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A characterization of weak convergence of weighted multivariate empirical processes

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1. Introduction

The characterization of weak convergence of the one-dimensional weighted empirical process indexed by points is obtained by CHIBISOV [5] and O'REILLY [11]. Later, SHORACK [16] and SHORACK and WELLNER [17] wanted to give a new, "elementary" proof of this so called Chibisov—O'Reilly theorem but their proofs were not correct wihout additional monotonicity conditions on the weight functions. This was pointed out in CSÖRGŐ, CSÖRGŐ, HORVÁTH and MASON [6] (pp. 25—27). SHORACK and WELLNER [17] also gave a characterization of weak convergence of the one-dimensional weighted empirical process indexed by rectangles. Their proof, however, is again only correct with an additional monotonicity condition on the weight function. Recently a new approximation of the empirical process is established in CSÖRGŐ CSÖRGŐ, HORVÁTH and MASON [7] which among others yields a proof of the Chibisov—O'Reilly theorem.

The aforementioned theorems can be generalized in two directions: (I) the case of dependent and/or non-identically distributed random variables and (II) the multivariate case. Case I has been studied by ALEXANDER [1], ALY, BEIRLANT and HORVÁTH [3] and BEIRLANT and HORVÁTH [4]. In our paper, which is a revision of the technical report EINMAHL, RUYMGAART and WELLNER [9], we study case II, i.e. we derive necessary and sufficient conditions on the weight functions for weak convergence of weighted multivariate empirical processes; these processes are indexed by quadrants (points) and rectangles respectively. Our main tools are exponential probability inequalities for the empirical process. The paper is a continuation of RUYMGAART and WELLNER [14], [15], where the basic tools are already presented but attention is focussed on strong convergence properties.

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During the preparation of the earlier version of this work we became aware of recent developments in this area, especially the work of ALEXANDER [1], already quoted before, on weighted empirical processes based on non-i.i.d. random elements and indexed by Vapnik—Chervonenkis classes of sets. Although his results are of impressive generality, also this author needs a rather unnatural monotonicity condition which we can avoid everywhere, i.e. though his theorems allow more general indexing classes, our theorems allow more general weight functions. Very recently, ALEXANDER [2] also obtained our (stronger) version of the multivariate characterization theorem for points.

In order to be more explicit we need to present the basic notation. Let $X_1^{(n)}, \ldots, X_n^{(n)}$, $n \in \mathbf{N}$, be a triangular array of i.i.d. random vectors that are uniformly distributed on $[0, 1]^d$, $d \in \mathbf{N}$. Adopting the notation in OREY and PRUITT [12] we shall write $x = \langle x_1, \ldots, x_d \rangle = \langle x_j \rangle = \langle x(j) \rangle \in \mathbf{R}^d$ if it is desirable to display the coordinates of x. If $x_j = \xi$ for all j we simply write $\langle \xi \rangle$. For $x, y \in \mathbf{R}^d$ we write $x \leqq y$ if $x_j \leqq y_j$ for all j and $x < y$ if $x \leqq y$ and $x \neq y$. It has some advantage to denote the half-open rectangles $(x_1, y_1] \times \ldots \times (x_d, y_d]$ by $R(x, y)$ rather than $(x, y]$. The classes

$$(1.1)\qquad \mathscr{R}_0 = \{R(\langle 0 \rangle, y)\colon R(\langle 0 \rangle, y) \subset [0, 1]^d\}, \quad \mathscr{R} = \{R(x, y)\colon R(x, y) \subset [0, 1]^d\},$$

of all half-open quadrants respectively rectangles in the unit square will play an important role. We will write $|t| = t_1 \times \ldots \times t_d$, $|dt|$ for Lebesgue measure on $[0, 1]^d$ and $|R|$ for the Lesbesgue measure of a rectangle R. Using this notation for the uniform underlying d.f. F we have

$$(1.2)\qquad F(t) = |t|, \; t \in [0, 1]^d.$$

Given any function $\Lambda\colon \mathbf{R}^d \to \mathbf{R}$ and an arbitrary rectangle $R = R(x, y)$ we write

$$(1.3)\qquad \Lambda\{R\} = \Lambda\{R(x, y)\} = \Delta_x^y \Lambda,$$

extending the difference operator Δ_x^y, usually applied only to distribution functions.

