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Lattice ordered binary systems

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By a *groupoid* we mean an *algebra* $(G, \cdot)=:\mathfrak{G}$ of *type* (2). By a *binary system* we mean a groupoid weaker than a *group*. Special binary systems are *semigroups, quasigroups,* and *loops*. The notion binary system was introduced by R. H. Bruck [13].

A binary system is called *partially (lattice-) ordered* if G is partially (lattice-) ordered by an order relation $\leqq$ satisfying:

(0) $$a \leqq b \to xa \leqq xb \,\&\, ax \leqq bx.$$

If $(G, \cdot, \leqq)$ is lattice-ordered we call $(G, \cdot, \leqq)$ briefly a *lattice groupoid*. By a *lattice semigroup* we mean a lattice groupoid satisfying $(ab)c=a(bc)$. Analogously we speak of a *lattice quasigroup* if all equations $ax=b$ and $ya=b$ have unique solutions $a\backslash b$ in the first and b/a in the latter case. Accordingly by a *lattice loop* we mean a lattice quasigroup with *unit* 1. A loop is said to have the *inverse property* if for each x there exists an x^{-1} such that for any a the identities $x^{-1}(xa)=a$ and dually $a=(ax)x^{-1}$ are valid. If $(G, \cdot)$ is an *inverse loop* we have in addition the equations $(x^{-1})^{-1}=x$ and $(xy)^{-1}=y^{-1}x^{-1}$, as is easily checked by the reader.

There is no lack of lattice quasigroups. To see this consider $(R^n, \leqq)$ with respect to $a\circ b:=a+2b$. Furthermore there is an abundance of lattice loops, since starting from a lattice quasigroup $(Q, \circ, \wedge, \vee)$ we get a lattice loop by putting $a\cdot b:=(a/x)\circ(y\backslash b)$, where x, y are fixed elements. And, above all, it should be emphasized that any free loop admits not only a lattice but even a total order [13, 22].

Lattice-ordered binary systems are *congruence distributive* in any case and *congruence permutable* in many cases. Thus the theory of lattice-ordered binary systems is rich from the purely algebraic point of view. On the other hand, however, there are not too many *lattice groupoid* results arising from order theoretic or combined apects although G. Birkhoff [5] and L. Fuchs [19] as well state problems of such type. Nevertheless, at least a fruitful lattice loop theory should be possible as indicated

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already in [5], and even suggested by results of EVANS and HARTMAN [17] who had a first breakthrough after several contributions of different authors like ZELINSKY [41], [42], Kaplansky, Ingraham and Birkhoff (cf. [5]), and ACZÉL [1].

An element a of a partially ordered binary system is called *positive* iff it satisfies $ax \geqq x \leqq xa$ $(\forall x \in G)$. The subset C^+ of all positive elements is called the *(positive) cone*. Dually *negative* elements and the *negative cone* are defined. Both the positive and the negative cone are closed under *multiplication, join* and *meet*, and if in addition a unit element is present the positive cone C^+ coincides with $\{x \mid x \geqq 1\}$, and the negative cone C^- is equal to the subset $\{x \mid x \leqq 1\}$.

The central structure of this paper is that of a *divisibility semiloop*, i.e. *a cancellation groupoid with unit* 1 whose *carrier* is *semilattice-ordered* such that $ax \leqq b \rightarrow \exists u$: $au=b$ and $ya \leqq b \rightarrow \exists v$: $va=b$. Hence a divisibility semiloop is a common abstraction of the lattice loop and the lattice loop cone.

It is a folklore today that any *lattice group* is a *quotient extension* of its cone such that the structure of the whole is completely determined by the structure of the cone. This is quite different in the lattice loop case where not even a *total and complete* order yields any connection between the positive and the negative cone. To verify this the reader may consider the real line with respect to $a \circ b := a+b$ if one of the components is not negative and $a \circ b := a-ab+b$ otherwise, [22]. Hence the situation seems to be hopeless. Nevertheless it is possible to prove a result shedding some light as far as isolated cones are considered, namely: The lattice loop cones are exactly the *positive* divisibility semiloops $(G=C^+)$, and every lattice loop cone is the cone even of an inverse lattice loop. This extends a theorem and answers a question of J. v. NEUMANN (cf. [4]).

Thus a chance might be given to settle general lattice loop problems via inverse lattice loops.

Given a lattice ordered binary system the first order problem to arise is the question what the *descending chain condition* (for *closed intervals*) is equivalent to from the purely algebraic point of view. Hence this question has been treated for different algebraic systems several times, especially for semigroups by ARNOLD [2], CLIFFORD [14], [15], LORENZEN [28] and others (cf. [20]), and for lattice groups by BIRKHOFF [4] and WARD [40]. But the problem remained open for lattice loops until EVANS [16] showed that lattice loops, satisfying the D.C.C. are abelian lattice groups with the *prime factorization property* (P.F.P.). This yields as a corollary that every lattice quasigroup with D.C.C. is the isotope of a free abelian group. See also TESTOV [38]. Therefore a similar investigation of divisibility semiloops is motivated, and it is by no means surprising that an analogue of Evans' theorem remains valid. However it is not the result by which Section 3 is legitimatized in the author's opinion, but the method of proof that justifies this part.

There are two natural generalizations of the D.C.C. and the P.F.P. respectively namely *completeness* (for closed intervals) on the one hand and *representability* on the other hand, i.e. the property to admit a *subdirect decomposition into totally ordered factors.*

As far as completeness is considered we shall prove that *power-associative* divisibility semiloops are *associative* and *commutative* thus carrying over IWASAWA's theorem [27] to our structure. Furthermore it is shown in Section 4 that completeness combined only with *mon*associativity is a too weak requirement with respect to the associativity or commutativity property.

As another topic in the context of completeness we take up the problem of characterizing divisibility semiloops admitting a *complete extension.* This has been done for lattice group like systems several times and it seems to the author that ARNOLD [2] and VAN DER WAERDEN [39] were the first to settle a problem of this type in general, followed by others like LORENZEN [29], CLIFFORD [14], [15], and EVERETT and ULAM [18], the first to treat a noncommutative case. But no nonassociative analysis was given before 1972 when P. A. HARTMAN [22], [23] settled the problem for partially ordered quasigroups and loops. Of course, there are further results, consult for instance [5] and [19], above all the initial contribution of RICHARD DEDEKIND (cf. [5]). Hence characterizing divisibility semiloops with complete extensions is a most natural additional step according to a long lasting development (Section 5).

Finally we turn to representable divisibility semiloops.

There are various results concerning lattice-ordered structures of such type, the historical one being Stone's celebrated decomposition theorem for boolean algebras, afterwards extended to distributive lattices (cf. [5]), for instance: LORENZEN [28], CLIFFORD [15], RIBENBOIM [32] (abelian lattice-ordered groups); LORENZEN [29], ŠIK [34], BANASCHEWSKI [3] (arbitrary lattice-ordered groups); SWAMY [37] (abelian residuated lattice-ordered semigroups); BOSBACH [8], [10] (complementary semigroups); TH. MERLIER [30] (abelian lattice-ordered monoids); FUCHS [20] (general lattice-ordered algebras); FUCHS [21] (positive abelian lattice-ordered monoids); BIRKHOFF and PIERCE [6] (lattice-ordered rings); EVANS and HARTMAN [17] (lattice-ordered loops).

But a general solution is still outstanding and also special problems have remained unsolved up to now although they were stated several times, like the lattice semigroup problem [19], [21] or the lattice groupoid and the lattice quasigroup problem [17]. Therefore Section 6 will be devoted not only to divisibility semiloops with a representation, but also to general lattice-ordered binary systems of this type, the principal result being a decomposition theorem that solves the problems mentioned above in a one cast manner.

The notation of this paper is standard in general, but sometimes $\hat{:}$ will stand for *"such that"* and $a\cdot bc$ for $a(bc)$. Consequently $a\cdot\cdot b\cdot cd$ f.i. will mean $a(b(cd))$.

The basic concepts of algebra and order theory are to be found in [5]. The later paragraphs are based only on Section 1.

Finally we give a most important hint. There will appear *dualities* of various kinds, for instance right/left dualities or $\geqq/\leqq$ dualities. Hence there will be propositions holding necessarily together with their dual. So the reader should realize this situation whenever it comes up. Nevertheless he will be requested from time to time to take that fact into account.

1. Divisibility semiloops

1.1. Definition. By a *divisibility semiloop* we mean an algebra $\mathfrak{G}:=(G, \cdot, \wedge, 1)$ of type $(2, 2, 0)$ satisfying

(DSL 1) $(G, \cdot)$ is a cancellation groupoid,
(DSL 2) 1 is unit of $(G, \cdot)$,
(DSL 3) $(G, \wedge)$ is a semilattice,
(DSL 4) $x(a\wedge b)\cdot y = xa\cdot y\wedge xb\cdot y$

(observe that (DSL 4) requires right- and left-distributivity because of axiom (DSL 2)),

(DSL 5) $ax\leqq b\rightarrow\exists u\colon au=b,\ ya\leqq b\rightarrow\exists v\colon va=b$

(observe furthermore that the negative cone of any divisibility semiloop is itself a positive divisibility semiloop with respect to $\vee$).
Classical examples of a divisibility semiloop are the lattice loop and the lattice loop cone. Therefore the divisibility semiloop is a common abstraction of these two structures.

For the sake of convenience we start from an arbitrary but fixed divisibility semiloop.

1.2. Lemma. $\forall a, b, x, y\colon a\leqq b\rightarrow ax\leqq bx\ \&\ ya\leqq yb$ *and*

$$ax\leqq bx \vee ya\leqq yb\rightarrow a\leqq b.$$

Proof. Obviously we may confine ourselves to the left-sided cases. But these follow by $a\leqq b\rightarrow ya\wedge yb=y(a\wedge b)=ya$ for the left-right direction and from $ya=ya\wedge \wedge yb\rightarrow ya=y(a\wedge b)\rightarrow a=a\wedge b$ otherwise.

1.3. Lemma. $b\geqq 1\ \&\ a''(a\wedge c)=a\rightarrow a\wedge bc=(a''\wedge b)(a\wedge c)$.

Proof. $b\geqq 1\rightarrow a\wedge bc=a''(a\wedge c)\wedge ba\wedge bc=(a''\wedge b)(a\wedge c)$.

As an immediate consequence of 1.3 we get.

1.3′. Lemma. $x \leqq bc \,\&\, b \geqq 1 \rightarrow x = x_b x_c \,\hat{:}\, x_b \leqq b \,\&\, x_c \leqq c$.

1.4. Proposition. $(a \wedge b)a' = a \rightarrow ba' = \sup(a, b) =: a \vee b$.

Proof. Suppose $(a \wedge b)a' = a \geqq (a \wedge b)1$. Then we can infer $a' \geqq 1$ and thereby: $ba' \geqq a$ & $ba' \geqq b$. On the other hand any c with $c \geqq a, b$ satisfies for some x the implication: $c = bx$ & $a = (a \wedge b)(a' \wedge x) \rightarrow a' = x \wedge a' \rightarrow a' \leqq x \rightarrow ba' \leqq bx = c$ which had to be proved. (Similarly one shows that $(a \vee b)a' = a$ & $(a \vee b)b' = b$ implies $ab' = a \wedge b$. This is possible by means of (DSL 5).)

1.5. Lemma. $x(a \vee b) \cdot y = xa \cdot y \vee xb \cdot y$.

Proof. Suppose $xa \vee xb = (xa)c$. Then by (DSL 5) there is an element u such that $xu = xa \vee xb$ from which follows $u \geqq a \vee b$ and thereby $x(a \vee b) = xa \vee xb$. The rest follows by duality.

1.6. Lemma. $(a \wedge b)a' = a \,\&\, (a \wedge b)b' = b \rightarrow (a \wedge b)a' \cdot b' = (a \wedge b)(a' \vee b')$.

Proof. $(a \wedge b)a' \cdot b' = ab' = a \vee b = (a \wedge b)(a' \vee b')$.

1.7. Corollary. $b \wedge c = 1 \vee b \vee c = 1 \rightarrow ab \cdot c = ac \cdot b = a \cdot bc$.

Proof. Indeed, $b \wedge c = 1 \rightarrow ab \wedge ac = a$ and $b \vee c = 1 \rightarrow ab \vee ac = a$.

1.8. Corollary. $a \wedge b = 1 \rightarrow ab = a \vee b = ba$.

1.9. Lemma. $ab = cd \rightarrow ab = (a \wedge c)(b \vee d) = (a \vee c)(b \wedge d)$.

Proof. $ab = cd \rightarrow ab \geqq (a \wedge c)b \vee (a \wedge c)d = (a \wedge c)(b \vee d)$

$$\& \, ab \leqq a(b \vee d) \wedge c(b \vee d) = (a \wedge c)(b \vee d).$$

1.10. Corollary. $a = (1 \wedge a)(1 \vee a) = (1 \vee a)(1 \wedge a)$.

1.11. Definition. By *the positive part of a* we mean the element $1 \vee a =: a^+$, by *the negative part of a* we mean the element $1 \wedge a =: a^-$. By a^* we denote the uniquely determined element x satisfying $a^- x = 1$, and we define dually $a^{\cdot}$, satisfying $a^{\cdot} a^- = 1$.

There is a series of crucial lemmata interlinking these notions.

1.12. Lemma. $ab = ab^+ \cdot b^- = ab^- \cdot b^+$.

Proof. Write $ab = a1 \cdot b = ab \cdot 1$ and apply Lemma 1.9.

1.13. Lemma. $a^+ \wedge a^* = 1$.

Proof. $a^+ = aa^* \,\&\, aa^* \wedge a^* = (a \wedge 1)a^* = 1$.

