} \Big\{\sum_{k=n_r}^{n_{r+1}-1} C_k\Big\}^{1/2},$$

where $O_\beta(1)$ does not depend on $\{C_k\}$ and as before $n_r = 2^{r-1}$.

In a similar way, we can obtain that for every sequence $\{B_i\colon i=0, 1, \ldots\}$ of nonnegative numbers we have

(4.15) $$\sum_{m=1}^{\infty} \Big\{\sum_{i=0}^{m} [\tau_{i0}^{m0}]^2 B_i\Big\}^{1/2} = O_\alpha(1) \sum_{r=0}^{\infty} \Big\{\sum_{i=n_r}^{n_{r+1}-1} B_i\Big\}^{1/2}.$$

Part 3. According to (4.15),

(4.16) $$\sum_{m=1}^{\infty} \mathscr{A}_{m0} < \infty.$$

Part 4. It remains to prove that

(4.17) $$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mathscr{A}_{mn} < \infty.$$

To this end, first we observe that

(4.18) $$\tau_{ik}^{mn} = \tau_{i0}^{m0} \tau_{0k}^{0n} \quad (i, k = 0, 1, \ldots;\ m, n = 1, 2, \ldots).$$

In particular, this implies that

$$\tau_{ik}^{mn} = 0 \quad \text{if} \quad i > m \quad \text{or} \quad k > n.$$

Then setting

(4.19) $$C_k = \sum_{i=0}^{m} [\tau_{i0}^{m0}]^2 a_{ik}^2 \quad (k = 0, 1, \ldots)$$

and

$$B_i = \sum_{k=n_r}^{n_{r+1}-1} a_{ik}^2 \quad (i = 0, 1, \ldots), \tag{4.20}$$

we can proceed as follows

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathscr{A}_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\{\sum_{i=0}^{m}\sum_{k=0}^{n}[\tau_{i0}^{m0}\tau_{0k}^{0n}]^2 a_{ik}^2\Big\}^{1/2} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\{\sum_{k=0}^{n}[\tau_{0k}^{0n}]^2 C_k\Big\}^{1/2} =$$

$$= O_\beta(1)\sum_{m=1}^{\infty}\sum_{r=0}^{\infty}\Big\{\sum_{k=n_r}^{n_{r+1}-1}\sum_{i=0}^{m}[\tau_{i0}^{m0}]^2 a_{ik}^2\Big\}^{1/2} = O_\beta(1)\sum_{m=1}^{\infty}\sum_{r=0}^{\infty}\Big\{\sum_{i=0}^{m}[\tau_{i0}^{m0}]^2 B_i\Big\}^{1/2} =$$

$$= O_\beta(1)O_\alpha(1)\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\Big\{\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=n_r}^{n_{r+1}-1} a_{ik}^2\Big\}^{1/2} < \infty,$$

the last inequality being (4.1). This proves (4.17).

Combining (4.4), (4.6), (4.13), (4.16) and (4.17) completes the proof of Theorem B.

Now we introduce the following notations:

$$m_q = \begin{cases} 2^{\sqrt{q-1}} & \text{if } q = 1, 2, \ldots, \\ 0 & \text{if } q = 0; \end{cases} \tag{4.21}$$

$$i_p = p^{1/(1-2\alpha)} \quad \text{if} \quad p = 0, 1, \ldots; \tag{4.22}$$

$$k_q = q^{1/(1-2\beta)} \quad \text{if} \quad q = 0, 1, \ldots. \tag{4.23}$$

We agree that if u and v are real numbers, $u \leqq v$ then by $\sum_{n=u}^{v}$ we mean the sum extended for all integers n such that $u \leqq n \leqq v$.

Theorem 3. *If*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{\sum_{i=m_p}^{m_{p+1}-1}\sum_{k=m_q}^{m_{q+1}-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.24}$$

then series (2.1) *is* $|C, 1/2, 1/2|$*-summable* μ*-a.e.*

Theorem 4. *If* $0 \leqq \alpha < 1/2$, $0 \leqq \beta < 1/2$, *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{\sum_{i=i_p}^{i_{p+1}-1}\sum_{k=k_q}^{k_{q+1}-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.25}$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Theorems 3 and 4 are the extensions of the corresponding theorems of LEINDLER and SCHWINN [7] from single to double orthogonal series.

Conditions (4.26) and (4.27) below imply the fulfilment of conditions (4.24) and (4.25), respectively, through an appropriate grouping and the Cauchy inequality (cf. [6]). In this way we obtain the following two corollaries.

