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On a geometric problem concerning discs

A. P. BOSZNAY and B. M. GARAY

Introduction

Let $(X, \|\cdot\|)$ be an n-dimensional real normed linear space, and let d be a metric defined on $B_X(0, 1) \equiv \{x \in X: \|x\| \leqq 1\}$ with the following properties:

(i) d is topologically equivalent with $\|\cdot\|$,

(ii) $d(x_1, x_2) = \|x_1 - x_2\|$ for all $1 = \|x_1\| = \|x_2\|$.

At first glance, one can conjecture that there will exist an $y^* \in B_X(0, 1)$ such that

$$\min_{x \in S_X(0,1)} d(y^*, x) \geqq 1.$$

In case of $n = 1$, this is an easy consequence of the triangle inequality.

The aim of this paper is to show that in general, this is not the situation. For arbitrary $n \geqq 2$, we construct an example d and $(X, \|\cdot\|)$ for which

$$\max_{y \in B_X(0,1)} \min_{x \in S_X(0,1)} d(y, x) < 1.$$

On the contrary, we prove that

$$\max_{y \in B_X(0,1)} \min_{x \in S_X(0,1)} d(y, x) \geqq \frac{1}{n}.$$

Results

Example. Let $n \geqq 2$. Then there exists a metric d on the n-dimensional euclidean unit ball $E(0, 1)$ such that d has properties (i) and (ii), and

$$\max_{y \in E(0,1)} \min_{x \in S(0,1)} d(y, x) < 1.$$

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Contsruction of d. Let us recall first that there exists a norm $\|\cdot\|$ on $\mathbf{R}^n \oplus \mathbf{R}$ with the following properties (here $|\cdot|$ denotes the euclidean norm in $\mathbf{R}^n$):

(a) $$\|(x, 0)\| = |x| \quad \text{for all} \quad x \in \mathbf{R}^n,$$

(b) $$\|0, \lambda)\| = |\lambda| \quad \text{for all} \quad \lambda \in \mathbf{R},$$

(c) for any projection $P\colon \mathbf{R}^n \oplus \mathbf{R} \to \mathbf{R}^n$ onto, there holds

$$\|P\| = \sup\{\|P(x, \lambda)\|\colon x \in \mathbf{R}^n, \lambda \in \mathbf{R}, \|(x, \lambda)\| \leqq 1\} \geqq 1 + \delta_0$$

for some fixed $\delta_0 > 0$.

Several types of such norms can be constructed. For example, the existence of such a norm is a consequence of [1].

We shall define now the metric d_α on the set $E(0, 1)$. For $y_1, y_2 \in E(0, 1)$, let

$$d_\alpha(y_1, y_2) = \|h_\alpha(y_1) - h_\alpha(y_2)\|,$$

where $h_\alpha\colon \mathbf{R}^n \to \mathbf{R}^n \oplus \mathbf{R}$ is defined by

$$h_\alpha(y) = \begin{cases} \left(y, \dfrac{\alpha}{\alpha - 1}(|y| - 1)\right) & \text{if} \quad \alpha \leqq |y| \leqq 1 \\ (y, \alpha) & \text{if} \quad |y| \leqq \alpha \end{cases}$$

and the contant $0 < \alpha < 1$ is to be specified later. Clearly, d_α has the desired properties (i) and (ii).

We shall show now that for all $y \in E(0, 1)$

(1) $$\min_{x \in S(0,1)} d_\alpha(y, x) < 1$$

provided that α is sufficiently small.

Firstly, let $|y| > \alpha$. For $x = \dfrac{y}{|y|}$, there holds $|x| = 1$ and

$$d_\alpha(y, x) = \|h_\alpha(y) - h_\alpha(x)\| = \left\| \left(y, \frac{\alpha}{\alpha - 1}(|y| - 1)\right) - \left(\frac{y}{|y|}, 0\right) \right\| =$$

$$= \left\| \left(0, \frac{\alpha}{\alpha - 1}(1 - |y|)\right) + \left(1 - \frac{1}{|y|}\right)(y, 0) \right\| \leqq$$

$$\leqq \frac{\alpha}{1 - \alpha}(1 - |y|) + \left(\frac{1}{|y|} - 1\right)|y| = \frac{1 - |y|}{1 - \alpha} < 1.$$

Secondly, let $|y| \leqq \alpha$. We have a $P\colon \mathbf{R}^n \oplus \mathbf{R} \to \mathbf{R}^n$ onto projection for which

$$\operatorname{Ker} P = \{\lambda(y, 1)\colon \lambda \in \mathbf{R}\}.$$

By (c), there exists an $(x, c)\in \mathbf{R}^n \oplus \mathbf{R}$ satisfying

$$\|(x, c)\| = 1, \quad \|P(x, c)\| \geqq 1+\delta_0. \tag{2}$$

Clearly we have $P(x, c)=(x-cy, 0)$, so

$$\|(x-cy, 0)\| = |x-cy| \geqq 1+\delta_0. \tag{3}$$

For $z=-\dfrac{x-cy}{|x-cy|}$ there holds $|z|=1$ and

$$d_\alpha(z, y) = \|h_\alpha(z)-h_\alpha(y)\| = \left\|\left(\frac{cy-x}{|x-cy|}, 0\right)-(y, \alpha)\right\| =$$

$$= \left\|\left(\frac{cy-x}{|x-cy|}, 0\right)-\alpha\left(\frac{y}{\alpha}, 1\right)\right\| =$$

$$= \left\|-\frac{\alpha}{c}(x, c)+\frac{c-|x-cy|\alpha}{c}\cdot\left(\frac{cy-x}{|x-cy|}, 0\right)\right\| \leqq$$

$$\leqq \frac{\alpha}{|c|}+\frac{|c-|x-cy|\alpha|}{|c|}.$$

At the last step, we have used (2).