1.14. Lemma. $c \leqq 1 \,\&\, b \wedge c^* = 1 \rightarrow a \cdot bc = ab \cdot c = ac \cdot b$.

Proof. $b \wedge c^* = 1 \rightarrow 1 \wedge cb = (1 \wedge c)(c^* \wedge b) = 1 \wedge c$ by the dual of Lemma 1.3. Thus, if moreover c is negative, we may infer $c = (cb)^-$ and $b = (cb)^+$ from which we get $a \cdot bc = ab \cdot c = ac \cdot b$ by Lemma 1.12.

1.15. Lemma. $u\wedge a^*=1\to a^-u\wedge 1=a^-\leftrightarrow 1\vee a^-u=u\to u\wedge a^*=1$.

This implies nearly immediately

1.13′. Lemma. $y\leqq 1\leqq x\,\&\,x\wedge y^*=1\,\&\,xy=a\to x=a^+\,\&\,y=a^-$.

Moreover 1.15 is essential for part (i) of the subsequent statement.

1.16. Lemma. (i) $(ab)^+=(1\vee a^+b^-)(1\vee a^-b^+)$, $(ab)^-=(1\wedge a^-b^+)(1\wedge a^+b^-)$,
(ii) $(a\wedge b)^+=a^+\wedge b^+\,\&\,(a\wedge b)^-=a^-\wedge b^-$,
(iii) $(a\vee b)^+=a^+\vee b^+\,\&\,(a\vee b)^-=a^-\vee b^-$.

Proof. Ad (i). By 1.14 we have

$$ab = a^+a^-\cdot b^+b^- = (a^+\cdot a^-b^+)b^- = \big(a^+(1\vee a^-b^+)\big)\big((1\wedge a^-b^+)\,b^-\big) =$$
$$= \big((1\wedge a^-b^+)\cdot a^+b^-\big)(1\vee a^-b^+),$$

from which (i) follows by repeating the method on the grounds of

$$u\wedge a^* = 1 = u\wedge b^* \to (1\vee a^-u)(1\vee b^-u) = uu = u\vee (a^-b^-\cdot u)\,u.$$

(We shall come back to this implication in Chapter 4.)

Ad (ii) & (iii). $x \leqq a \leftrightarrow 1\wedge x \leqq 1\wedge a\,\&\,1\vee x \leqq 1\vee a$ and $(a^-\wedge b^-)^* = a^*\vee b^*$ and $(a^-\vee b^-)^* = a^*\wedge b^*$ by 1.9.

1.17. Lemma. $a\wedge b^{:}=1\leftrightarrow a\wedge b^*=1$.

Proof. $a\wedge b^{:}=1\to ab^-\wedge 1=b^-\to a(b^-b^*)\wedge b^*=1\to a\wedge b^*=1$.

We now introduce two further operations.

1.18. Definition. x is called *the right complement* $a*b$ *of* a *in* b if $(a\wedge b)x=b$. Dually we define *the left complement* $b:a$ *of* a *in* b.

Because of $(a\wedge b)(a*b\wedge b*a)=a\wedge b$ we get immediately $a*b\wedge b*a=1$. Next we have

1.19. Lemma. $a\wedge b=a/(b*a)=(b:a)\backslash b$ *and* $a\vee b=a(a*b)=(a:b)\,b$.

1.20. Lemma. $a\leqq b\to x*a\leqq x*b\,\&\,a*x\geqq b*x$.

Furthermore we obtain

1.21. Lemma. $a*(b\vee c)=a*b\vee a*c$.

Proof. $a(a*b\vee a*c)=a(a*b)\vee a(a*c)=a\vee b\vee a\vee c=a\vee(b\vee c)$.

1.22. Lemma. $(a\wedge b)*c=a*c\vee b*c$.

Proof. $(a\wedge b)*c\geqq a*c\vee b*c\,\&\,(a\wedge b)(a*c\vee b*c)\geqq(a\wedge b)\vee c$.

1.23. Lemma. $a*(b\wedge c)=a*b\wedge a*c$.

Proof. We have $a*(b\wedge c)\leqq a*b\wedge a*c$ and

$$(a\wedge b\wedge c)(a*b\wedge a*c) \leqq (a\wedge b)(a*b)\wedge(a\wedge c)(a*c) = b\wedge c$$

whereby $a*b\wedge a*c\leqq a*(b\wedge c)$.

1.24. Lemma. $(a\vee b)*c=a*c\wedge b*c$.

Proof. We have $(a\vee b)*c\leqq a*c\wedge b*c$ and

$$(a\vee b)(a*c\wedge b*c) \leqq a(a*c)\vee b(b*c) = a\vee b\vee c$$

whereby $a*c\wedge b*c\leqq(a\vee b)*c$.

The reader should check that 1.21 through 1.24 remain valid if we replace $*$ by $\backslash$ and : by $/$, provided the "results" under consideration do exist. Now, applying Lemma 1.23 we are able to prove

1.25. Proposition. $(G, \wedge, \vee)$ *is distributive.*

Proof. $a\vee(b\wedge c)=a(a*(b\wedge c))=a(a*b)\wedge a(a*c)=(a\vee b)\wedge(a\vee c)$ (and, alternatively, by applying 1.24, $a\wedge(b\vee c)=a/((b\vee c)*a)=a/(b*a)\vee a/(c*a)=$ $=(a\wedge b)\vee(a\wedge c)$).

In the remainder of this section special situations are considered with respect to later paragraphs.

1.26. Definition. We say that *a covers b* if a satisfies $a>b$ and no element of G lies strictly between a and b. By an *atom* we mean any p which covers 1.

1.27. Lemma. *Every atom is prime, i.e. every atom satisfies the implication* $p\leqq a^+b^+\rightarrow p\leqq a^+ \vee p\leqq b^+$.

Proof. $p\leqq a^+b^+ \,\&\, p\nleqq b^+\rightarrow p=(p\wedge a^+)(p\wedge b^+)=p\wedge a^+$ by (1.3).

Recall that the standard meaning of p^n is $(\cdots((pp)p)p\cdots)$.

1.28. Lemma. *Every atom p satisfies* $ap\cdot p^n=a\cdot pp^n$.

Proof. $ap\cdot p^n=a\cdot qp^n$ & $p\neq q$ implies $ap\cdot p^n=aq\cdot p^n$ because of Lemma 1.7, since $ap\cdot p^n$ covers ap^n, whence q is an atom.

1.29. Corollary. *The natural powers of any atom p form a subsemigroup.*

Proof. This is easily shown by induction on the grounds of 1.28.

1.30. Lemma. *Every atom satisfies* $px=1\leftrightarrow xp=1$.

Proof. We prove the left-right direction: 1 covers x and moreover we have $x \leqq 1 \leqq p$ & $x \leqq xp \leqq p$ whence we can infer

$$1 \wedge xp = 1 \rightarrow xp = 1 \quad \text{because of} \quad xp < p$$

$$\vee\, 1 \wedge xp = x \rightarrow 1 \vee xp = p \rightarrow xp = px = 1.$$

We are now turning to rules relevant for Section 4.

1.31. Lemma. *Let the right inverses a^r and b^r exist. Then $a \wedge b$ and $a \vee b$ are right invertible, too, and they satisfy the formulas*

$$(a \wedge b)^r = a^r \vee b^r \quad \textit{and} \quad (a \vee b)^r = a^r \wedge b^r.$$

Proof. $aa^r = 1 = bb^r \rightarrow (a \wedge b)(a^r \vee b^r) = 1 = (a \vee b)(a^r \wedge b^r)$.

Furthermore we shall need some implications for *orthogonal pairs* a, b, i.e. pairs with $a \wedge b = 1 \leftrightarrow: a \perp b$. Here we obtain:

1.32. Lemma. *If $\mathfrak{G}$ is positive, i.e. $G = G^+$, then*

$$a \perp b \rightarrow a * bc = b(a * c) \ \&\ cb : a = (c : a)b.$$

Proof. Making use of 1.3 and 1.7 we get

$$a \perp b \rightarrow (a \wedge bc)\big(b(a * c)\big) = (a \wedge c)\big(b(a * c)\big) = b \cdot (a \wedge c)(a * c) = bc$$

and the rest follows by duality.

1.33. Lemma. *If $\mathfrak{G}$ is positive, then*

$$a \perp c \rightarrow ab * c = b * c = ba * c \,\&\, c : ab = c : b = c : ba.$$

Proof. $a \perp c \rightarrow (ab \wedge c)x = c \rightarrow (b \wedge c)x = c$ by Lemma 1.3, and the rest follows by duality.

1.34. Lemma. *If $\mathfrak{G}$ is positive, then*

$$a \perp b \rightarrow xa * xb = b \,\&\, bx : ax = b.$$

Proof. $a \perp b \rightarrow (xa \wedge xb)y = xb \rightarrow x(a \wedge b) \cdot y = xb \rightarrow y = b$, the rest following by duality.

1.35. Lemma. *If $\mathfrak{G}$ is positive and associative then $\mathfrak{G}$ satisfies*

(i) $ab * c = b * (a * c)$,

(ii) $a * (b : c) = (a * b) : c$,

(iii) $a * bc = (a * b)\big((b * a) * c\big)$.

Proof. These formulas were developed already in earlier papers of the author but for the sake of selfcontainedness we give short proofs in spite of this.

Ad (i): $abx \geqq c \leftrightarrow bx \geqq a*c \leftrightarrow x \geqq b*(a*c)$,
Ad (ii): $ax \geqq b:c \leftrightarrow axc \geqq b \leftrightarrow cx \geqq a:b$,
Ad (iii): $ax \geqq bc \leftrightarrow x = (a*b)y \leftrightarrow a(a*b)y = b(b*a)y \geqq bc$.

Henceforth we consider (conditionally) complete divisibility semiloops. Here we obtain analogously to the finite case:

1.36. Lemma. *If $\mathfrak{G}$ is complete then $\mathfrak{G}$ satisfies the equation:*
(i) $x(\bigvee a_i)\cdot y = \bigvee(xa_i\cdot y)\,\&\, x(\bigwedge a_i)\cdot y = \bigwedge(xa_i\cdot y)$, *implying*
(ii) $x\diagdown(\bigvee a_i) = \bigvee(x\diagdown a_i)\,\&\, x\diagdown(\bigwedge a_i) = \bigwedge(x\diagdown a_i)$ *and*
(iii) $(\bigvee a_i)\diagdown x = \bigwedge(a_i\diagdown x)\,\&\,(\bigwedge a_i)\diagdown x = \bigvee(a_i\diagdown x)$, *implying*
(iv) $a\wedge\bigvee b_i = \bigvee(a\wedge b_i)\,\&\, a\vee\bigwedge b_i = \bigwedge(a\vee b_i)$.

Proof. The proof is left to the reader since it is analogous to the corresponding proofs of the finite cases. (Of course, (ii) and (iii) are valid as far as the objects under consideration do exist.)

Finally we remark

1.37. Lemma. *$\mathfrak{G}$ is already complete if its (positive) cone is complete. More precisely:* $s \leqq a_i \rightarrow \bigwedge(1\vee a_i)\cdot\bigwedge(1\wedge a_i) = \bigwedge a_i$.

Proof. This is an immediate consequence of $x \leqq a_i$ if and only if $1\vee x \leqq 1\vee a_i$ & $(1\wedge x)^* \geqq (1\wedge a_i)^*$ which implies for lower bounded sets a_i $(i\in I)$ the formula stated above.

2. Lattice loop cones

The structure of a lattice group is completely determined (up to isomorphism) by the structure of its cone. The question arises whether the same is true in the lattice loop case. Obviously the situation is pleasant as far as the underlying lattice is considered (1.31). But it was already shown in the introduction, that non-isomorphic lattice loops may have isomorphic cones. Hence the question is reduced to the problem whether it is possible to characterize those divisibility semiloops which admit some lattice loop extension. To this end we start from a positive divisibility semiloop $\mathfrak{C}$.

2.1. Definition. By L we denote the set of all *orthogonal pairs* $(a|b)$ $(a\perp b,$ $a, b\in C)$. Furthermore $\mathfrak{L}$ will symbolize the structure $(L, \circ, \wedge)$ the operations of which are defined by

$$(a|b)\circ(c|d) := ((a:d)(b*c)|(d:a)(c*b))$$

and

$$(a|b)\wedge(c|d) := (a\wedge c|b\vee d).$$

Obviously $\circ$ is defined in a right left dual manner. This means: a proposition and its proof remain true if $(x|y)$ is replaced by $(y|x)$ and $a*b$ by $b:a$, $c:d$ by $d*c$. Furthermore by Lemma 1.3 $\circ$ is an operation.

2.2. Lemma. $(L, \wedge)$ *is a semilattice.*

Proof. We have to show $a \perp b$ & $c \perp d \to a \wedge c \perp b \vee d$, which follows from $(a \wedge c) \wedge (b \vee d) = (a \wedge c \wedge b) \vee (a \wedge b \wedge d)$

2.3. Lemma. $\mathfrak{L}$ *satisfies*

$$(a|b) \leqq (c|d) \to (a|b)\circ(x|y) \leqq (c|d)\circ(x|y) \;\&\; (x|y)\circ(a|b) \leqq (x|y)\circ(a|b).$$

Proof. This is an immediate consequence of Lemma 1.20.