Corollary 2. *If*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{(p+1)(q+1)\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1}a_{ik}^2\}^{1/2}<\infty, \tag{4.26}$$

then series (2.1) *is* $|C, 1/2, 1/2|$*-summable* μ*-a.e.*

Corollary 3. *If* $0\leqq\alpha<1/2$, $0\leqq\beta<1/2$, *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\{2^{p(1-2\alpha)}2^{q(1-2\beta)}\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1}a_{ik}^2\}^{1/2}<\infty, \tag{4.27}$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Corollaries 2 and 3 as well as Theorem 5 below are the extensions of the corresponding theorems of LEINDLER [5] from single to double orthogonal series.

Theorem 5. *If* $-1<\alpha<0$, $-1<\beta<0$, *and condition* (4.27) *is satisfied, then series* (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Proofs of Theorems 3 and 4. We follow the scheme of the proof of Theorem B, changing it only at the reference numbers indicated by $*$ or $**$. Instead of (4.1), (4.2), (4.8)—(4.12) we have to take (4.24), (4.21), (4.8*)—(4.12*) and (4.25), (4.22)—(4.23), (4.8**)—(4.12**), respectively, and the proofs run along the same line as the proof of Theorem B. The $*$ estimates below are valid for $\beta=1/2$, while the $**$ estimates are valid for $0\leqq\beta<1/2$, but some of them remain valid for $\beta>-1$ too.

The appropriate estimates are the following:

$$m_{q+1}-m_q=O\left(\frac{m_q}{\log m_q}\right) \tag{4.8*}$$

and

$$k_{q+1}-k_q=O_\beta(k_q^{2\beta}) \tag{4.8**}$$

(this latter estimate holds true for $\beta>-1$);

$$\sum_{n=\max(m_q,k)+1}^{m_{q+1}}(n-k)^{-1}=O(\log m_q) \tag{4.9*}$$

and

$$\sum_{n=\max(k_q,k)+1}^{k_{q+1}}(n-k)^{2\beta-2}=O_\beta(1); \tag{4.9**}$$

$$(n-k)^{-1}\leqq(m_q-m_{r+1})^{-1} \tag{4.10*}$$

and

(4.10**) $$(n-k)^{2\beta-2} \leqq (k_q-k_{r+1})^{2\beta-2};$$

(4.11*) $$(m_q-m_{r+1})^{-1/2} \leqq \begin{cases} 2r^{1/4}(q-1-r)^{-1/2}m_{r+1}^{-1/2} & \text{if} \quad r+2 \leqq q \leqq r+r^{1/2}, \\ 2m_q^{-1/2} & \text{if} \quad r+r^{1/2} < q; \end{cases}$$

and

(4.11**) $$(k_q-k_{r+1})^{\beta-1} = \begin{cases} O_\beta(1)\,(q-1-r)^{\beta-1}k_{r+1}^{2\beta(\beta-1)} & \text{if} \quad r+2 \leqq q \leqq 2r+1, \\ O_\beta(1)\,k_q^{\beta-1} & \text{if} \quad 2r+1 < q; \end{cases}$$

finally, for $\beta=1/2$,

(4.12*) $$\Sigma = O(1)\sum_{r=4}^{\infty} m_r\Big\{\sum_{k=m_r}^{m_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=r+2}^{\infty} m_q^{-1/2}(m_q-m_{r+1})^{-1/2}\log^{-1} m_q =$$

$$= O(1)\sum_{r=4}^{\infty} r^{1/4}m_r^{1/2}\Big\{\sum_{k=m_r}^{m_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=r+2}^{r+r^{1/2}} m_q^{-1/2}(q-1-r)^{-1/2}\log^{-1} m_q +$$

$$+O(1)\sum_{r=4}^{\infty} m_r\Big\{\sum_{k=m_r}^{m_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=r+r^{1/2}+1}^{\infty} m_q^{-1}\log^{-1} m_q =$$

$$= O(1)\sum_{r=4}^{\infty} r^{-1/4}\Big\{\sum_{k=m_r}^{m_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=1}^{r^{1/2}} q^{-1/2} +$$

$$+O(1)\sum_{r=4}^{\infty} m_r\Big\{\sum_{k=m_r}^{m_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=r+1}^{\infty} m_q^{-1}\log^{-1} m_q < \infty,$$

while for $0<\beta<1/2$,

(4.12**) $$\Sigma = O_\beta(1)\sum_{r=1}^{\infty} k_r\Big\{\sum_{k=k_r}^{k_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=r+2}^{\infty} k_q^{\beta-1}(k_q-k_{r+1})^{\beta-1} =$$

$$= O_\beta(1)\sum_{r=1}^{\infty} k_r^{\beta(2\beta-1)}\Big\{\sum_{k=k_r}^{k_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=r+2}^{2r+1} (q-1-r)^{\beta-1} +$$

$$+O_\beta(1)\sum_{r=1}^{\infty} k_r\Big\{\sum_{k=k_r}^{k_{r+1}-1} a_{0k}^2\Big\}^{1/2} \sum_{q=2r+2}^{\infty} k_q^{2\beta-2} < \infty,$$

and for $\beta=0$,

(4.13**) $$\sum_{n=1}^{\infty} \mathscr{A}_{0n} = \sum_{n=1}^{\infty}\Big\{\sum_{k=0}^{n}[\tau_{0k}^{0n}]^2 a_{0k}^2\Big\}^{1/2} = \sum_{n=1}^{\infty}|a_{0n}| < \infty.$$

These inequalities completes the proof of Theorems 3 and 4.