The weight functions will be always restricted to the class

$$(1.4)\qquad \mathscr{Q}^* = \{q\colon [0, 1] \to [0, \infty) \text{ with } q \text{ continuous and non-decreasing, } q > 0 \text{ on } (0, 1]\}.$$

The subclasses that will appear in our characterization are

$$(1.5)\qquad \mathscr{Q}_0 = \Big\{q \in \mathscr{Q}^*\colon \int_0^1 \sigma^{-1} \exp\big(-\lambda q^2(\sigma)/\sigma\big)\, d\sigma < \infty \text{ for all } \lambda > 0\Big\},$$

$$(1.6)\qquad \mathscr{Q}_k = \Big\{q \in \mathscr{Q}^*\colon q(\sigma)/\sqrt{\sigma(\log(1/\sigma))^k} \to \infty \text{ as } \sigma \downarrow 0\Big\}, \quad k \in \mathbf{N}.$$

Occosionally it will be convenient to use

$$(1.7)\qquad \mathscr{Q} = \{q \in \mathscr{Q}^*\colon (\cdot)^{-1/2} q(\cdot) \text{ non-increasing on } (0, 1]\}.$$

The (reduced multivariate) empirical process (indexed by points) is defined by

$$U_n(t) = n^{1/2}(\hat{F}_n(t) - |t|), \quad t \in [0, 1]^d, \tag{1.8}$$

where the empirical d.f. $\hat{F}_n$ is based on $X_1^{(n)}, \ldots, X_n^{(n)}$ and defined by $n\hat{F}_n(t) = \# \{1 \leqq i \leqq n: X_i^{(n)} \in R(\langle 0 \rangle, t)\}$, $t \in [0, 1]^d$. It is well-known that $U_n \to_d U$, as $n \to \infty$, where U denotes the standard tied-down d-parameter Brownian motion. The so called Skorokhod construction ensures the existence of processes, equal in law to the U_n and U above and all defined on the same probability space, for which this convergence in distribution may be even replaced by almost sure convergence in the supremum norm. Without loss of generality we can and will assume that the present U_n and U are obtained from the Skorokhod construction so that we have

$$\sup_{t \in [0,1]^d} |U_n(t) - U(t)| \to_{\text{a.s.}} 0, \quad \text{as} \quad n \to \infty. \tag{1.9}$$

In view of (1.3) it will be clear that we even have

$$\sup_{R \in \mathscr{R}} \left|U_n\{R\} - U\{R\}\right| \to_{\text{a.s.}} 0, \quad \text{as} \quad n \to \infty. \tag{1.10}$$

It is the purpose of this paper to give necessary and sufficient conditions on the weight functions q and $\tilde{q}$ in order that

$$\sup_{R \in \mathscr{C}} \left|U_n\{R\} - U\{R\}\right| / q(|R|)\tilde{q}(1 - |R|) \to_p 0, \quad \text{as} \quad n \to \infty, \tag{1.11}$$

where either $\mathscr{C} = \mathscr{R}_0$ (Section 2) or $\mathscr{C} \subset \mathscr{R}$ (Section 3).