2.4. Lemma. $(a|b)\circ(c|d) = ((a|b)\circ(c|1))\circ(1|d) = (a|1)\circ((1|b)\circ(c|d))$.

Proof. By 1.32 and 1.33

$$((a:d)(b*c)|(d:a)(c*b)) = (a(b*c):d|(d:a)(c*b)) =$$
$$= (a(b*c):d|(d:a(b*c))(c*b)) = (a(b*c)|c*b)\circ(1|d) =$$
$$= ((a|b)\circ(c|1))\circ(1|d),$$

the rest following by duality.

2.5. Lemma. $((a|b)\circ(1|x))\circ(x|1) = (a|b) = (1|x)\circ((x|1)\circ(a|b))$.

Proof. We have

$$((a:x)|(x:a)\,b)\circ(x|1) = ((a:x)((x:a)\,b*x)|x*(x:a)\,b) =$$
$$= ((a:x)((x:a)\,b*(x:a)(a \wedge x))|(x:a)(a \wedge x)*(x:a)\,b) = ((a:x)(x \wedge a)|b) = (a|b)$$

by 1.34, the rest following by duality.

2.6. Lemma. $((a|b)\circ(x|1))\circ(1|x) = (a|b) = (x|1)\circ((1|x)\circ(a|b))$.

Proof. We have

$$(a(b*x)|(x*b)\circ(1|x)) = (a(b*x):x|(x:a(b*x))(x*b)) =$$
$$= (a(b*x):(b \wedge x)(x*b)|((x:(b*x)):a)(x*b)) = (a|(x \wedge b)(x*b)) = (a|b)$$

by 1.34, the rest following by duality.

2.7. Lemma. $((a|b)\circ(x|y))\circ(y|x) = (a|b) = (x|y)\circ((y|x)\circ(a|b))$.

Proof. We have

$$((a|b)\circ(x|y))\circ(y|x) = ((((a|b)\circ(x|1))\circ(1|y))\circ(y|1))\circ(1|x) =$$
$$= ((a|b)\circ(x|1))\circ(1|x) = (a|b),$$

the rest following by duality.

2.8. Lemma. *$(a|b)\circ(x|y)\doteq(c|d)$ and $(u|v)\circ(a|b)\doteq(c|d)$ have uniquely determined solutions.*

Proof. Apply Lemma 2.7. It follows that $(x|y)=((b|a)\circ(c|d))$ in the first case and $(u|v)=((c|d)\circ(b|a))$ in the second case are the only solutions.

2.9. Lemma. $(a|b)\circ(1|1)=(a|b)=(1|1)\circ(a|b)$.

Proof. $(a:1|1*b)=(a|b)=(1*a|b:1)$.

2.10. Lemma. $(a|1)\circ(b|1)=(ab|1)$ *and* $(a|1)\wedge(b|1)=(a\wedge b|1)$.

Proof. Obvious.

Hence summarizing the lemmata proven so far we get

2.11. Proposition. *A partially ordered groupoid is the cone of some lattice loop if and only if it is a positive divisibility semiloop.*

2.12. Definition. By an *inverse* loop we mean a loop having the inverse property, i.e. satisfying $\forall a\, \exists a^{-1}$: $a^{-1}(ab)=b=(ba)a^{-1}$.

Obviously inverse loops satisfy $xx^{-1}=1=x^{-1}x$ and furthermore one can infer $(xy)^{-1}=y^{-1}x^{-1}$, since $(xy)y^{-1}=x\to y^{-1}=(xy)^{-1}x\to y^{-1}x^{-1}=(xy)^{-1}$. In general a lattice loop is far from being inverse. However we can prove

2.13. Proposition. *Any lattice loop cone is the cone of an inverse lattice loop.*

Proof. We define $(x|y)^{-1}:=(y|x)$. Then the assertion is proven by Lemma 2.7.

Let us consider now the extension $\mathfrak{L}$ of the cone $\mathfrak{C}$. We shall show that $\mathfrak{L}$ is uniquely determined up to isomorphism provided inverse lattice loops are considered. Furthermore we shall prove some other extension properties concerning congruence relations and order.

2.14. Proposition. *$\mathfrak{L}$ is uniquely determined provided inverse extensions are considered.*

Proof. Let $\mathfrak{J}$ denote an inverse lattice loop. Then by Lemma 1.16 we can infer $ab^{-1}\cdot cd^{-1}=a(1\vee b^{-1}c)\cdot(1\wedge b^{-1}c)d$ and by the rules of lattice loop arithmetic we get $1\vee a^{-1}b=a*b$ since $a(1\vee a^{-1}b)=a\vee b$, and $1\vee ba^{-1}=b:a$ by duality. Thus $1\wedge a^{-1}b=(1\vee b^{-1}a)^{-1}=(b*a)^{-1}$ and $1\wedge ba^{-1}=(a:b)^{-1}$ by duality, whence

$$ab^{-1}=(1\vee ab^{-1})(1\wedge ab^{-1})=(a:b)(b:a)^{-1}.$$

But applying these formulas and 1.16 we obtain

$$ab^{-1}\cdot cd^{-1} = a(1\vee b^{-1}c)\cdot(1\wedge b^{-1}c)\,d^{-1} = a(b*c)\cdot(c*b)^{-1}d^{-1} =$$
$$= (c*b)^{-1}\cdot\big(a(b*c)\cdot d^{-1}\big) = (c*b)^{-1}\cdot\big(ad^{-1}\cdot(b*c)\big) =$$
$$= (c*b)^{-1}\cdot\big((a:d)(d:a)^{-1}\cdot(b*c)\big) = (a:d)(b*c)\cdot(c*b)^{-1}(d:a)^{-1} =$$
$$= (a:d)(b*c)\cdot\big((d:a)(c*b)\big)^{-1}.$$

Hence the function $(a|b)\to ab^{-1}$ is an isomorphism of $\mathfrak{L}$ and $\mathfrak{J}$ if the cone $\mathfrak{C}$ is isomorphic to the cone of $\mathfrak{J}$.

We now turn to elementary algebraic properties like associativity, commutativity, etc., the first result of this type being nearly obvious:

2.15. Lemma. *If $\mathfrak{C}$ is commutative then $\mathfrak{L}$ is commutative, too.*

Proof. If $\mathfrak{C}$ is commutative then $x:y$ is equal to $y*x$ which yields

$$(a|b)\circ(c|d) = \big((a:d)(b*c)|(d:a)(c*b)\big) =$$
$$= \big((b*c)(d*a)|(c*b)(a*d)\big) = (c|d)\circ(a|b).$$

A loop $\mathfrak{L}$ is called *monassociative* if every $a\in L$ generates a subsemigroup of $(L,\cdot)$. A loop is called *power-associative* if every $a\in L$ generates a subgroup of $(L,\cdot,\diagdown,\diagup)$.

2.16. Lemma. *If $\mathfrak{C}$ is monassociative then $\mathfrak{L}$ is power-associative.*

Proof. By Lemma 1.3 we get $(a|b)^n=(a^n|b^n)$ $(n\in\mathbf{N})$ and by the inverse property we have $(a|b)^{-n}=((a|b)^{-1})^n$.

2.17. Lemma. *If $\mathfrak{C}$ is associative then $\mathfrak{L}$ is associative, too.*

Proof. We show

$$\big((a|1)\circ(c|d)\big)\circ(1|v) = (a|1)\circ\big((c|d)\circ(1|v)\big),$$
$$\big((1|b)\circ(c|d)\big)\circ(1|v) = (1|b)\circ\big((c|d)\circ(1|v)\big),$$
$$\big((1|b)\circ(c|d)\big)\circ(u|1) = (1|b)\circ\big((c|d)\circ(u|1)\big).$$

(Observe that line 3 can be considered as a dual of line 1, since putting $a\cdot b:=ba$ we get a dual divisibility semiloop with $(a|b)\bullet(c|d)=(c|d)\circ(a|b)$. Hence line 3 results from line 1 for the dual structure.)

Equivalently

$$\big((a:d)\,c:v|(v:(a:d)\,c)(d:a)\big) = \big((a:(v:c)\,d)(c:v)|(v:c)\,d:a\big),$$
$$\big((b*c):v|(v:(b*c))\,d(c*b)\big) = \big(b*(c:v)|(v:c)\,d((c:v)*b)\big),$$

and

$$\big((b*c)(d(c*b)*u)|u*d(c*b)\big) = \big(b*c(d*u)|(u*d)(c(d*u)*b)\big).$$

But lines 1 and 3 follow from Lemma 1.35 and its duals, and the left components of the second equation are equal because of 1.35, too. So it remains to show

$$(v:(b*c))\,d(c*b)*(v:c)\,d((c:v)*b) = 1,$$
$$(v:(b*c))\,d(c*b):(v:c)\,d((c:v)*b) = 1.$$

Now, the second equation is the right-left dual of the first one. Therefore it suffices to settle the first case. Here we obtain:

$$\begin{aligned}(v:(b*c))\,d(c*b)*(v:c)\,d((c:v)*b) &= \\ = d(c*b)*(((v:c)*(v:(b*c)))*d((c:v)*b)) &= \\ = d(c*b)*(((c:v)*(c:(b*c)))*d((c:v)*b)) &= \\ = d(c*b)*d(((c:v)*(c:(b*c)))*((c:v)*b)) &= \\ = (c*b)*(((c\wedge b)*(c:v))*((c\wedge b)*b)) &= \\ = (c*b)*(((c\wedge b)*(c:v))*(c*b)) &= 1.\end{aligned}$$

The second, third, and fourth equalities follow from 1.35, 1.32 and 1.19, 1.35, respectively. Hence the proof is completed by

$$\begin{aligned}((a|b)\circ(c|d))\circ(u|v) &= (((a|1)\circ((1|b)\circ(c|d)))\circ(u|1))\circ(1|v) = \\ &= ((a|1)\circ(((1|b)\circ(c|d))\circ(u|1)))\circ(1|v) = \\ &= (a|1)\circ((((1|b)\circ(c|d))\circ(u|1))\circ(1|v)) = \\ &= (a|1)\circ(((1|b)\circ((c|d)\circ(u|1)))\circ(1|v)) = \\ &= (a|1)\circ((1|b)\circ(((c|d)\circ(u|1))\circ(1|v))) = (a|b)\circ((c|d)\circ(u|v)).\end{aligned}$$

We continue our investigation by two further results concerning the order relation.

2.18. Lemma. *If $\mathfrak{C}$ is totally ordered then $\mathfrak{Q}$ is totally ordered, too.*

Proof. $a\leqq b\rightarrow(a|b)=(1|b)$ and $a\geqq b\rightarrow(a|b)=(a|1)$. Furthermore we get $(a|1)\geqq\geqq(1|b)$ for all $a, b\in C$.

2.19. Lemma. *If $\mathfrak{C}$ is completely ordered then $\mathfrak{Q}$ is completely ordered, too,*

Proof. Apply Lemma 1.37.

Finally we consider congruences. Here we can show

2.20. Proposition. *The congruences of $(C, \cdot, *, :)$ are uniquely extended to $\mathfrak{Q}$.*

Proof. Let $\equiv$ be a congruence of $(C, \cdot, *, :)$. We define $(a|b)\equiv(c|d)$ iff $a\equiv c$ & $b\equiv d$. This provides a congruence on $\mathfrak{Q}$ as is easily checked by the reader. On the other hand for any extension ϱ of $\equiv$ from $(C, \cdot, *, :)$ to $\mathfrak{Q}$ we get $(a|b)\,\varrho\,(c|d)\leftrightarrow ad\equiv bc$ which implies $a\equiv c$ & $b\equiv d$ because of Lemma 1.3.

3. The chain condition

Obviously a divisibility semiloop satisfies the descending chain condition for any $[a, b)$ iff it satisfies the ascending chain condition for any $(a, b]$. Hence we may speak of models with chain condition (C.C.). Suppose in this section that $\mathfrak{G}$ has the C.C.-property. Then every positive element a is a product of atoms since otherwise there would be a minimal one to fail, a contradiction. Furthermore for every $a > 1$ and arbitrary atom p there exists a maximal number $p(a)$ such that $p^{p(a)} \leqq a$. Finally for any pair of different atoms p, q we get $p^m \perp q^n$ $(m, n \in \mathbf{N})$ because of 1.3, and thereby $p^m \cdot q^n = p^m \vee q^n$. This provides a uniquely determined prime factorization for any positive $a \in G$ (see f.i. [16]).

The purpose of this paragraph is to show that C.C. implies commutativity and associativity. This is nearly obvious for C^+ and by duality also for C^- (consult 1.29 and the remark above). But the general case requires some additional calculation.

3.1. Lemma. *Let $\bar{q}$ be the right inverse of q and let p, q be two atoms. Then every p^m commutes with every $\bar{q}^n$.*

Proof. It suffices to prove $p\bar{p} = 1 \rightarrow p^m \cdot \bar{p}^m = 1$, because of 1.14, 1.30. But this is shown by induction since 1.28 implies $p^m p \cdot \bar{p}\bar{p}^m = p^m(p\bar{p} \cdot \bar{p}^m)$.

3.2. Lemma. *If $\mathfrak{G}$ satisfies* C.C. *then $\mathfrak{G}$ is associative and commutative.*

Proof. By 3.1 and the distributivity laws we get $a^+ \cdot b^- = b^- \cdot a^+$ whence $a^+ \cdot b = a^+ b^+ \cdot b^- = b^- \cdot b^+ a^+ = b \cdot a^+$ and dually $a \cdot b^- = b^- \cdot a$. Hence we obtain $a \cdot b = a^- \cdot a^+ b = ba^+ \cdot a^- = ba$. Furthermore we have $ab^- \cdot c^- = a \cdot b^- c^-$. Thus we get $ab \cdot c = (a^+ b^+ \cdot a^- b^- \cdot \cdot c^-)c^+ = c^+(a^+ b^+ \cdot a^- b^- c^-) = c^+ a^+ b^+ \cdot a^- b^- c^- = a \cdot bc$.