Proof of Theorem 5. We use notation (4.2) and follow the pattern of the proof of Theorem B again. By (4.8) and (4.27),

$$\Sigma_2 = \sum_{q=0}^{\infty}\Big\{(n_{q+1}-n_q)\sum_{n=n_q+1}^{n_{q+1}} n^{-2\beta}a_{0n}^2\Big\}^{1/2} \leqq \sum_{q=0}^{\infty}\Big\{(n_{q+1}-n_q)\,n_{q+1}^{-2\beta}\sum_{n=n_q+1}^{n_{q+1}} a_{0n}^2\Big\}^{1/2} =$$

$$= O_\beta(1)\sum_{q=0}^{\infty}\Big\{2^{q(1-2\beta)}\sum_{n=2^{q-1}+1}^{2^q} a_{0n}^2\Big\}^{1/2} < \infty$$

and

$$\Sigma_1 = \sum_{q=1}^{\infty} \{(n_{q+1}-n_q) \sum_{n=n_q+1}^{n_{q+1}} \sum_{r=0}^{q} \sum_{k=n_r}^{\min(n_{r+1},n)-1} k^2 n^{-2\beta-2}(n-k)^{2\beta-2} a_{0k}^2\}^{1/2} =$$

$$= O_\beta(1) \sum_{q=1}^{\infty} \{2^{-q(1+2\beta)} \sum_{r=0}^{q} \sum_{k=n_r}^{n_{r+1}-1} k^2 a_{0k}^2 \sum_{n=\max(n_q,k)+1}^{n_{q+1}} (n-k)^{2\beta-2}\}^{1/2} =$$

$$= O_\beta(1) \sum_{q=2}^{\infty} \{2^{-q(1+2\beta)} \sum_{r=0}^{q-2} 2^{2r} \sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2 \sum_{n=2^{q-1}+1}^{2^q} (n-2^r)^{2\beta-2}\}^{1/2} +$$

$$+ O_\beta(1) \sum_{q=1}^{\infty} \{2^{-q(1+2\beta)} \sum_{r=q-1}^{q} 2^{2q} \sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\}^{1/2} =$$

$$= O_\beta(1)(1 + \sum_{q=2}^{\infty} \{2^{-q(1+2\beta)} \sum_{r=0}^{q-2} 2^{2r} \sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2 2^{q(2\beta-1)}\}^{1/2}) =$$

$$= O_\beta(1)(1 + \sum_{r=0}^{\infty} 2^r \{\sum_{k=n_r}^{n_{r+1}-1} a_{0k}^2\}^{1/2} \sum_{q=r}^{\infty} 2^{-q}) < \infty.$$

These calculations show that (4.13) is satisfied.

In the above manner (cf. Remark in the proof of Theorem B), we can conclude that if $\{C_k\colon k=0, 1, \ldots\}$ is a sequence of nonnegative numbers, then

$$\sum_{n=1}^{\infty} \{\sum_{k=0}^{n} [\tau_{0k}^{0n}]^2 C_k\}^{1/2} = O_\beta(1) \sum_{r=0}^{\infty} \{2^{r(1-2\beta)} \sum_{k=n_r}^{n_{r+1}-1} C_k\}^{1/2} \tag{4.14*}$$

and if $\{B_i\colon i=0, 1, \ldots\}$ is a sequence of nonnegative numbers, then

$$\sum_{m=1}^{\infty} \{\sum_{i=0}^{m} [\tau_{i0}^{m0}]^2 B_i\}^{1/2} = O_\alpha(1) \sum_{r=0}^{\infty} \{2^{r(1-2\alpha)} \sum_{i=n_r}^{n_{r+1}-1} B_i\}^{1/2}. \tag{4.15*}$$

The latter inequality implies the fulfilment of (4.16).