Since for $R = R(\langle 0 \rangle, t) \in \mathscr{R}_0$ we have $U_n\{R(\langle 0 \rangle, t)\} = U_n(t)$ and $|R(\langle 0 \rangle, t)| = |t|$, the random variable in (1.11) could as well be represented by means of the time points $t \in [0, 1]^d$ instead of the quadrants. More generally, a similar remark holds true for $R = R(s, t) \in \mathscr{R}$ provided we allow the time points to be of dimension $2d$. Let us write $\bar{s} = \langle \bar{s}_j \rangle = \langle 1 - s_j \rangle$ and note that

$$\begin{aligned} F\{R(s, t)\} &= P(X_i^{(n)} \in R(s, t)) = \\ &= P(1 - X_{i,1}^{(n)} \leqq \bar{s}_1, \ldots, 1 - X_{i,d}^{(n)} \leqq \bar{s}_d, X_{i,1}^{(n)} \leqq t_1, \ldots, X_{i,d}^{(n)} \leqq t_d) \equiv \\ &\equiv \bar{F}(\bar{s}, t) = \begin{cases} |t + \bar{s} - 1| = |t - s|, & \text{for} \quad s < t, \quad s, t \in [0, 1]^d, \\ 0, \quad \textit{if} \quad s < t & \text{is not fulfilled;} \end{cases} \end{aligned} \tag{1.12}$$

cf. KIEFER and WOLFOWITZ [10]. Let $\bar{U}_n$ denote the reduced empirical process based on the vectors $(1 - X_{i,1}^{(n)}, \ldots, 1 - X_{i,d}^{(n)}, X_{i,1}^{(n)}, \ldots, X_{i,d}^{(n)})$ in $[0, 1]^{2d}$, for $i = 1, \ldots, n$. Now it suffices for our purposes to consider

$$\frac{\bar{U}_n(\bar{s}, t)}{q(|t - s|)\tilde{q}(1 - |t - s|)} \quad \text{instead of} \quad \frac{U_n\{R(s, t)\}}{q(|R(s, t)|)\tilde{q}(1 - |R(s, t)|)}. \tag{1.13}$$

This will be called the point representation for rectangles.

To conclude this section we present, in the next paragraph, our basic inequality which can be found in RUYMGAART and WELLNER [14], [15]. The main results are presented in Section 2 and 3. They are derived under the assumption that the d.f. of the $X_i^{(n)}$ is uniform. We conjecture, however, that extension to the case that F has a density w.r.t. to Lebesgue measure that is bounded away from 0 and ∞ is possible. Let ψ: $[0, \infty)\to[0, \infty)$ be the decreasing function defined by

$$\psi(\lambda) = 2\lambda^{-2}\int_0^\lambda \log(1+\sigma)\,d\sigma,\ \lambda > 0;\ \psi(0) = 1. \tag{1.14}$$

See SHORACK and WELLNER [17] for elementary properties of ψ.

Theorem 1.1 (basic inequality). *Let $R\in\mathscr{R}$ with $|R|\leqq 1/2$. Then we have*

$$P(\sup_{S\subset R}|U_n\{S\}| \geqq \lambda) \leqq 2^{2d+4}\exp\left(\frac{-\lambda^2}{32|R|}\psi\left(\frac{\lambda}{4|R|n^{1/2}}\right)\right),\quad \lambda\geqq 0, \tag{1.15}$$

where $s\in\mathscr{R}$.

2. Weight functions for quadrants (points)

We first derive a useful inequality that should be compared with Inequality 1.1 in SHORACK and WELLNER [17]; see also RUYMGAART and WELLNER [14] (Corollary 2.3). For the proof a special countably infinite partition of $(0, 1]^d$ will be used that becomes arbitrarily fine near the lower boundary of this set. This kind of partition is motivated by O'REILLY [11]; see also SHORACK and WELLNER [17]. This partition is the collection of rectangles

$$\mathscr{P} = \{R(\langle(1/2)^{k(j)}\rangle, \langle(1/2)^{k(j)-1}\rangle)\colon \langle k(j)\rangle\in\mathbf{N}^d\}. \tag{2.1}$$

For any $R(a, b)\in\mathscr{P}$ we have the useful property

$$\frac{|a|}{|b|} = \frac{(1/2)^{k(1)+\dots+k(d)}}{(1/2)^{(k(1)-1)+\dots+(k(d)-1)}} = (1/2)^d = \theta(d) = \theta\in(0, 1); \tag{2.2}$$

notice that θ is independent of the particular rectangle in the partition.