Summarizing the preceding remarks and results we get

3.3. Theorem. *A divisibility semiloop satisfies the chain condition for closed intervals* $[a, b]$ *if and only if it is a direct sum of copies of* $(\mathbf{Z}, +, \min)$ *and* $(\mathbf{N}^0, +, \min)$ *respectively.*

4. Complete divisibility semiloops

In this section we shall prove that power-associative complete divisibility semiloops are even associative and commutative. This was done for loops with the real line as underlying lattice by Aczél [1], and for totally ordered loops in general by Hartman [22].

4.1. Definition. $\mathfrak{G}$ is called *power-associative* if any element a generates a subsemigroup and any pair a^-, a^* generates a subgroup of $(G, \cdot)$.

4.2. Definition. Extending the relation $\perp$, henceforth by $u \perp x$ we shall mean $u^+u^* \wedge x^+x^* = 1$. Furthermore $U^\perp$ will denote the set of all x satisfying $u \perp x$, where u is running through U.

It is easily checked by Lemma 1.16 and Lemma 1.31 that $U^\perp$ is a multiplicatively closed sublattice of $\mathfrak{C}$.

4.3. Lemma. *Let $\mathfrak{C}_1 \times \mathfrak{C}_2$ be a direct decomposition of $(C^+, \cdot, \wedge, \vee)$. Then $\mathfrak{C}_1^\perp \times \mathfrak{C}_2^\perp$ is a direct decomposition of $\mathfrak{G}$.*

Proof. We denote $C_1^\perp$ by G_2 and $C_2^\perp$ by G_1. Then every element a is a product of type a_1a_2 where the indices indicate the components G_1, G_2. To see this we consider a^-. There is a decomposition $a^* = a_1^* a_2^*$ and we have $a^- a_1^* \leqq 1$ and $a^- a_2^* \leqq 1$ whence there are elements a_1^{*l} and a_2^{*l} with $(a_1^{*l} \cdot a_2^{*l}) \cdot (a_1^* \cdot a_2^*) = 1$. Hence $a_1^{*l} a_2^{*l}$ is equal to a^- and by definition a_1^{*l} and a_2^{*l} are contained in G_1 and G_2 respectively. But this yields

$$a^+a^- = a_1^+a_2^+ \cdot a_1^{*l}a_2^{*l} = a_1^+(a_2^+ \cdot a_1^{*l}a_2^{*l}) = a_1^+(a_1^{*l} \cdot a_2^+a_2^{*l}) = a_1^+a_1^{*l} \cdot a_2^+a_2^{*l}$$

by means of 1.14, 1.17, 1.3, and, applying 1.14, 1.3, we obtain furthermore

$$a_1a_2 = b_1b_2 \to a_1^+a_2^+ \cdot a_1^-a_2^- = b_1^+b_2^+ \cdot b_1^-b_2^- \to$$
$$\to a_1^+a_2^+ = b_1^+b_2^+ \,\&\, a_1^-a_2^- = b_1^-b_2^- \to a_1^+ = b_1^+ \ldots a_2^- = b_2^-,$$

since $a_1^- a_2^- \cdot a_1^* a_2^* = 1$, which implies $a_1^+ a_2^+ \perp (a_1^- a_2^-)^*$.

Hence G may be considered as the cartesian product of G_1 and G_2. We now show that the operations $\cdot$ and $\wedge$ may be carried out pointwise. First of all we recall $a_1a_2 = a_1^+ a_2^+ \cdot a_1^- a_2^-$ which was stated above on the grounds of Lemma 1.14. This implies with respect to multiplication

$$a \cdot b_1b_2 = (a \cdot b_1^+b_2^+) \cdot b_1^-b_2^- = (a \cdot b_2^+b_1^+) \cdot b_2^-b_1^- =$$
$$= (ab_1^+ \cdot b_2^+)b_1^- \cdot b_2^- = (ab_1^+ \cdot b_1^-)b_2^+ \cdot b_2^- = ab_1 \cdot b_2 = ab_2 \cdot b_1$$

(in the third step 1.7 was applied), from which it follows that

$$a_1a_2 \cdot b_1b_2 = (a_1a_2 \cdot b_1)b_2 = (a_1b_1 \cdot a_2)b_2 = a_1b_1 \cdot a_2b_2.$$

Recall now $a_1^+ a_2^+ = a_1^+ \vee a_2^+$ and $a_1^- a_2^- = a_1^- \wedge a_2^-$ (1.8). One can infer:

$$a_1a_2 \wedge b_1b_2 = (a_1^+a_2^+ \wedge b_1^+b_2^+) \cdot (a_1^-a_2^- \wedge b_1^-b_2^-) =$$
$$= (a_1^+ \wedge b_1^+)(a_2^+ \wedge b_2^+) \cdot (a_1^- \wedge b_1^-)(a_2^- \wedge b_2^-) =$$
$$= (a_1^+ \wedge b_1^+)(a_1^- \wedge b_1^-) \cdot (a_2^+ \wedge b_2^+)(a_2^- \wedge b_2^-) = (a_1 \wedge b_1) \cdot (a_2 \wedge b_2).$$

Thus out proof is complete.

4.4. Lemma. *Let $\mathfrak{G}$ be complete and $a \not\leqq b$ & $b \not\leqq a$. Then there is a direct decomposition $\mathfrak{G} = \mathfrak{G}_1 \times \mathfrak{G}_2$ with $\bar{a}_1 \leqq \bar{b}_1$ & $\bar{a}_2 \leqq \bar{b}_2$.*

Proof. By Lemma 4.2 it suffices to verify the assertion for positive divisibility semiloops. In this case we define $C_1:=(a*b)^{\perp}$ and $C_2:=C_1^{\perp}$. Then C_1 and C_2 are 1-disjoint and every c has a decomposition c_1c_2 with $c_1=\mathrm{Sup}\,(x|x\leqq c\ \&\ x\in C_1)$. (This idea seems to go back to RIESZ [33]. See also BIRKHOFF [4].) Observe: $y\in C_1\rightarrow \rightarrow c_1\cdot(c_2\wedge y)=c_1\cdot 1$. Furthermore this decomposition is unique and the operations may be carried out pointwise since $a\wedge b=1\rightarrow a\cdot b=a\vee b$.

Now we are ready to prove:

4.5. Theorem. *A power-associative and complete divisibility semiloop $\mathfrak{L}$ is associative and commutative. But if a complete divisibility semiloop is only monassociative it need neither be associative nor commutative even though $\mathfrak{G}$ should be a complete totally ordered loop.*

Proof. We shall verify our assertion by constructing a series of models and specializing the situation until $ab\cdot c\neq a\cdot bc$ leads to a contradiction.

By Lemma 4.3 we may start from a model $\mathfrak{G}_1$ with $ab\cdot c<a\cdot bc$ for some triple a, b, c. Furthermore, by the same lemma, we may suppose that a, b and c are strictly positive or negative, and that $\{a, b, c\}$ is totally ordered. We consider $1<t\leqq d:=ab\cdot c*a\cdot bc$ and some $x>1$. There exists a natural number n such that $t^n\leqq x\ \&\ t^{n+1}\nleqq x$, since otherwise $\mathrm{Sup}\,(t^n|n\in\mathbf{N})=:\Omega$ would exist and satisfy $\Omega t=\Omega$, a contradiction. Hence in any case there exists a model $\bar{\mathfrak{G}}_1$, with $\bar{t}^n\leqq\bar{x}<\bar{t}^{n+1}$ satisfying $\bar{a}\bar{b}\cdot\bar{c}<\bar{a}\cdot\bar{b}\bar{c}$ because of $\bar{1}<\bar{x}*\bar{t}^{n+1}\leqq\bar{a}\bar{b}\cdot\bar{c}*\bar{a}\cdot\bar{b}\bar{c}$.

Consequently we may suppose a model $\mathfrak{G}_2$ containing a triple u, v, w with $uv\cdot w<u\cdot vw$ and $1<s\leqq uv\cdot w*u\cdot vw$ such that $\{s^n|n\in\mathbf{Z}\}\cap\{u, v, w\}$ is totally ordered: Apply the method above successively to $a\vee a^*$, $b\vee b^*$, $c\vee c^*$. None of these elements is equal to 1 and if for instance a is (strictly) negative, then according to (DSL 5) $\bar{r}:=\bar{t}\wedge\bar{a}^*$ is invertible whence we can continue the procedure with $\bar{r}$ satisfying $\bar{1}<\bar{r}\leqq\bar{d}$. So in $\mathfrak{G}_2$ we have $1<s\leqq uv\cdot w*u\cdot vw<s^3$. But this implies that the proof is complete if we deduce $1<g^4\leqq xy\cdot z*x\cdot yz$ for some triple x, y, z in some model $\mathfrak{H}$.

To this end we start w.l.o.g. from $s^n<u\vee u^*=:\hat{u}<s^{n+1}$. This leads to $1<u*s^{n+1}=:f<s$ and further to $f(f*s)=s$ whence we get one of the three relations $1<f^2\leqq s$ or $1<(f\wedge(f*s))^2\leqq s$ or $f^2\nleqq s\ \&\ f\perp f*s$. Obviously in the first two cases there is some f_1 in G_2 satisfying the inequality $1<f_1^2\leqq s$ in $\mathfrak{G}_2$. We now show that also the third case provides some model of this type. Indeed, $f\perp f*s$ implies $s\nleqq f^2$ since $f(f*s)\leqq ff$ would yield $1<f*s<f$. Hence we get $f^2\nleqq s\nleqq f^2$, and thereby a direct decomposition $\mathfrak{G}_2=\bar{\mathfrak{G}}_2\times\bar{\bar{\mathfrak{G}}}_2$ with $\bar{f}^2\geqq\bar{s}$ in $\bar{\mathfrak{G}}_2$ and $\bar{\bar{f}}^2\leqq\bar{\bar{s}}$ in $\bar{\bar{\mathfrak{G}}}_2$. Suppose now that $\bar{\bar{f}}$ is equal to $\bar{\bar{1}}$. Then $\bar{f}$ is different from $\bar{1}$ and hence $\bar{\mathfrak{G}}_2$ is a model satisfying $\bar{f}*\bar{s}=\bar{f}*\bar{s}\wedge\bar{f}^2=\bar{1}$ whence we get $\bar{f}=\bar{s}$ and thereby $\bar{u}=\bar{s}^n$. Hence continuing the procedure with $\bar{v}$ or $\bar{w}$ in the role of u (above), in any case we arrive

at a direct factor $\mathfrak{G}'$ of $\mathfrak{G}$ with $1'<f'^2 \leqq s' \leqq u'v' \cdot w' * u' \cdot v'w'$. Therefore starting from this new situation with f' in the role of s we finally do obtain a model $\mathfrak{H}$ with a triple x, y, z satisfying the inequality $1<g^4 \leqq xy \cdot z * x \cdot yz$, a contradiction.

Hence $\mathfrak{G}$ is associative and in the same manner one verifies that $\mathfrak{G}$ is also commutative.

It remains to show that there are complete totally ordered loops which are neither associative nor commutative. To this end we consider the real line with respect to some derived operations:

(i) We define $a \circ b := a+b$ except for the case $a \leqq 0 \leqq b$, where we put $a \circ b := a+b/2$ if $a+b/2 \leqq 0$ & $a \circ b := 2a+b$ otherwise. This provides a monassociative but non-associative and non-commutative complete and totally ordered loop. Observe:

$$((-1)\circ 2)\circ(-1) = -1 \neq -1/2 = (-1)\circ\big(2\circ(-1)\big).$$

(ii) We define $a \circ b := a+b$, except for the case $a, b \leqq 0$, where we put $a \circ b := a-ab+b$ [22]. This provides a commutative monassociative but non-associative complete and totally ordered loop. Observe:

$$\big(1\circ(-1)\big)\circ(-1) = -1 \neq -2 = 1\circ\big((-1)\circ(-1)\big).$$

5. Completion

The goal of this section is a characterization of divisibility semiloops admitting a complete extension. Nearly obviously such models have to satisfy for lower bounded subsets A the implications

(i) $x, y \,|_l A \;\&\; x \diagdown A \downarrow y \diagdown A \rightarrow x = y$,
(ii) $x, y \,|_r A \;\&\; A/x \downarrow A/y \rightarrow x = y$,
(iii) $A \,|_l x, y \;\&\; A \diagdown x \uparrow A \diagdown y \rightarrow x = y$,
(iv) $A \,|_r x, y \;\&\; x/A \uparrow y/A \rightarrow x = y$,

where $|_l$ and $|_r$ stand for *left-divisor* and *right-divisor* respectively, and $\downarrow$ and $\uparrow$ stand for *coinitial* and *cofinal* respectively. For instance (i) follows from $x \diagdown A \downarrow y \diagdown A \rightarrow x \diagdown \bigwedge A = \bigwedge x \diagdown A = \bigwedge y \diagdown A = y \diagdown \bigwedge A$.

Thus a characterization of models with complete extensions is given provided that (i) through (iv) guarantee such an extension. In order to verify this we start by giving some symbols and notions. Henceforth (A) will denote the set of all upper bounds of A and dually $[A]$ will stand for the set of all lower bounds of A. Furthermore by p we shall mean a multiplication polynomial in one variable, i.e. a polynomial of type $\ldots a_4\big((a_2(xa_1))a_3\big)\ldots$. (Recall that $\mathfrak{G}$ has a unit.) Consequently $p(A)$ will denote the set of all $p(a)$ $(a \in A)$.