As to the fulfilment of (4.17), we use notation (4.19) and set

$$B_i = \sum_{k=n_r}^{n_{r+1}-1} k^{1-2\beta} a_{ik}^2 \quad (i = 0, 1, \ldots). \tag{4.20*}$$

We proceed as follows (cf. (4.18))

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mathscr{A}_{mn} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{\sum_{k=0}^{n} [\tau_{0k}^{0n}]^2 \sum_{i=0}^{m} [\tau_{i0}^{m0}]^2 a_{ik}^2\}^{1/2} =$$

$$= O_\beta(1) \sum_{m=1}^{\infty} \sum_{q=0}^{\infty} \{2^{q(1-2\beta)} \sum_{k=n_q}^{n_{q+1}-1} \sum_{i+0}^{m} [\tau_{i0}^{m0}]^2 a_{ik}^2\}^{1/2} =$$

$$= O_\beta(1) \sum_{m=1}^{\infty} \sum_{q=0}^{\infty} \{\sum_{i=0}^{m} [\tau_{i0}^{m0}]^2 \sum_{k=n_q}^{n_{q+1}-1} k^{1-2\beta} a_{ik}^2\}^{1/2} =$$

$$= O_\beta(1)O_\alpha(1)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{2^{p(1-2\alpha)}\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=n_q}^{n_{q+1}-1} k^{1-2\beta}a_{ik}^2\Big\}^{1/2} =$$

$$= O_\beta(1)O_\alpha(1)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{2^{p(1-2\alpha)}2^{q(1-2\beta)}\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=n_q}^{n_{q+1}-1} a_{ik}^2\Big\}^{1/2} < \infty,$$

completing the proof of Theorem 5.

The following three theorems cover the so-called "mixed" cases. We remind notations (4.2), (4.21)—(4.23).

Theorem 6. *If* $\alpha>1/2$, $\beta=1/2$ *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=m_q}^{m_{q+1}-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.28}$$

or if $\alpha>1/2$, $0\leqq\beta<1/2$ *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=k_q}^{k_{q+1}-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.29}$$

or if $\alpha>1/2$, $-1<\beta<0$ *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{2^{q(1-2\beta)}\sum_{i=n_p}^{n_{p+1}-1}\sum_{k=n_q}^{n_{q+1}-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.30}$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Theorem 7. *If* $\alpha=1/2$, $0\leqq\beta<1/2$ *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{\sum_{i=m_p}^{m_{p+1}-1}\sum_{k=k_q}^{k_{q+1}-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.31}$$

or if $\alpha=1/2$, $-1<\beta<0$ *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{2^{q(1-2\beta)}\sum_{i=m_p}^{m_{p+1}-1}\sum_{k=n_q}^{n_{q+1}-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.32}$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Theorem 8. *If* $0\leqq\alpha<1/2$; $-1<\beta<0$ *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{2^{q(1-2\beta)}\sum_{i=i_p}^{i_{p+1}-1}\sum_{k=n_q}^{n_{q+1}-1} a_{ik}^2\Big\}^{1/2} < \infty, \tag{4.33}$$

then series (2.1) *is* $|C, \alpha, \beta|$*-summable* μ*-a.e.*

Combining the proofs of Theorem B and Theorems 3—5 yields Theorem 6, combining those of Theorems 3 and 4 yields Theorem 7, while combining those of Theorems 4 and 5 yields Theorem 8.

As an example, we sketch the proof for the case $\alpha>1/2$ and $\beta=1/2$. Similarly to (4.14), for any sequence $\{C_k:\ k=0, 1, \dots\}$ of nonnegative numbers we have

$$\sum_{n=1}^{\infty}\Big\{\sum_{k=0}^{n}[\tau_{0k}^{0n}]^2 C_k\Big\}^{1/2} = O_\beta(1)\Big\{\sum_{k=m_q}^{m_{q+1}-1} C_k\Big\}^{1/2}. \tag{4.14**}$$

Furthemore, we have (4.15).

Assume (4.28) is satisfied. First, setting $C_k=a_{0k}^2$ and $B_i=a_{i0}^2$ we can derive (4.13) and (4.16). Second, using notation (4.19) and setting

$$B_i = \sum_{k=m_q}^{m_{q+1}-1} a_{ik}^2 \tag{4.20**}$$

we can conclude (4.17). So, applying Theorem 1 provides the first statement in Theorem 6.

The next two corollaries of Theorems 6 and 7 can be deduced via the Cauchy inequality.

Corollary 4. *If* $\alpha>1/2$, $\beta=1/2$ *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{(q+1)\sum_{i=2^{p-1}}^{2^p-1}\ \sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2\Big\}^{1/2}<\infty, \tag{4.34}$$

or if $\alpha>1/2$, $-1<\beta<1/2$ *and condition* (4.30) *is satisfied, then series* (2.1) *is* $|C, \alpha, \beta|$-*summable* μ-*a.e.*

Corollary 5. *If* $\alpha=1/2$, $-1<\beta<1/2$ *and*

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{(p+1)2^{q(1-2\beta)}\sum_{i=2^{p-1}}^{2^p-1}\ \sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2\Big\}^{1/2}<\infty, \tag{4.35}$$

or if $-1<\alpha<1/2$, $-1<\beta<1/2$ *and condition* (4.27) *is satisfied, then series* (2.1) *is* $|C, \alpha, \beta|$-*summable* μ-*a.e.*

Corollaries 4 and 5 as well as Corollaries 2 and 3 were proved by Ponomarenko and Timan [11] for the two-dimensional trigonometric system.