In case of $0<\alpha<\dfrac{|c|}{|x-cy|}$, we have by (3)

$$\frac{\alpha}{|c|}+\frac{|c-|x-cy|\alpha|}{|c|} = 1-\frac{|x-cy|-1}{|c|}\cdot\alpha < 1-\delta_0\frac{\alpha}{|c|},$$

so

$$d_\alpha\left(\frac{-x+cy}{|x-cy|}, y\right) < 1-\delta_0\frac{\alpha}{|c|}. \tag{4}$$

Pick a $\beta>0$. By elementary compactness arguments, $x\in\mathbf{R}^n$ and $c\in\mathbf{R}$ (satisfying condition (2)) can be chosen so that

$$c_1 < |c| < c_2 \quad\text{and}\quad |x-cy| < c_3, \tag{5}$$

for some fixed $c_1, c_2, c_3>0$ whenever $|y|\leqq\beta$.

It is clear that (5) implies (1) provided that

$$\alpha < \min\{\beta, c_1/c_3\}.$$

Theorem. *Let $(X, \|\cdot\|)$ be a real n-dimensional linear space, d a metric on $B_X(0, 1)$ with properties* (i) *and* (ii). *Then there exists an $y^*\in B_X(0, 1)$ such that*

$$\min_{x\in B_X(0,1)} d(y^*, x) \geqq \frac{1}{n}.$$

We shall need the following two lemmas.

Lemma 1. *Let* $(X, \|\cdot\|)$ *and* $(X_1, \|\cdot\|_1)$ *be n-dimensional real normed linear spaces. Then there exists a* $T\colon X \to X_1$ *linear onto operator such that* $\|T^{-1}\| \leqq 1$, *and* $\|T\| \leqq n$.

Lemma 2. *Let* $(Z, \|\cdot\|_\infty)$ *be the n-dimensional* l_∞ *space,* $Y_1 \subset B_X(0, 1)$ *and let us assume we have a nonexpansive mapping* $g\colon Y_1 \to \{z \in Z\colon \|z\| \leqq r\}$ $(r > 0$ *arbitrary). Then there exists a nonexpansive*

$$\tilde{g}\colon B_X(0, 1) \to \{z \in Z\colon \|z\| \leqq r\} \quad \textit{with} \quad \tilde{g}|_{B_X(0,1)} = g.$$

(A special case of [2] p. 48. Theorem 11.2.)

Now, let us prove the theorem. First, by Lemma 1, there exists a $T\colon X \to Z$ linear onto mapping such that $\|T\| \leqq n$, $\|T^{-1}\| \leqq 1$. Let us introduce now the metric d^* on $B_X(0, 1)$ as follows:

$$d^*(y_1, y_2) = n \cdot d(y_1, y_2). \tag{6}$$

Clearly $T = g$ restricted to the set $S_X(0, 1)$ is nonexpansive from (Y_1, d^*) to $Z_1 = \{z \in Z\colon 1 \leqq \|z\|_\infty \leqq n\}$. So, using Lemma 2, we have a

$$\tilde{g}\colon B_X(0, 1) \to \{z \in Z\colon \|z\|_\infty \leqq n\}$$

nonexpansive extension of g.

Since $T^{-1}\tilde{g}$ maps $B_X(0, 1)$ into itself and $T^{-1}\tilde{g}$ restricted to $S_X(0, 1)$ is the identity, it follows from Borsuk's nonrecractibility theorem that $0_X \in T^{-1}\tilde{g}\ (B_X(0, 1))$. Consequently, $0_Z \in \tilde{g}(Y)$. Clearly,

$$\min_{z_1 \in Z_1} \|0_Z - z_1\|_\infty \geqq 1,$$

so, for arbitrary element y^* of $\tilde{g}^{-1}(0_Z)$, there holds

$$\min_{z_1 \in Z_1} d^*(y^*, \tilde{g}^{-1}(z_1)) \geqq 1,$$

and this implies

$$\min_{x \in S_X(0,1)} d^*(y^*, z) \geqq 1.$$

Using (6), we obtain the desired result.

Remark 1. Instead of $1/n$ we can write 1 in the Theorem provided that $(X, \|\cdot\|) = (Z, \|\cdot\|_\infty)$.

Remark 2. Considerations similar to the ones used in the paper play an interesting role in the theory of Liapunov functions [3], and of metrics of Liapunov type [4].

Remark 3. The infinite dimensional analog of the Theorem does not hold. There exist examples d with

$$\inf_{x \in S_X(0,1)} \sup_{y^* \in Y} d(y^*, x) = 0,$$

for arbitrary $(X, \|\cdot\|)$ real, infinite dimensional, normed linear space, where d has properties (i) and (ii).

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BUDAPEST UNIVERSITY OF TECHNOLOGY
FACULTY OF MECHANICAL ENGINEERING
DEPARTMENT OF MATHEMATICS
H—1521 BUDAPEST, HUNGARY