As an immediate consequence of (DSL 5) we notice that $p^{-1}(v)$ exists if there is an a such that $v \geqq p(a)$.

5.1. Definition. A subset A of G is called a *t-ideal* if A contains all elements c with $v \geqq p(A) \rightarrow v \geqq p(c)$.

It is easily checked that t-ideals are lattice ideals. Furthermore the reader straightforwardly verifies that G is a t-ideal and that the intersection of all t-ideals containing $A \neq \emptyset$ is a t-ideal, too. This yields that there is a smallest t-ideal $\bar{A}$ containing $A \neq \emptyset$ and moreover the definition $\bar{A} \cdot \bar{B} = \overline{AB}$ provides a unique multiplication since $\bar{A} = \bar{C}$ & $\bar{B} = \bar{D}$ implies $v \geqq p(AB) \leftrightarrow v \geqq p(CD)$. Henceforth we shall denote $\bar{A}$ also by $\mathbf{A}$.

Let us suppose now that the set X of elements x with $\mathbf{A}x \subseteqq \mathbf{B}$ is not empty. Then $X =: \mathbf{A} * \mathbf{B}$ is a t-ideal which follows from the following implication:

$$v \geqq p(X) \rightarrow v \geqq p(c) \quad \text{implies} \quad w \geqq q(\mathbf{B}) \rightarrow w \geqq q(AX) \rightarrow w \geqq q(Ac),$$

which implies $Ac \subseteqq B$.

5.2. Lemma. $\mathfrak{G}$ *satisfies* $\mathbf{A} = [(A)]$.

Proof. Obviously $\mathbf{A}$ is contained in $[(A)]$. Furthermore any $c \in [(A)]$ satisfies the implication $v \geqq p(A) \rightarrow p^{-1}(v) \geqq A \rightarrow p^{-1}(v) \geqq c \rightarrow v \geqq p(c)$ whence each c of $[(A)]$ is contained in $\mathbf{A}$.

5.3. Lemma. $\mathbf{a} := \bar{a}$ *is equal to the set of all x below a. Hence $\mathfrak{G}$ is embedded in the structure formed by the t-ideals with respect to $\cdot$ and inclusion.*

Proof. Left to the reader.

5.4. Lemma. $\mathfrak{G}$ *satisfies* $\mathbf{A} \cdot \mathbf{X} \subseteqq \mathbf{b} \rightarrow \mathbf{A} \cdot (\mathbf{A} * \mathbf{b}) = \mathbf{b}$.

Proof. By assumption $\mathbf{A} * \mathbf{b}$ exists. We suppose $\mathbf{A} \cdot (\mathbf{A} * \mathbf{b}) \leqq c \leqq b$. Then there exists an element v with $A \cdot v \leqq c \leqq b$, whence there is also an element u with $\mathbf{A} \leqq u$ & $us = b$. But for any such u we get:

$$us = b \rightarrow \mathbf{A}s \leqq b \rightarrow \mathbf{A}s \leqq c \rightarrow \mathbf{A} \leqq c/s = u_c |_l b.$$

Hence for any u with $A \leqq u$ we find an u_c with $A \leqq u_c$ such that $us = b$ implies $u_c s = c$. But this means that the set U of all u with $\mathbf{A} \leqq u$ & $u|_l b$ satisfies $U \searrow b \uparrow U \searrow c$ which yields $c = b$.

5.5. Lemma. $\mathfrak{G}$ *satisfies* $\mathbf{A} \subseteqq \mathbf{B} \rightarrow \mathbf{A} \cdot (\mathbf{A} * \mathbf{B}) = \mathbf{B}$.

Proof. Consider an arbitrary element $b \in \mathbf{B}$. Then the t-ideal $\mathbf{A}_b$ generated by all $a \wedge b$ $(a \in A)$ satisfies $\mathbf{A}_b \cdot \mathbf{X}_b = \mathbf{b}$ for $\mathbf{X}_b = \mathbf{A}_b * \mathbf{b}$. We consider the t-ideal $\mathbf{X}$

generated by all $\mathbf{X}_b$. Then $\mathbf{A}\cdot\mathbf{X}\supseteq\mathbf{B}$ is obvious and moreover for any pair a, x ($a\in A$, $x\in X_b$) we can infer

$$(a\wedge b)x \leqq b \to x \leqq a*b \to ax \leqq a(a*b) = a\vee b\in\mathbf{B},$$

whence $\mathbf{A}\cdot\mathbf{X}$ is also contained in $\mathbf{B}$.

5.6. Lemma. (i), ..., (iv)$\Rightarrow\mathbf{a}\cdot\mathbf{X}\subseteq\mathbf{B}\to\exists\mathbf{Z}$: $\mathbf{a}\cdot\mathbf{Z}=\mathbf{B}$.

Proof. By 5.5 there is a t-ideal $\mathbf{Y}$ with $(\mathbf{a}\cdot\mathbf{X})\cdot\mathbf{Y}=\mathbf{B}$, and for every pair x, y ($x\in\mathbf{X}$, $y\in\mathbf{Y}$) there exists an element z with $(ax)y\leqq az=b\in\mathbf{B}$ since $ax\in\mathbf{B}$ & $(ax)y\in\mathbf{B}$ implies $(ax)(1\vee y)\in\mathbf{B}$. Hence the t-ideal $\mathbf{Z}$ generated by these elements z satisfies $\mathbf{a}\cdot\mathbf{Z}=\mathbf{B}$.

5.7. Lemma. (i), ..., (iv)$\Rightarrow s\geqq A$ & $\mathbf{A}\cdot\mathbf{X}=\mathbf{A}\cdot\mathbf{Y}\to\mathbf{X}=\mathbf{Y}$.

Proof. Suppose $v\geqq\mathbf{X}$. It follows $\mathbf{A}\cdot v\geqq\mathbf{A}\cdot y$ for all $y\in\mathbf{Y}$, and thereby $\mathbf{A}\cdot\overline{(v\vee y)}=\mathbf{A}\cdot\mathbf{v}=:\mathbf{B}$ (5.2). But this yields $\mathbf{B}/v=\mathbf{B}/(y\vee v)$ whence we get $v=y\vee v$. It follows $v\geqq\mathbf{Y}$ and thereby $\mathbf{X}\supseteq\mathbf{Y}$. Thus the proof is complete by duality.

5.8. Lemma. (i), ..., (iv)$\Rightarrow\mathbf{a}\cdot\bigwedge\mathbf{X}_i=\bigwedge(\mathbf{a}\cdot\mathbf{X}_i)$.

Proof. By 5.6 there is a t-ideal $\mathbf{Z}$ with $\mathbf{a}\cdot\mathbf{Z}=\bigwedge(a\cdot\mathbf{X}_i)$ $(i\in I)$. Furthermore by 5.2 the t-ideal generated by all $a\vee b$ ($a\in\mathbf{A}$, $b\in\mathbf{B}$) satisfies $\overline{\{a\vee b\mid a\in\mathbf{A}, b\in\mathbf{B}\}}=\overline{\{\mathbf{A},\mathbf{B}\}}$. Consequently for upper bounded t-ideals $\mathbf{A}$ the following implication holds: $\mathbf{A}\cdot\mathbf{X}\subseteq\mathbf{A}\cdot\mathbf{Y}\to\mathbf{X}\subseteq\mathbf{Y}$. Thus $\mathbf{Z}$ is contained in every $\mathbf{X}_i$, which implies the assertion.

Once more we emphasize that we consider a proposition to be proven once its dual has been verified.

Up to now we have been concerned with t-ideals. But obviously there is a dual notion, called v-ideal, which is defined by writing (in 5.1) the symbol $\leqq$ instead of the symbol $\geqq$. We shall denote v-ideals by $\underline{A}$ or A. The proofs, however, given here so far do not carry over in any case since the structure under consideration is not $\geqq/\leqq$-dual. Nevertheless the reader will easily verify that the part up to 5.2 (excluded) can straightforwardly be dualized. Thus there is a product $\mathsf{A}\circ\mathsf{B}=\overline{\underline{AB}}$ and a right-quotient $\mathsf{A}*\mathsf{B}:=\{x|\mathsf{A}x\subseteq\mathsf{B}\}$ (a left-quotient $\mathsf{B}:\mathsf{A}:=\{x|x\mathsf{A}\subseteq\mathsf{B}\}$).

We now return to the t-ideal-extension of $\mathfrak{G}$. We wish to show that (DSL 5) is valid. To this end we denote the principal t-ideal $\mathbf{t}$ also by t, the t-ideals in general by lower case greek letters. Furthermore we shall write (α) for $\{v|v\in G$ & $v\geqq\alpha\}$ and define $[\alpha]$ dually. Thus we consider an upper-continuous cut extension Σ of $\mathfrak{G}$ satisfying:

$$x\alpha\leqq\beta\to\beta=x\varkappa\,\&\,\alpha x\leqq\beta\to\beta=\lambda x \quad\text{and}\quad a\bigwedge\beta_i=\bigwedge(a\beta_i).$$

5.9. Lemma. *There are no other (lower bounded) v-ideals of* $\mathfrak{G}$ *than the subsets* (α) *of* Σ, *which means in particular that* $\mathsf{A}=([A])$.

Proof. Consider a lower bounded A with $\bigwedge A=\alpha$. Then $\mathbf{A}\subseteqq(\alpha)$ is valid since Σ is a cut extension, and $(\alpha)\subseteqq\mathbf{A}$ follows from $t\leqq p(A)\rightarrow t\leqq p(\alpha)$ (5.8)$\rightarrow t\leqq$ $\leqq p(c)$ $(c\geqq\alpha)$.

5.10. Lemma. *If* $\mathbf{B}$ *is contained in* $\mathbf{A}$ *then* $\mathbf{A}$ *is a left (right) divisor of* $\mathbf{B}$.

Proof. Consider a fixed $b\in B$. Then, for $C=A\wedge b$, $\mathbf{A}$ is equal to $\mathbf{C}$. Let X_b be the set of all x satisfying $Ax\geqq b$ and suppose $b\leqq c\leqq AX_b$. We abbreviate Inf(A) by α. It follows $Ax\geqq b\rightarrow\alpha x\geqq b\rightarrow\alpha x\geqq c$. But according to our previous remark there are elements β, γ such that $\alpha\beta=b$ & $\alpha\gamma=c$, whence $x\geqq\beta\rightarrow\alpha x\geqq b\rightarrow$ $\rightarrow\alpha x\geqq c\rightarrow x\geqq\gamma$. This yields $\beta=\gamma$ from which results $b=c$. Therefore any d with $d\leqq AX_b$ satisfies $d\vee b=b$. Hence the ideal $\mathbf{X}$ generated by all X_b satisfies $\mathbf{A}\circ\mathbf{X}=\mathbf{B}$.

So far we have shown that the v-ideals form a lower continuous extension of $\mathfrak{G}$ with respect to $\leqq:=\supseteqq$. We shall now show that Σ and the v-ideal extension are isomorphic. Doing this we shall implicitly verify, too, that there is a complete extension satisfying also axiom (DSL 5) which results from $\mathbf{A}\cdot\mathbf{B}\subseteqq\mathbf{C}\rightarrow\mathbf{A}\cdot\mathbf{B}\subseteqq\mathbf{c}$ $(c\geqq\mathbf{C})$ (cf. 5.4) by lower continuity.

5.11. Lemma. *Σ satisfies* $\bigwedge(\alpha)\circ\bigwedge(\beta)=\bigwedge(\alpha\beta)$.

Proof. Define $\alpha\circ\beta=\gamma$ if $(\alpha)\circ(\beta)=(\gamma)$. Then $\alpha\circ d$ and αd are equal because of Lemma 5.8. Suppose now $\alpha\beta\leqq c$ and $s\leqq ab$ for all $a,b\in(\alpha)\times(\beta)$ and $c=\alpha\circ\gamma$. Then $\alpha\circ c_i=\alpha c_i\geqq\alpha\beta\rightarrow c_i\geqq\beta$ for all $c_i\geqq\gamma$, whence we get by assumption $s\leqq\alpha\circ c_i$ and hereby furthermore $s\leqq\alpha\circ\gamma=c$.

5.12. Proposition. *A divisibility semiloop satisfying* (i), ..., (iv) *has a cut extension isomorphic to the lower bounded v-ideal extension if* $\leqq:=\supseteqq$, *as well as to the upper bounded t-ideal extension if* $\leqq:=\subseteqq$.

Proof. By 5.11 $[(A)]\rightarrow(A)$ is a homomorphism, and by definition this mapping is bijective.

Thus summarizing we can state:

5.13. Theorem. *A divisibility semiloop admits a complete (cut-) extension if and only if it satisfies the conditions* (i) *through* (iv).

Let now $\mathfrak{G}$ be a divisibility semiloop satisfying (i) through (iv), and let Σ be its cut extension in the sense of above. Then we can show in addition:

5.14. Corollary. *If* $\mathfrak{G}$ *is power-associative, then* Σ *is power-associative, too.*

Proof. If c is equal to a product built by factors a_i $(1\leqq i\leqq n)$ satisfying $a_i\leqq\alpha$ we can infer $c\leqq(a_1\vee\ldots\vee a_n)^n\leqq\alpha^n$.

5.15. Corollary. *If $\mathfrak{G}$ is a lattice loop then Σ is a lattice loop, too. If in addition $\mathfrak{G}$ is inverse then Σ is inverse, too.*

Proof. $\alpha \leqq a$ & $b \leqq \beta \to \alpha(a \diagdown b) \leqq \beta$, and starting from $\alpha = \bigvee a_i$ $(i \in I,\ a_i \in G)$ we get:

$$(ba_i)a_i^{-1} = b \to \bigvee ba_i \bigwedge a_i^{-1} = (b\alpha)\alpha^{-1} = b,$$

from which the general inverse property follows by upper continuity.