We remind that a double sequence $\{\lambda_{ik}:\ i, k=0, 1, \dots\}$ of numbers is said to be nondecreasing if

$$\lambda_{ik} \leqq \min\{\lambda_{i+1,k}, \lambda_{i,k+1}\}$$

and to be nonincreasing if

$$\lambda_{ik} \geqq \max\{\lambda_{i+1,k}, \lambda_{i,k+1}\}\quad (i, k = 0, 1, \dots).$$

In Corollaries 6 and 7 below, let $\{\lambda_{ik}\}$ be a nondecreasing sequence of positive numbers such that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(i+1)(k+1)\lambda_{ik}}<\infty, \tag{4.36}$$

or equivalently,

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{\lambda_{2^p,2^q}}<\infty.$$

Applying the Cauchy inequality to series (4.1), (4.26), (4.27) and then to series (4.34), (4.30) and (4.35) results in the following two corollaries.

Corollary 6. *If* $\alpha>1/2$, $\beta>1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}<\infty,$$

or if $\alpha=1/2$, $\beta=1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}\log(i+2)\log(k+2)<\infty,$$

or if $-1<\alpha<1/2$, $-1<\beta<1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}(i+1)^{1-2\alpha}(k+1)^{1-2\beta}<\infty,$$

then series (2.1) *is* $|C,\alpha,\beta|$*-summable* μ*-a.e.*

Corollary 7. *If* $\alpha>1/2$, $\beta=1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}\log(k+2)<\infty,$$

or if $\alpha>1/2$, $-1<\beta<1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}(k+1)^{1-2\beta}<\infty,$$

or if $\alpha=1/2$, $-1<\beta<1/2$ *and*

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}a_{ik}^2\lambda_{ik}(k+1)^{1-2\beta}\log(i+2)<\infty,$$

then series (2.1) *is* $|C,\alpha,\beta|$*-summable* μ*-a.e.*

Corollary 6 is the extension of the corresponding results of UL'JANOV [15, pp. 46—37 and 51—52] from single to double orthogonal series.

5. Application of Theorem 2: Necessary conditions for $|C, \alpha, \beta|$-summability of orthogonal series

The sufficient conditions (4.24), (4.25) and (4.27)—(4.32) are the best possible. To see this, we consider the special case where the double sequence $\{|a_{ik}|: i, k=0, 1, \ldots\}$ is nonincreasing. Then (4.24) is equivalent to (4.26), and both are equivalent to the condition

$$(5.1)\qquad S_1 = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(p+1)^{1/2}(q+1)^{1/2}2^{p/2}2^{q/2}|a_{2^p,2^q}| < \infty;$$

while (4.25), (4.27) and (4.33) are also equivalent to each other, and each of them is equivalent to the condition

$$(5.2)\qquad \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{p(1-\alpha)}2^{q(1-\beta)}|a_{2^p,2^q}| < \infty \quad (-1 < \alpha, \beta < 1/2).$$

Similarly, in the special case where $\{|a_{ik}|\}$ is nonincreasing in k for each fixed i both (4.28) and (4.34) are equivalent to the condition

$$(5.3)\qquad \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(q+1)^{1/2}2^{q/2}\Big\{\sum_{i=2^{p-1}}^{2^p-1}a_{i,2^q}^2\Big\}^{1/2} < \infty;$$

while both (4.29) and (4.30) are equivalent to the condition

$$(5.4)\qquad S_4 = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}2^{q(1-\beta)}\Big\{\sum_{i=2^{p-1}}^{2^p-1}a_{i,2^q}^2\Big\}^{1/2} < \infty \quad (-1 < \beta < 1/2).$$

Furthermore, in the special case where again the double sequence $\{|a_{ik}|\}$ is nonincreasing, each of the conditions (4.31), (4.32) and (4.35) is equivalent to

$$(5.5)\qquad \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(p+1)^{1/2}2^{p/2}2^{q(1-\beta)}|a_{2^p,2^q}| < \infty \quad (-1 < \beta < 1/2).$$

As an illustration, we show the equivalence in two cases.

Case 1. The equivalence of (4.24), (4.26) and (5.1). We remind notation (4.21).