For any $0<\alpha\leqq\beta\leqq 1$ let us introduce the subclass

$$\mathscr{P}_{\alpha,\beta} = \{R(a, b)\in\mathscr{P}\colon |b| \geqq \alpha, |a| < \beta\}, \tag{2.3}$$

consisting of all rectangles having a non-empty intersection with the set $\{t\in[0, 1]^d\colon \alpha\leqq|t|\leqq\beta\}$. The inclusions

$$\{\alpha \leqq |t| \leqq \beta\} \subset \bigcup_{R\in\mathscr{P}_{\alpha,\beta}} R \subset \{\theta\alpha \leqq |t| \leqq \beta/\theta\} \tag{2.4}$$

are immediate.

Inequality 2.1. *Let us choose any* $0<\alpha\leqq\beta\leqq\theta/2=1/2^{d+1}$. *For any* $q\in\mathcal{Q}$ *and* $\lambda\geqq 0$ *we have*

$$(2.5)\qquad P\Big(\sup_{\alpha\leqq|t|\leqq\beta}|U_n(t)|/q(|t|)\geqq\lambda\Big)\leqq$$
$$\leqq 2^{3d+4}\int_{\theta\alpha}^{\beta/\theta}\frac{(\log 1/\sigma)^{d-1}}{\sigma}\exp\left(\frac{-\theta\lambda^2q^2(\sigma)}{32\sigma}\psi\left(\frac{\lambda q(\alpha)}{4\alpha n^{1/2}}\right)\right)d\sigma.$$

Proof. It follows from the monotonicity of q and from Theorem 1.1 that

$$(2.6)\qquad P\Big(\sup_{\alpha\leqq|t|\leqq\beta}|U_n(t)|/q(|t|)\geqq\lambda\Big)\leqq P\Big(\max_{R(a,b)\in\mathcal{P}_{\alpha,\beta}}\sup_{t\in R(a,b)}|U_n(t)|/q(|a|)\geqq\lambda\Big)\leqq$$
$$\leqq\sum_{R(a,b)\in\mathcal{P}_{\alpha,\beta}}P\Big(\sup_{t\in R(a,b)}|U_n(t)|\geqq\lambda q(|a|)\Big)\leqq$$
$$\leqq 2^{2d+4}\sum_{R(a,b)\in\mathcal{P}_{\alpha,\beta}}\exp\left(\frac{-\lambda^2q^2(|a|)}{32|b|}\psi\left(\frac{\lambda q(|a|)}{4|b|n^{1/2}}\right)\right).$$

In view of (2.2) and because $(\cdot)^{-1/2}q(\cdot)$ is non-increasing we may bound the first factor in the exponent in (2.6) below by

$$(2.7)\qquad \lambda^2q^2(|a|)/32|b|\geqq\theta\lambda^2q^2(|t|)/32|t|,\quad\text{for}\quad t\in R(a,b).$$

Using the monotonicity of q and ψ and $q\in Q$, the second factor in the exponent in (2.6) may be bounded below by

$$(2.8)\qquad \psi\big(\lambda q(|a|)/4|b|n^{1/2}\big)\geqq\psi\big(\lambda q(\alpha)/4\alpha n^{1/2}\big),\quad\text{for}\quad R(a,b)\in\mathcal{P}_{\alpha,\beta}.$$

When we use

$$(2.9)\qquad 1=2^d/|b|\int_{R(a,b)}|dt|\leqq 2^d\int_{R(a,b)}1/|t|\,|dt|,\quad\text{for}\quad R(a,b)\in\mathcal{P},$$

at the transition from summation to integration we find, by combining (2.4), (2.6)—(2.8) that

$$(2.10)\qquad P\Big(\sup_{\alpha\leqq|t|\leqq\beta}|U_n(t)|/q(|t|)\geqq\lambda\Big)\leqq$$
$$\leqq 2^{3d+4}\int_{\{\theta\alpha\leqq|t|\leqq\beta/\theta\}}\frac{1}{|t|}\exp\left(\frac{-\theta\lambda^2q^2(|t|)}{32|t|}\psi\left(\frac{\lambda q(\alpha)}{4\alpha n^{1/2}}\right)\right)|dt|.$$