5.16. Corollary. *A lattice group admits a complete extension if and only if it is archimedean.*

Proof. Obviously the condition is necessary. On the other hand, if $\mathfrak{G}$ is a lattice group, (i) through (iv) are satisfied if $Ax \downarrow A \to x = 1$ and its left dual are valid. But this is a consequence of the archimedean property, since

$$Ax\downarrow A \geqq s \to Ax^{-n}\downarrow A \downarrow Ax^n \to x^{-n} \leqq s^{-1}a \geqq x^n \quad (a \in A,\ n \in \mathrm{N}) \to$$

$$\to (x^*)^n \leqq s^{-1}a \,\&\, (x^+)^n \leqq s^{-1}a,$$

by application of Lemma 1.3. Hence Σ is a complete lattice group since associativity follows from $\mathbf{A} \circ \mathbf{B} = \underline{AB}$.

6. Congruences

In this section we are interested in cancellative congruences of an underlying divisibility semiloop $\mathfrak{G}$. The reader will easily remember that there was given a first result already in Section 4, namely the direct decomposition extension result of Lemma 4.2. The main purpose of this section is to analyze under what conditions $\mathfrak{G}$ is representable, that is, is a subdirect product of totally ordered factors.

Observe that cancellative congruences are also $*$, : congruences.

6.1. Lemma. *If U is the positive part of the class $1 \equiv$ of some cancellative congruence then U is a multiplicatively closed convex subset satisfying*

(i) $aU = Ua$, (ii) $ab \cdot U = a \cdot bU$, (iii) $U \cdot ab = Ua \cdot b$.

Proof. $u \in U$ implies $a \equiv au = va \to v \equiv 1$, and

$$ab \equiv ab \cdot u = a \cdot bv \to bv \equiv b1 \to v \equiv 1, \quad ab \equiv a \cdot bu = ab \cdot v \to \ldots \to v \equiv 1,$$

whence (i) through (iii) are satisfied, the rest being obvious.

Every multiplicatively closed convex positive subset of G containing 1 and satisfying (i) through (iii) will be called a *kernel*.

6.2. Lemma. *If U is a kernel then* $x \equiv y\ (U)$ *iff* $x \leqq yu$ & $y \leqq xv$ *for some* $u, v \in U$ *defines a cancellative congruence such that the positive part of* $1 \equiv$ *coincides with* U.

Proof. Straightforward by definition.

Thus we getas a first result.

6.3. Proposition. In every divisibility semiloop $\mathfrak{G}$ the cancellative congruence relations $\equiv$ are uniquely represented by the kernels U via the following definition: $a \equiv b\ (U)$ iff $a \leqq bu$ & $b \leqq au$.

Hint. $a \equiv b \rightarrow a \leqq b(a * b \vee b * a)$ & $b \leqq a(a * b \vee b * a)$ $(a * b,\ b * a \equiv 1)$.

Evans and Hartman [17] gave a characterization of lattice loops admitting a subdirect decomposition into totally ordered ones. This result can be extended to divisibility semiloops. To this end we consider two orthogonal elements a, b. By (DSL 5) they obviously satisfy the equivalence

$$a \wedge (bx \cdot y)/xy = 1 \leftrightarrow a \cdot xy \wedge bx \cdot y = xy \leftrightarrow ((a \cdot xy)/y)/x \wedge b = 1.$$

Hence requiring the first equality means requiring: $u \perp v$ implies u and $(vx \cdot y)/xy$ are orthogonal, too. And the validity of the third equality means: if u, v are orthogonal then u and $((v \cdot xy)/y)/x$ are orthogonal, too. So, if $u \wedge v = 1$ and $U = (u^{\perp})^{+}$, we can deduce from the validity of each of these equalities

$$(Ux \cdot y)/xy \subseteqq U, \quad \text{whence} \quad Ux \cdot y \subseteqq U \cdot xy,$$

and

$$(((U \cdot xy)/y)/x) \subseteqq U, \quad \text{whence} \quad U \cdot xy \subseteqq Ux \cdot y.$$

Similarly we get $Ux = xU$ from $u \wedge v = 1 \rightarrow u \wedge (xv)/x = 1$.

6.4. Theorem. *A divisibility semiloop* $\mathfrak{G}$ *is representable if and only if it satisfies the conditions*

(i) $(a * b) \cdot xy \wedge (b * a)x \cdot y = xy,$

(ii) $xy \cdot (a * b) \wedge x \cdot y(b * a) = xy,$ and

(iii) $x \cdot (a * b) \wedge (b * a) \cdot x = x.$

Proof. Obviously a and b are orthogonal iff $a * b = b$ & $b * a = a$. Hence the conditions above require that the positive part of any $u^{\perp}$ forms a kernel. Suppose now that U is maximal in the set of kernels $M \not\ni c$. Then $\mathfrak{G}/U =: \mathfrak{H}$ is totally ordered since otherwise $\mathfrak{H}$ would contain a pair p, q with $p * q \neq 1 \neq q * p$. But then $U_1 := ((p * q)^{\perp})^{+}$ and $U_2 := (U_1^{\perp})^{+}$ would be two kernels satisfying $U_1 \cap U_2 = \{1\}$, although U_1 and U_2 differ from $\{1\}$ by construction. Therefore the conditions under consideration are sufficient.

On the other hand our conditions are necessary as is easily checked by the reader.

By 6.4. the subdirect products of totally ordered divisibility semiloops are characterized in a classical manner. But it is obvious that this method relies strongly on (DSL 5) and $a*b \perp b*a$. Hence, in order to find a method working also in more general cases, we have to leave orthogonality conditions and to look for $\cdot/\leqq$-conditions. This will be done in the remainder of this section.

Nearly immediately we get:

6.5. Theorem. *A divisibility semiloop $\mathfrak{G}$ is representable if and only if it satisfies the condition*

(0) $$p(a)\wedge q(b) \leqq p(b)\vee q(a)$$

for any pair p, q of multiplication polynomials.

Proof. Obviously condition (0) is necessary. So let condition (0) be satisfied. Then putting $(c^+x\cdot y)/xy := c^+\theta$ we infer for orthogonal elements a, b,

$$a\wedge b\theta \leqq b\vee a\theta \rightarrow a\wedge b\theta = (a\wedge b\theta)\wedge(b\vee a\theta) =$$
$$= (a\wedge b\theta\wedge b)\vee(a\wedge b\theta\wedge a\theta) = (1\wedge b\theta)\vee(a\wedge 1) = 1,$$

whence (i) is valid. And in an analogous manner one can deduce (ii) and (iii).

We now show that the condition (0) provides a key for solving the problems stated by Fuchs and Evans & Hartman. To this end we shall leave the group oriented standpoint and exploit the lattice-order of the underlying structure. Moreover for the sake of economy we shall start more generally.

6.6. Definition. Let $\mathfrak{A} := (A, \wedge, \vee, f_i)$ be an algebra such that $\wedge$ and $\vee$ provide a lattice order and the f_i are of arity n_i. Then $\mathfrak{A}$ is called a *lattice-ordered algebra* if each operation is isotone at each place. If each operation even distributes over meet and join at each place we call $\mathfrak{A}$ a distributive lattice-ordered algebra.

Examples of lattice-ordered algebras are the lattice groupoids satisfying the $\cdot/\wedge$- or the $\cdot/\vee$-distributivity laws. Hence lattice quasigroups and thereby lattice loops and lattice groups are lattice-ordered algebras in the above sense. However, there remains an inaccuracy. For example, given a lattice group, what are the fundamental operations? Obviously $^{-1}$ is antitone. On the other hand lattice quasigroups satisfy

$$x\diagdown(a\wedge b) = x\diagdown a\wedge x\diagdown b \quad \& \quad (a\wedge b)\diagup x = a\diagup x\wedge b\diagup x$$

and

$$x\diagdown(a\vee b) = x\diagdown a\vee x\diagdown b \quad \& \quad (a\vee b)\diagup x = a\diagup x\vee b\diagup x.$$

So we may regard lattice quasigroups, lattice loops, and lattice groups as lattice-ordered algebras by defining $l_x(a) := x\diagdown a$ and $r_x(a) := a\diagup x$ and considering $\mathfrak{G}$ as an algebra $(G, \cdot, \wedge, \vee, l_x, r_x)$ $(x\in G)$.

6.7. Definition. Let $\mathfrak{A}$ be a lattice-ordered algebra. A *term* is called *linearly composed* if it is a variable or if it is of the special type $f(x_1, \ldots, q(x, y_1, \ldots, y_m), \ldots, x_n)$ where f is a fundamental operation and $q(x, y_1, \ldots, y_m)$ is (already) linearly composed.

6.7 provides a set of terms with a "*starting variable*" x such that in the case of a distributive lattice-ordered algebra the arising *polynomial functions* $\bar{p}(x)$ of type $p(x, c_1, \ldots, c_n)$ $(c_i \in A)$ satisfy the distributivity laws $\bar{p}(a \wedge b) = \bar{p}(a) \wedge \bar{p}(b)$ and $\bar{p}(a \vee b) = \bar{p}(a) \vee \bar{p}(b)$. To emphasize that $\bar{p}(x)$ stems from a term built up without $\wedge$ and $\vee$ we write also $\tilde{p}(x)$. Now we are ready to show

6.8. Theorem. *A lattice-ordered algebra $\mathfrak{A}$ is representable iff it is distributive and satisfies*

$$(\tilde{0}) \qquad \tilde{p}(a) \wedge \tilde{q}(b) \leqq \tilde{p}(b) \vee \tilde{q}(a),$$

which can be unified to the condition

$$(\bar{0}) \qquad \bar{p}(a) \wedge \bar{q}(b) \leqq \bar{p}(b) \vee \bar{q}(a).$$

Proof. Obviously $(\bar{0})$ is necessary and a fortiori $(\bar{0})$ implies $(\tilde{0})$. Moreover $(\bar{0})$ yields $f(\ldots a \wedge a \ldots) \wedge f(\ldots b \wedge b \ldots) \leqq f(\ldots a \wedge b \ldots) \vee f(\ldots b \wedge a \ldots)$, whence f distributes over meet, and join which is shown similarly. The reader should notice that $(\bar{0})$ follows nearly immediately from $(\tilde{0})$ if $\mathfrak{G}$ is distributive. Hint: write $\bar{p}$ and $\bar{q}$ as meets of joins of $\sim$-functions.

We now prove that distributivity together with $(\tilde{0})$ provides a representation. To this end we may start from $r < s$ in order to construct a totally ordered homomorphic image $\bar{A}$ satisfying $\bar{r} < \bar{s}$. By Zorn's Lemma, we see that there is a maximal lattice ideal M, containing r but avoiding s. Furthermore it is well known that such an M is $\wedge$-prime $(a \wedge b \in M \to a \in M \vee b \in M)$, since $(A, \wedge, \vee)$ is distributive. (Otherwise there would be a pair u, v with $u \wedge v \in M$ & $u, v \notin M$ which would lead to $U := \{x \mid x \wedge v \in M\}$, $V := \{y \mid u \wedge y \in M \ (\forall u \in U)\}$ with $b \in U \cap V \subseteq M$.) We define

$$a \equiv b :\Leftrightarrow \tilde{p}(a) \in M \leftrightarrow \tilde{p}(b) \in M.$$

(Obviously we could define this congruence relation also by $V := A - M$ and it is easily checked by the reader that there is a dual proof w.r.t. this prime filter V.) This is a congruence as is easily shown in the groupoid case and analogously proven in the general case. Furthermore we obtain in $\bar{\mathfrak{A}} := \mathfrak{A}/\equiv$

$$\bar{u} \leqq \bar{v} \Leftrightarrow \tilde{p}(v) \in M \to \tilde{p}(u) \in M$$

since

$$\bar{u} \leqq \bar{v} \Rightarrow \bar{u} = \bar{u} \wedge \bar{v} \Rightarrow \tilde{p}(u) \in M \leftrightarrow \tilde{p}(u) \wedge \tilde{p}(v) \in M \Rightarrow$$

$$\Rightarrow \tilde{p}(v) \in M \to \tilde{p}(u \wedge v) \in M \to \tilde{p}(u) \in M$$

and

$$\tilde{p}(v)\in M \to \tilde{p}(u)\in M \Rightarrow \tilde{p}(u\wedge v)\in M \to \tilde{p}(u)\in M\vee\tilde{p}(v)\in M \Rightarrow$$
$$\Rightarrow \tilde{p}(u\wedge v)\in M \to \tilde{p}(u)\in M \Rightarrow \bar{u} \leqq \bar{v}.$$

Hence $\bar{a}$ and $\bar{b}$ are incomparable if and only if there are linearly composed polynomial functions $\tilde{p}(x)$, $\tilde{q}(x)$ satisfying

$$\tilde{p}(a)\notin M, \quad \tilde{p}(b)\in M, \quad \tilde{q}(a)\in M, \quad \tilde{q}(b)\notin M.$$

But this is excluded by $(\tilde{0})$, since otherwise we could infer

$$\tilde{p}(a)\wedge\tilde{q}(b)\notin M \quad \& \quad \tilde{p}(b)\vee\tilde{q}(a)\in M,$$

contradicting $\tilde{p}(a)\wedge\tilde{q}(b)\leqq\tilde{p}(b)\vee\tilde{q}(a)$. Hence $\mathfrak{A}$ is totally ordered.

Theorem 6.8 yields a series of special results.

6.9. Corollary. *An abelian lattice monoid $\mathfrak{M}$ is representable if and only if the underlying lattice is distributive and if furthermore multiplication distributes over meet and join* [30].