First, we show that (4.26) implies (4.24) without any restriction. By the Cauchy inequality,

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{\sum_{i=m_p}^{m_{p+1}-1}\sum_{k=m_q}^{m_{q+1}-1}a_{ik}^2\Big\}^{1/2} \leqq 2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(m+1)^{1/2}(n+1)^{1/2}\times$$

$$\times\Big\{\sum_{p:\,2^{m-1}\leqq m_p<2^m}\ \sum_{q:\,2^{n-1}\leqq n_q<2^n}\ \sum_{i=m_p}^{m_{p+1}-1}\sum_{k=m_q}^{m_{q+1}-1}a_{ik}^2\Big\}^{1/2},$$

since for every $m=1, 2, \ldots$ the number of those integers for which $2^{m-1} \leqq m_p < 2^m$ is less than $2m$. Taking into account that the quadruple sum in the last square root does not exceed the double sum

$$\sum_{i=2^{m-1}}^{2^{m+1}-1} \sum_{k=2^{n-1}}^{2^{n+1}-1} a_{ik}^2,$$

we get implication (4.26)⇒(4.24).

Second, if we use the monotonicity of $\{|a_{ij}|\}$ we can immediately see that

$$\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \{(p+1)(q+1) \sum_{i=2^{p-1}}^{2^p-1} \sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2\}^{1/2} \leqq$$

$$\leqq \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (p+1)^{1/2}(q+1)^{1/2} 2^{p/2} 2^{q/2} |a_{2^{p-1}, 2^{q-1}}|,$$

which shows implication (5.1)⇒(4.26).

Third, we show implication (4.24)⇒(5.1) in the monotonic case. Again by the Cauchy inequality,

$$S_1 = O(1) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=2^{p-1}}^{2^p-1} \sum_{n=2^{q-1}}^{2^q-1} |a_{mn}| \times$$

$$\times (m+1)^{-1/2}(n+1)^{-1/2} \log^{1/2}(m+2) \log^{1/2}(n+2) =$$

$$= O(1) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |a_{mn}|(m+1)^{-1/2}(n+1)^{-1/2} \log^{1/2}(m+2) \log^{1/2}(n+2) =$$

$$= O(1) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=m_p}^{m_{p+1}-1} \sum_{n=m_q}^{m_{q+1}-1} |a_{mn}| \times$$

$$\times (m+1)^{-1/2}(n+1)^{-1/2} \log^{1/2}(m+2) \log^{1/2}(n+2) =$$

$$= O(1) \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \{\sum_{m=m_p}^{m_{p+1}-1} \sum_{n=m_q}^{m_{q+1}-1} a_{mn}^2\}^{1/2} R_{pq},$$

where by (4.8*),

$$R_{pq} = \{\sum_{m=m_p}^{m_{p+1}-1} \sum_{n=m_q}^{m_{q+1}-1} (m+1)^{-1}(n+1)^{-1} \log(m+2) \log(n+2)\}^{1/2} \leqq$$

$$\leqq \{(m_{p+1}-m_p)(m_{q+1}-m_q)(m_p+1)^{-1}(m_q+1)^{-1} p^{1/2} q^{1/2}\}^{1/2} = O(1).$$

This proves implication (4.24)⇒(5.1).

Case 2. The equivalence of (4.29), (4.30), and (5.4). This time we use notation (4.23).

First, we show that (4.30) implies (4.29) without any restriction. By the Cauchy inequality,

$$S_4 = \sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q:2^{n-1}\leqq k_q<2^n}\Big\{\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=k_q}^{k_{q+1}-1} a_{ik}^2\Big\}^{1/2} \leqq$$

$$\leqq \sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\Big\{\sum_{q:2^{n-1}\leqq k_q<2^n}\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=k_q}^{k_{q+1}-1} a_{ik}^2\Big\}^{1/2}\times\Big\{\sum_{q:2^{n-1}\leqq k_q<2^n} 1\Big\}^{1/2}.$$

Since the number of those integers q for which $2^{n-1}\leqq k_q<2^n$ is $O_\beta(2^{n(1-2\beta)})$ thus

$$S_4 = O_\beta(1)\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\Big\{2^{n(1-2\beta)}\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{n-1}}^{2^{n+1}-1} a_{ik}^2\Big\}^{1/2}.$$

This proves implication (4.30)⇒(4.29).

Second, using the monotonicity of $\{|a_{ik}|\}$ we can easily get implication (5.4)⇒ (4.30) as follows

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{2^{q(1-2\beta)}\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=2^{q-1}}^{2^q-1} a_{ik}^2\Big\}^{1/2} \leqq \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{q(1-\beta)}\Big\{\sum_{i=2^{p-1}}^{2^p-1} a_{i,2^{q-1}}^2\Big\}^{1/2}.$$

Third, we show implication (4.29)⇒(5.4) in the monotonic case. By the Cauchy inequality again,

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} 2^{q(1-\beta)}\Big\{\sum_{i=2^{p-1}}^{2^p-1} a_{i,2^q}^2\Big\}^{1/2} = O_\beta(1)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=2^{q-1}}^{2^q-1} k^{-\beta}\Big\{\sum_{i=2^{p-1}}^{2^p-1} a_{ik}^2\Big\}^{1/2} =$$

$$= O_\beta(1)\sum_{p=0}^{\infty}\sum_{k=0}^{\infty} k^{-\beta}\Big\{\sum_{i=2^{p-1}}^{2^p-1} a_{ik}^2\Big\}^{1/2} = O_\beta(1)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{k=k_q}^{k_{q+1}-1} k^{-\beta}\Big\{\sum_{i=2^{p-1}}^{2^p-1} a_{ik}^2\Big\}^{1/2} =$$

$$= O_\beta(1)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\Big\{\sum_{k=k_q}^{k_{q+1}-1} k^{-2\beta}\Big\}^{1/2}\Big\{\sum_{i=2^{p-1}}^{2^p-1}\sum_{k=k_q}^{k_{q+1}-1} a_{ik}^2\Big\}^{1/2}.$$

Since (4.8**) holds true for $\beta>-1$ we have

$$\sum_{k=k_q}^{k_{q+1}-1} k^{-2\beta} \leqq (k_{q+1}-k_q)k_q^{-2\beta} = O_\beta(1),$$

proving implication (4.29)⇒(5.4).

After these preliminaries, the point is that if $\{|a_{ik}|\}$ is nonincreasing in a certain sense indicated above, then conditions (5.1)—(5.5) are not only sufficient, but also necessary for the a.e. $|C, \alpha, \beta|$-summability of series (2.1), for a fixed pair of α and β in the appropriate domain, if all ONS φ are considered.

To go into details, the case $\min(\alpha,\beta)>1/2$ was studied in [9] without any additional restriction on $\{|a_{ik}|\}$. Theorem C obtained there extends the corresponding results of BILLARD [2] ($\alpha=1$) and GREPACHEVSKAJA [4] ($\alpha>1/2$) from single to double orthogonal series.

Theorem C. *If* $\alpha>1/2$, $\beta>1/2$ *and condition* (4.1) *is not satisfied, then the two-dimensional Rademacher series* (2.7) *is not* $|C, \alpha, \beta|$*-summable a.e.*

The following theorems cover various cases in the domain $-1<\min(\alpha, \beta)\leqq 1/2$.

Theorem 9. *If the double sequence* $\{|a_{ik}|\}$ *is nonincreasing and condition* (5.1) *is not satisfied, then series* (2.7) *is not* $|C, 1/2, 1/2|$*-summable a.e.*

Theorem 10. *If* $-1<\alpha<1/2$, $-1<\beta<1/2$, *the double sequence* $\{|a_{ik}|\}$ *is nonincreasing, and condition* (5.2) *is not satisfied, then series* (2.7) *is not* $|C, \alpha, \beta|$*-summable a.e.*

Theorems 9 and 10 are the extensions of the corresponding results of GREPACHEVSKAJA [4] from the one-dimensional Rademacher system to the two-dimensional one. Theorem 10 for two-dimensional trigonometric series was proved by PONOMARENKO and TIMAN [11], assuming that $\{a_{ik}\}$ is a nonincreasing sequence of nonnegative numbers.

Serving as a pattern, we present here the proof of Theorem 9. In this case, t_{ik}^{mn} is defined by (2.8) for $\alpha=\beta=1/2$.

First, we check that condition (2.6) is satisfied. This is simple by the means of estimates (4.18), (4.7), and the corresponding estimate on τ_{ik}^{m0} all applied in the case $\alpha=\beta=1/2$.

Second, we verify that condition (2.5) is not satisfied. Thus, we can apply Theorem 2 and conclude the statement of Theorem 9. In fact, again by (4.18), (4.7) and its symmetric counterpart as well as by the monotonicity of $\{|a_{ik}|\}$,

$$(5.6)\qquad S_{11}=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}2^{p/2}2^{q/2}p^{1/2}q^{1/2}|a_{2^p,2^q}|=$$

$$=O(1)\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{m=2^{p-1}}^{2^p-1}\sum_{n=2^{q-1}}^{2^q-1}|a_{mn}|m^{-1/2}n^{-1/2}\log^{1/2}(m+1)\log^{1/2}(n+1)=$$

$$=O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mn}|m^{-1/2}n^{-1/2}\log^{1/2}(m+1)\log^{1/2}(n+1)=$$

$$=O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mn}|m^{-3/2}n^{-3/2}\Big\{\sum_{i=m/2}^{m}i^2(m+1-i)^{-1}\sum_{k=n/2}^{n}k^2(n+1-k)^{-1}\Big\}^{1/2}=$$

$$=O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big\{\sum_{i=0}^{m}\sum_{k=0}^{n}a_{ik}^2 i^2 m^{-3}(m+1-i)^{-1}\times$$

$$\times k^2 n^{-3}(n+1-k)^{-1}\Big\}^{1/2}=O(1)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathcal{A}_{mn}$$

(cf. notation (4.3)).