Proof. Since $\mathfrak{M}$ is an abelian monoid we may confine ourselves to the proof of $xa\wedge yb\leqq xb\vee ya$ which follows by

$$(xa\wedge yb)\wedge(xb\vee ya) = (xa\wedge yb\wedge xb)\vee(xa\wedge yb\wedge ya) =$$
$$= \big(xa\wedge(y\wedge x)\,b\big)\vee\big((x\wedge y)\,a\wedge yb\big) = xa\wedge(xa\vee yb)\wedge(x\wedge y)(a\vee b)\wedge yb = xa\wedge yb.$$

(Obviously, all we need is a common unit for any pair a, b.)

6.10. Corollary. *A lattice semigroup $\mathfrak{S}=(S,\,\cdot,\,\wedge,\,\vee)$ is representable if and only if the lattice $(S,\,\wedge,\,\vee)$ is distributive, multiplication distributes over meet and join and in addition $\mathfrak{S}$ satisfies the inclusion*

(S0) $$xay\wedge ubv \leqq xby\vee uav,$$

for each quadruple x, y, u, v taken from S^1.

Proof. The laws under consideration guarantee $\tilde{p}(a)\wedge\tilde{q}(b)\leqq\tilde{p}(b)\vee\tilde{q}(a)$ as is easily seen.

6.11. Corollary. *A lattice loop $\mathfrak{L}=(L,\,\cdot,\,\wedge,\,\vee)$ is representable if and only if $\mathfrak{L}$ satisfies the equations*

(EH) $$x(a*b)\wedge(b*a)x = x, \quad (a*b)\cdot xy\wedge(b*a)x\cdot y = xy,$$
$$xy\cdot(a*b)\wedge x\cdot y(a*b) = xy \quad [17].$$

Proof. It was already shown in Section 1 that multiplication and join distribute over meet and join. Furthermore the conditions are necessary. So it remains to show that they are sufficient. Obviously this was done already by 6.4. But we wish to give a direct proof of (EH)$\to(\tilde{0})$.

To this end we consider $\mathfrak{L}$ as a lattice-ordered algebra $(L, \cdot, \wedge, \vee, l_s, r_s)$ $(s \in L)$. We have to show

$$\tilde{p}(a) \wedge \tilde{q}(b) \leqq \tilde{p}(b) \vee \tilde{q}(a).$$

Here, by the rules of loop arithmetic we may suppose $\tilde{p}$ to be the identity mapping and furthermore we can transform the general problem to the proof of

$$(a:b)\,u \wedge (b:a)\,\theta \leqq (b:a)\,u \vee (a:b)\,\theta$$

where θ is an inner mapping and u is equal to some $(r(1))^r$. So we may start from $a \perp b$, $\tilde{p}(x) = xu$ and $\tilde{q}(y) = y\theta$, which leads to

$$au \wedge b\theta = x_a x_u \overset{\cdot}{:} x_a \leqq a \;\&\; x_u \leqq u,$$

$$au \wedge b\theta = (x_a \wedge x_a x_u)(x_u \vee 1) \leqq 1 \vee u \leqq a\theta \vee bu$$

since $a \perp b\theta$ and $a\theta \perp b$. (Recall: if $a \perp b$ implies $a \perp b\theta$ for the generating inner mappings θ then $a \perp b$ implies $a \perp b\theta$ for all inner mappings θ.)

On the grounds of the preceding theorem one can start from (EH) and prove the subdirect decomposition theorem for lattice loops by deducing $(\tilde{0})$ and applying Theorem 6.8. But one has to notice that the proof given above applies the inner mapping theorem which tells that the group of inner mappings is generated by $((* \cdot xy)/y)/x$, $xy \diagdown (x \cdot y*)$ and $(x \cdot *)/x$, see for instance [13].

Furthermore, applying 4.3 (and 1.29) we get as a special result

6.12. Corollary. *Any complete divisibility semiloop* $(L, \cdot, \leqq, 1)$ *is representable, and if moreover the chain condition for closed intervals is satisfied,* $(L, \leqq, 1)$ *is a direct sum of atomic chains* (recall 3.3).

Lattice quasigroups or lattice rings are not lattice-ordered algebras in the sense of Definition 6.6. But sometimes a given structure can be turned to a lattice-ordered algebra as was shown for instance for lattice quasigroups by splitting right and left division into a set of operators. This idea might be fruitful also in other situations. For example, consider a lattice semigroup $\mathfrak{S}$. Then by splitting its multiplication into operators m_x with $m_x(a) := xa$ any left congruence of $\mathfrak{S}$ becomes a congruence of $(S, \wedge, \vee, m_x)$ $(x \in S)$ and vice versa any congruence of $(S, \wedge, \vee, m_x)$ $(x \in S)$ may be considered as a left congruence of $\mathfrak{S}$. This enables us to develop also results based on left congruences, the most important being:

6.13. Corollary. *Any distributive lattice monoid $\mathfrak{S}$ is a lattice monoid of chain endomorphisms* [9].

Proof. Consider $\mathfrak{S}$ as a lattice-ordered algebra $(S, \wedge, \vee, m_x)$. This structure satisfies condition $(\tilde{0})$ which is shown by copying the proof of 6.9. Hence there are enough totally ordered residue systems which can be added to a chain C of left classes of $\mathfrak{S}$ on which the elements of S act from the left. Thus $\mathfrak{S}$ can be embedded into the lattice semigroup of all order endomorphisms of C.

As an immediate consequence of 6.13 we get the celebrated theorem of HOLLAND [25]:

6.14. Corollary. *Any lattice group is a lattice group of chain automorphisms* [25].

We now turn to lattice rings. A ring is called partially ordered with respect to $\leqq$ if it satisfies

$$a \leqq b \rightarrow x+a \leqq x+b \quad \text{and} \quad 0 \leqq a, b \rightarrow 0 \leqq ab.$$

A partially ordered ring is called a lattice ring if $\leqq$ defines a lattice order. Obviously multiplication is not isotone. On the other hand multiplication is completely determined once it is defined on the positive cone. Hence any homomorphic image is completely determined by the image of the cone. So it makes sense to consider a lattice ring $\mathfrak{R}$ as an algebra $(R, +, \wedge, \vee, r_x, l_x)$ where $r_x(a) := ax^+$ and $l_x(a) := x^+a$. Then $\mathfrak{R}$ is a lattice-ordered algebra but $\mathfrak{R}$ need not be distributive since l_x and r_x need not distribute over $\wedge$ and $\vee$. (Consider for instance the ring of 2×2-matrices over the real field with respect to $A \leqq B$ if $a_{ik} \leqq b_{ik}$, $1 \leqq i \leqq 2$, $1 \leqq k \leqq 2$.) To yield this we look for a further condition. Here we succeed by considering the positive cone of $\mathfrak{R}$.

6.15. Lemma. *Let $\mathfrak{R}$ be a lattice ring. Then $(R, +, \wedge, \vee, l_x, r_x)$ is a distributive lattice-ordered algebra in the above sense iff it satisfies*

(L) $$c^+(a*b)\wedge c^+(b*a) = 0 = (a*b)c^+\wedge(b*a)c^+.$$

Proof. Suppose that (L) is valid and that c is positive. Then we obtain, for example:

$$\begin{aligned} ca\wedge cb &= c((a\wedge b)+a*b)\wedge c((a\wedge b)+b*a) = \\ &= (c(a\wedge b)+c(a*b))\wedge(c(a\wedge b)+c(b*a)) = \\ &= c(a\wedge b)+(c(a*b)\wedge c(b*a)) = c(a\wedge b) \end{aligned}$$

and thereby

$$\begin{aligned} ca\vee cb &= (ca+cb)-(ca\wedge cb) = c(a+b)-c(a\wedge b) = \\ &= c((a+b)-(a\wedge b)) = c(a\vee b). \end{aligned}$$

Hence, applying Theorem 6.8 we get:

6.16. Corollary. *A lattice ring is a function ring (is representable) iff it satisfies the conditions* (L) *and* $(\tilde{0})$, *briefly* $(\mathrm{L}, \tilde{0})$.

Corollary 6.16 characterizes the function ring along the lines of this paper. This was done by a different condition in a basic paper published by BIRKHOFF and PIERCE [6], and by a further condition in FUCHS [19] where also the equivalence of these two conditions is proved. To this equivalence proof we now add a further one by showing

$$\text{(BP)} \qquad a\perp b \rightarrow c^+a\perp b \quad \& \quad ac^+\perp b$$

(Birkhoff—Pierce) and condition $(L, \tilde{0})$ to be equivalent.

6.17. Remark. There is a short direct proof of $(BP)\leftrightarrow(L, \tilde{0})$.

Proof. We shall treat the associative case. However, the reader should notice that associativity is by no means essential, only pleasant for the demonstration.

Let $\mathfrak{R}$ satisfy (BP). Then (L) is obvious. Furthermore it is easy to see that the polynomials in $(\tilde{0})$ are of type $c_1^+xc_2^+ + s$. Hence, after some simple calculation $(\tilde{0})$ is reduced to

$$(c_1ac_2+u)\wedge d_1bd_2 \leqq (c_1bc_2+u)\vee d_1ad_2$$

for positive elements c_1, c_2, d_1, d_2 and orthogonal pairs a, b. But because of (BP) we may omit c_1ac_2 on the left side (apply Lemma 1.3). Hence condition $(\tilde{0})$ is satisfied, too.

Let now $\mathfrak{R}$ satisfy $(L, \tilde{0})$. Then (BP) follows by

$$c^+a\wedge b \leqq c^+b\vee a \rightarrow c^+a\wedge b = (c^+a\wedge b\wedge c^+b)\vee(c^+a\wedge b\wedge a) =$$

$$= (c^+(a\wedge b)\wedge b)\vee(c^+a\wedge 0) = 0.$$

We turn to complementary semigroups $(S, \cdot, *, :)$. Complementary semigroups were introduced in [7] as monoids satisfying $aS=Sa$ in which for any pair a, b there exist uniquely determined elements $a*b$ and $b:a$ such that $b|ax\Leftrightarrow a*b|x$ and $b|xa\Leftrightarrow b:a|x$. Complementary semigroups are partially ordered with respect to $a\leqq b\Leftrightarrow: a|b$, and $a\leqq b$ is equivalent to $b*a=1$ and to $a:b=1$ as well. Furthermore $(S, \leqq)$ forms a semilattice under $a\vee b:=a(a*b)=(b:a)a$. In addition the following distributivity laws hold:

$$a(b\vee c) = ab\vee ac \quad \& \quad (a\vee b)c = ac\vee bc$$

and

$$a*(b\vee c) = a*b\vee a*c \quad \& \quad (a\vee b):c = a:c\vee b:c.$$

Therefore, defining operators c_x^* and $c_x^:$ by $c_x^*(a)=x*a$ and $c_x^:(a)= a:x$, any complementary semigroup may be considered as a distributive $\vee$-semilattice-ordered algebra $(S, \cdot, c_x^*, c_x^:, \vee)$. However, we have to show that the congruences of $(S, \cdot, c_x^*, c_x^:)$ are congruences of $(S, \cdot, *, :)$ as well. Here we succeed by the formula $a*(b:c)=(a*b):c$ which results from

$$x \geqq a*(b:c) \leftrightarrow ax \geqq b:c \leftrightarrow axc \geqq b \leftrightarrow x \geqq (a*b):c.$$

To see this, let $\equiv$ be a congruence of $(S, \cdot, c_x^*, c_x^{:})$. Then we have

$$a \equiv b \rightarrow a*b \equiv 1 \equiv b*a \quad (\leftrightarrow b:a \equiv 1 \equiv a:b) \rightarrow a \equiv a(a*b) = b(b*a) \equiv b,$$

and thereby

$$a \equiv b \rightarrow a*b \equiv 1 \rightarrow (a*c):(b*c) = a*(c:(b*c)) \equiv 1.$$

Hence, by duality we get $(b*c):(a*c)\equiv 1$ which leads to $a*c\equiv b*c$.

Special complementary semigroups are the lattice group cones under $a*b:=1\vee\vee a^{-1}b$ and $b:a:=1\vee ba^{-1}$ on the one hand, and the brouwerian semilattices on the other hand.

Complementary semigroups need not be $\wedge$-closed, but products of totally ordered complementary semigroups necessarily satisfy $a*b \perp b*a$ which is equivalent to $a:b \perp b:a$ and also to $a:(b*a)\vee b:(a*b)=a\wedge b$. Moreover, in this case further distributivity laws hold, namely:

$$a(b\wedge c) = ab\wedge ac \quad \& \quad (a\wedge b)c = ac\wedge bc,$$

$$a*(b\wedge c) = a*b\wedge a*c \quad \& \quad (a\wedge b):c = a:c\wedge b:c,$$

and

$$a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c).$$

Therefore complementary semigroups with a representation may be regarded as distributive lattice-ordered algebras $(S, \cdot, c_x^*, c_x^{:})$, and we get as an immediate consequence

6.18. Corollary. *A complementary semigroup is representable if and only if the following implication holds:*

$(0^\vee)$ $$x \leqq \tilde{p}(a), \tilde{q}(b) \rightarrow x \leqq \tilde{p}(b)\vee \tilde{q}(a).$$

Proof. $x\leqq a*b,\ b*a \rightarrow x\leqq a*a\vee b*b=1.$

This corollary provides as a further characterization

6.18′. Corollary. *A complementary semigroup is representable if and only if it satisfies the equation*

(0^c) $$(a*b)*x\vee(c*(b*a)c\vee c(b*a):c)*x = x \quad [8].$$

Proof. (a) Axiom $(0^\vee)$ implies nearly immediately $(0^\perp)$ $c*(a*b)c \perp b*a \perp \perp c(a*b):c$. Hence (0^c) can be inferred from

$(0^{c\prime})$ $$(a*b\wedge(c*(b*a)c\vee c(b*a):c))*x = x.$$

(b) Axiom (0^c) implies $a*b\perp b*a$ whence $(S,\leqq)$ is $\wedge$-closed, and we observe that

(i) $x*yz\leqq(x*y)z$ and $zy:x\leqq z(y:x)$

holds in any case, that

(ii) $ca^{\perp}=a^{\perp}c$

holds according to $(0^{\perp})$, and that

(iii) any $\tilde p(a)$ can be extended to some $\dots x_5((x_3*(x_1a)x_2:x_4)\dots$.