Similarly, we can obtain that

$$(5.7)\qquad S_{01} = \sum_{q=1}^{\infty} 2^{q/2} q^{1/2} |a_{0,2^q}| = O(1) \sum_{n=1}^{\infty} \mathscr{A}_{0n}$$

and

$$(5.8)\qquad S_{10} = \sum_{p=1}^{\infty} 2^{p/2} p^{1/2} |a_{2^p,0}| = O(1) \sum_{m=1}^{\infty} \mathscr{A}_{m0}.$$

Collecting (5.6)—(5.8) we find that

$$S_1 = |a_{00}| + S_{01} + S_{10} + S_{11} = O(1) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \mathscr{A}_{mn}$$

(see also (4.6)). Since, by assumption $S_1 = \infty$ condition (2.5) cannot be satisfied either. Applying Theorem 2 gives the statement of Theorem 9.

The last two theorems in this Section are concerned with the "mixed" cases.

Theorem 11. *Assume that the sequence* $\{|a_{ik}|\}$ *is nonincreasing in k for each fixed i. If* $\alpha > 1/2$, $\beta = 1/2$ *and condition* (5.3) *is not satisfied, or if* $\alpha > 1/2$, $-1 < \beta < 1/2$ *and condition* (5.4) *is not satisfied, then series* (2.7) *is not* $|C, \alpha, \beta|$*-summable a.e.*

Theorem 12. *If* $\alpha = 1/2$, $-1 < \beta < 1/2$, *the sequence* $\{|a_{ik}|\}$ *is nonincreasing, and condition* (5.5) *is not satisfied, then series* (2.7) *is not* $|C, \alpha, \beta|$*-summable a.e.*

Theorems 10—12 can be proved in a similar fashion to as Theorem 9 is proved above on the basis of Theorem 2.

6. Generalized $|C, \alpha, \beta|_l$-summability of orthogonal series

Let $l \geqq 1$ be a real number. Following Flett [3], series (2.1) is said to be $|C, \alpha, \beta|_l$*-summable* at x if

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+1)^{l-1} (n+1)^{l-1} |\Delta_{mn}^{\alpha\beta}(x)|^l < \infty,$$

where $\Delta_{mn}^{\alpha\beta}(x)$ is defined in (3.1) with the matrix given by (2.8). The case $l=1$ gives back the ordinary $|C, \alpha, \beta|$-summability. Using the same techniques which occur in the proofs of Theorems 3—12 and Corollaries 2—7, we can derive both necessary and sufficient conditions on the a.e. $|C, \alpha, \beta|_l$-summability of series (2.1). Here we present only three samples of these extensions. We use the notation $m_p = 2^{(p-1)^{1-1/2}}$.

Theorem 3*. *If* $1 \leqq l \leqq 2$ *and*

$$(6.1)\qquad \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \Big\{ \sum_{i=m_p}^{m_{p+1}-1} \sum_{k=m_q}^{m_{q+1}-1} a_{ik}^2 \Big\}^{l/2} < \infty,$$

then series (2.1) *is* $|C, 1/2, 1/2|_l$*-summable* μ*-a.e.*

Corollary 6*. *Let* $1 \leqq l \leqq 2$ *and* $\{\lambda_{ik}\}$ *be a nondecreasing sequence of positive numbers satisfying the condition*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(i+1)(k+1)\lambda_{ik}^{l}} < \infty. \tag{6.2}$$

If $\alpha > 1/2$, $\beta > 1/2$ *and*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^2 \lambda_{ik}^{2-l} < \infty;$$

or if $\alpha = 1/2$, $\beta = 1/2$ *and*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^2 \lambda_{ik}^{2-l} \log(i+2) \log(k+2) < \infty;$$

or if $-1 < \alpha < 1/2$, $-1 < \beta < 1/2$ *and*

$$\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik}^2 \lambda_{ik}^{2-l} (i+1)^{1-2\alpha} (k+1)^{1-2\beta} < \infty,$$

then series (2.1) *is* $|C, \alpha, \beta|_l$*-summable* μ*-a.e.*

We note that in case $l=2$ condition (6.2) can be dropped.

Theorem 9*. *Let* $1 \leqq l \leqq 2$. *If the sequence* $\{|a_{ik}|\}$ *is nonincreasing and condition* (6.1) *is not satisfied, then series* (2.7) *is not* $|C, 1/2, 1/2|$*-summable a.e.*

Theorems 3*, 9* and Corollary 6* are the extensions of the corresponding theorems of the second named author [12] and Spevakov [13], respectively, from single orthogonal series to double ones.

On closing, we mention that our results can be extended in a natural way to d-multiple orthogonal series with $d \geqq 3$, too.

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