Hence we may start from a pair $\tilde p(a)$, $\tilde q(b)$ with $a\perp b$. But, applying (i) and (ii) again and again this leads to $\tilde p(a)\wedge\tilde q(b)\leqq a^*\tilde p(1)\wedge b^*\tilde q(1)$ with $a^*\perp b^*$, hence

$$\tilde p(a)\wedge\tilde q(b)=x_ax_p=x_bx_q\colon\ x_a\leqq a^*,\quad x_p\leqq\tilde p(1),\quad x_b\leqq b^*,\quad x_q\leqq\tilde q(1),$$

which yields

$$\tilde p(a)\wedge\tilde q(b)=(x_a\wedge x_b)(x_p\vee x_q)\leqq\tilde p(1)\vee\tilde q(1)\leqq\tilde p(b)\vee\tilde q(a).$$

The method of proof shows that a lattice group is already representable if $a^{\perp}c\subseteqq ca^{\perp}$. To see this look at (S0) in 6.10. Furthermore we see that (0^c) is equivalent to $a\perp b\rightarrow(a*c)*c\perp b$ & $c:(c:a)\perp b$, since $a*bx\leqq(a*b)((b*a)*x)$ & $cb:a\leqq$ $\leqq(c:(a:b))(b:a)$.

As an immediate consequence we get

6.19. Corollary. *An abelian complementary semigroup is representable if and only if it satisfies* $a*b\perp b*a$.

Since 6.19 is a direct consequence no proof is needed. But it should be mentioned that in the commutative case $a*b\perp b*a\rightarrow(0^{\vee})$ has a short proof by the formulas $(a*b)*(a*c)=(b*a)*(b*c)$ and $ab*c=b*(a*c)$.

Next, applying 6.19 to boolean algebras $(B,\vee,*)$ (where $a*b:=a'\wedge b$), we can state the celebrated theorem of Stone:

6.20. Corollary. *Any boolean algebra is a subdirect product of 2-element ones, and hence a field of sets* [36].

In a similar manner one shows that normally residuated lattices [12] are distributive lattice-ordered algebras whence 6.8 applies also to these structures. Furthermore one easily sees that dually residuated semigroups [37] may be regarded as extended complementary semigroups by adding $a*b:=0\vee b-a$. Therefore we get

6.21. Corollary. *A dually residuated (commutative) semigroup is representable if and only if it satisfies* $a-b\wedge b-a\leqq 0$ [37].

We consider cone algebras $(C,*,:)$. They were introduced in [11] and turned out to be $*$, :-subalgebras of some lattice group cone $(P,\cdot,*,:)$. Any cone algebra is $\wedge$-closed with respect to $a\wedge b:=a:(b*a)=(b:a)*b$ but a cone algebra need not form a lattice. However $a\vee b$ is contained in C if $\{a, b\}$ is upper bounded, and $ab\in C$ implies that the elements x and y with $xa=ab=by$ are contained in C. So we may apply 6.18 once a prime filter is guaranteed containing b yet not containing a, whenever $a\not\geqq b$. But this is an easy consequence of maximality, since given a filter F maximal with respect to not containing a we get

$$x\vee y\in F \rightarrow x\wedge f_1 \geqq a \,\&\, y\wedge f_2 \geqq a \rightarrow (x\vee y)\wedge(f_1\wedge f_2) \geqq a,$$

a contradiction. Thus we are led to

6.22. Corollary. *A cone algebra is representable if and only if it satisfies*

(C0) $$a*b \perp a:b.$$

(Observe that this condition is equivalent to $(a\wedge b)^2=a^2\wedge b^2$ in lattice group cones and lattice groups as well, and observe furthermore that this equation is equivalent to $aa\wedge bb\leqq ab\vee ab$.)

Proof. Any complementary semigroup satisfies $cb:a=(c:(a:b))(b:a)$, and the method of 6.18 works also in the present case which is shown by cone algebra technique. Hence by the last footnote it suffices to prove the implication $a\perp b\rightarrow a\perp \perp c:(c:b)$. But this can be done as follows: $a\perp b$ implies

$$c:(c:a)*(a\wedge c:(c:b)) \leqq (a\wedge c)\wedge(c:(c:a))*(c:(c:b)) =$$
$$= (c:a)*(c:b)\wedge(c:a):(c:b) = 1,$$

whence

$$a\wedge c:(c:b) = a\wedge c:(c:b)\wedge c:(c:a) \leqq c:(c:(b\wedge a)) = 1.$$

Final remark. Obviously the principle of 6.8 works whenever a partially ordered algebra — this may be an arbitrary algebra with respect to $=$ — has enough *order ideals (order filters)*, i.e. o-ideals (o-filters), M satisfying

(P) $$\bar{p}(b)\in M \,\&\, \bar{q}(a)\in M \rightarrow \bar{p}(a)\in M \wedge \bar{q}(b)\in M$$

If M is a *prime ideal* in the sense of (P) then $A-M$ is a *prime filter* in the sense of (P) and vice versa, and we see nearly immediately that the set of prime ideals (prime filters) is closed under intersections and unions of chains of prime ideals (prime filters).

Let us suppose now that $\mathfrak{A}$ has enough prime ideals. Then the partially ordered algebra $\mathfrak{A}$ is representable and hence admits an extension to some representable distributive lattice-ordered algebra $\mathfrak{B}$. Therefore we should check how artificial this condition is. To this end we present some applications which lead to well known results.

6.23. Example. Any partially ordered set is a subdirect product of 2-element chains, since any $(a]$ is prime with respect to the identity operator.

6.24. Example. Any $\vee$-semilattice is a subdirect product of 2-element chains, since any $(a]$ is prime in the sense of (P).

6.25. Example. A partially ordered abelian group $\mathfrak{G}$ is representable if and only if it is *semiclosed*, i.e. iff it satisfies, for any $n \in \mathrm{N}$, the implication

$$\text{(SC)} \qquad a^n \geqq 1 \rightarrow a \geqq 1.$$

(The first proof of this result seems to be due to Clifford [15]. Another proof was given by Dieudonné in 1941, cf. [19].)

Proof. Obviously (SC) is necessary. Suppose now that (SC) is satisfied and $a \neq b$. The set N of strictly negative elements is closed under multiplication, and it is easily shown that ab^{-1}, N and $a^{-1}b, N$ cannot both generate a submonoid (with respect to multiplication). Hence there is a maximal subsemigroup $\mathfrak{M}$ containing N and w.l.o.g. ab^{-1} but not containing 1. We show that M is a prime ideal in the sense of (P).

(i) M is an o-ideal, since $u < v \in M$ implies $uv^{-1} < 1$ & $v \in M$ from which it follows that $(uv^{-1})v = u \in M$.

(ii) M is prime, since $ax, by \in M$ and $ay, bx \notin M$ would yield a $k \in \mathrm{N}$ with $a^{-k}y^{-k}, b^{-k}x^{-k} \in M$ whence $a^{-k}b^k$ and $a^k b^{-k}$ would both belong to M, a contradiction.

References

[1] J. Aczél, Quasigroups, nets and nomograms, *Adv. Math.*, **1** (1965), 383—450.

[2] I. Arnold, Ideale in kommutativen Halbgruppen, *Math. Sb.*, **36** (1929), 401—407.

[3] B. Banaschewski, On lattice-ordered groups, *Fund. Math.*, **55** (1964), 113—122.

[4] G. Birkhoff, Lattice-ordered groups, *Ann. Math.*, **43** (1942), 298—331.

[5] G. Birkhoff, *Lattice Theory*, 3rd ed., Coll. Publ., vol. 25, Amer. Math. Soc. (Providence, R. I., 1967).

[6] G. Birkhoff, R. S. Pierce, Lattice-ordered rings, *Ann. Acad. Bras. d. Cienc.*, **22** (1956), 41—69.

[7] B. Bosbach, Komplementäre Halbgruppen. Axiomatik und Arithmetik, *Fund. Math.*, **64** (1969), 257—287.

[8] B. Bosbach, Komplementäre Halbgruppen. Eine Darstellungstheorie, *Math. Ann.*, **179** (1968), 1—14.

[9] B. Bosbach, Schwache Teilbarkeitshalbgruppen, *Semigroup Forum*, **12** (1976), 119—135.

[10] B. Bosbach, Teilbarkeitshalbgruppen mit vollständiger Erweiterung, *J. Algebra*, **83** (1983), 237—255.

[11] B. Bosbach, Concerning cone algebras, *Algebra Universalis*, **15** (1982), 58—66.

[12] B. Bosbach, Residuation groupoids and lattices, *Studia Sci. Math.*, **13** (1978), 433—457.

[13] R. H. Bruck, *A Survey of Binary Systems*, Ergebnisse der Mathematik, Neue Folge, Heft 20, Springer (Berlin, 1958).

[14] A. H. Clifford, Arithmetic and ideal theory in commutative semigroups, *Ann. Math.*, **39** (1938), 594—610.

[15] A. H. Clifford, Partially ordered abelian groups, *Ann. Math.*, **41** (1940), 465—473.
[16] T. Evans, Lattice-ordered loops and quasigroups, *J. Algebra*, **16** (1970), 218—226.
[17] T. Evans, P. A. Hartman, Varieties of lattice-ordered algebras, *Algebra Universalis*, **17** (1983), 376—392.
[18] C. J. Everett, S. Ulam, On ordered groups, *Trans. Amer. Math. Soc.*, **57** (1945), 208—216.
[19] L. Fuchs, *Teilweise geordnete algebraische Strukturen*, Studia Mathematica, Vandenhoek & Rupprecht (Göttingen, 1966).
[20] L. Fuchs, On partially ordered algebras. I, *Coll. Math.*, **13** (1966), 116—130; II, *Acta Sci. Math.*, **16** (1965), 35—41.
[21] L. Fuchs, A remark on lattice ordered semigroups, *Semigroup Forum*, **7** (1974), 372—374.
[22] P. A. Hartman, *Ordered Quasigroups and Loops*, Ph. D. Thesis, Emory University, 1971.
[23] P. A. Hartman, Integrally closed and complete ordered quasigroups and loops, *Proc. Amer. Math. Soc.*, **33** (1972), 250—256.
[24] O. Hölder, Die Axiome der Quantität und die Lehre vom Maß, *Ber. d. Sächs. Ges. d. Wiss. Leipzig, Math. Phys. Cl.*, **53** (1901), 1—64.
[25] W. C. Holland, The lattice-ordered group of automorphisms of an ordered set, *Michigan Math. J.*, **10** (1963), 399—408.
[26] K. Iséki, Structure of special ordered loops, *Portugal Math.*, **10** (1951), 81—83.
[27] K. Iwasawa, On the structure of conditionally complete lattice groups, *Japan. J. Math.*, **18** (1943), 777—789.
[28] P. Lorenzen, Abstrakte Begründung der multiplikativen Idealtheorie, *Math. Z.*, **45** (1939), 533—553.
[29] P. Lorenzen, Über halbgeordnete Gruppen, *Math. Z.*, **52** (1950), 483—526.
[30] Th. Merlier, Sur les demi-groupes réticulés et les *o*-demigroupes, *Semigroup Forum*, **2** (1971), 64—70.
[31] N. Naik, B. L. N. Swamy, S. S. Misra, Lattice-ordered commutative loops, *Math. Sem. Notes Kobe Univ.*, **8** (1980), 57—71.
[32] P. Ribenboim, Un théorème de réalisation des groupes réticulés, *Pacific J. Math.*, **10** (1960), 305—308.
[33] F. Riesz, Sur la theorie générale des operations linéaires, *Ann. Math.*, **41** (1940), 174—206.
[34] F. Šik, Über subdirekte Summen geordneter Gruppen. *Czechoslovak Math. J.*, **10** (1960), 400—424.
[35] F. A. Smith, A subdirect decomposition of additively idempotent semirings, *J. Natur. Sci. Math.*, **7** (1967), 253—257.
[36] M. H. Stone, The theory of representations for boolean algebras, *Trans. Amer. Math. Soc.*, **40** (1936), 37—111.
[37] K. L. N. Swamy, Dually residuated lattice ordered semigroups, *Math. Ann.*, **167** (1966), 71—74.
[38] V. A. Testov, On the theory of lattice-ordered quasigroups, in: *Webs and Quasigroups*, Kalinin University (1981); pp. 153—157.
[39] B. L. van der Waerden, Zur Produktzerlegung der Ideale in ganzabgeschlossenen Ringen, *Math. Ann.*, **101** (1929), 293—308.
[40] M. Ward, Residuated distributive lattices, *Duke Math. J.*, **6** (1940), 641—651.
[41] D. Zelinsky, Non associative valuations, *Bull. Amer. Math. Soc.*, **54** (1948), 175—183.
[42] D. Zelinsky, On ordered loops, *Amer. J. Math.*, **70** (1948), 681—697.

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