To complete the proof we use the change of variables $\sigma=s_1=|t|$, $s_2=t_2$, ..., $s_d=t_d$ with Jacobian $\big(\prod_{j=2}^{d}s_j\big)^{-1}$ to compute the integral on the right hand side of (2.10). This yields as an upper bound for the right hand side of (2.10)

$$(2.11)\qquad \int_{\theta\alpha}^{\beta/\theta}\left(\int_{\sigma}^{1}\cdots\int_{\sigma}^{1}\frac{1}{s_2\dots s_d}ds_2\dots ds_d\right)\frac{1}{\sigma}\exp\left(\frac{-\theta\lambda^2q^2(\sigma)}{32\sigma}\psi\left(\frac{\lambda q(\alpha)}{4\alpha n^{1/2}}\right)\right)d\sigma,$$

which is easily seen to be equal to the expression on the right in (2.5).

Theorem 2.1. *Let* $F(t)=|t|$, $t\in[0,1]^d$, $d\in\mathbf{N}$, *and* $q\in\mathcal{Q}^*$. *Then we have*

$$\sup_{0\leq|t|\leq 1} |U_n(t)-U(t)|/q(|t|)\to_p 0, \quad as \quad n\to\infty, \tag{2.12}$$

if and only if $q\in\mathcal{Q}_{d-1}$.

Proof. The theorem is well-known for $d=1$: see O'Reilly [11]. Hence we assume $d\geq 2$. The notation

$$g(\sigma) = q(\sigma)/\sqrt{\sigma(\log 1/\sigma)^{d-1}}, \quad \sigma>0, \tag{2.13}$$

will be used in both parts of the proof.

($\Leftarrow$) Suppose that $q\in\mathcal{Q}_{d-1}$. Following Shorack and Wellner [17] (p. 649) we can and will assume without loss of generality that

$$g(\cdot)\leq\sqrt{(\log 1/(\cdot))^{d-1}} \quad \text{and} \quad g\downarrow \quad \text{on} \quad (0,1] \quad (\text{hence } q\in\mathcal{Q}). \tag{2.14}$$

For any $0<\delta\leq(1/2)^{d+1}$ we have

$$\sup_{0\leq|t|\leq 1} |U_n(t)-U(t)|/q(|t|)\leq\sum_{k=1}^{5} Y_{nk}, \tag{2.15}$$

where, with $\alpha_n=q^2(1/n)$, $\beta_n=(d-1)!\cdot(n(\log n)^{d-1})^{-1}$ and $\gamma\in(0,\infty)$, the r.v.'s Y_{nk} are given by

$$Y_{n1}=\sup_{0\leq|t|\leq\beta_n/\gamma} |U_n(t)|/q(|t|), \tag{2.16}$$

$$Y_{n2}=\sup_{\beta_n/\gamma\leq|t|\leq\alpha_n} |U_n(t)|/q(|t|), \tag{2.17}$$

$$Y_{n3}=\sup_{\alpha_n\leq|t|\leq\delta} |U_n(t)|/q(|t|), \tag{2.18}$$

$$Y_{n4}=\sup_{0\leq|t|\leq\delta} |U(t)|/q(|t|), \tag{2.19}$$

$$Y_{n5}=\sup_{0\leq|t|\leq 1} |U_n(t)-U(t)|/q(\delta). \tag{2.20}$$

It will be shown that for any $\varepsilon>0$ and each $k=1,\ldots,5$ there exist $\gamma=\gamma(\varepsilon)$, $\delta=\delta(\varepsilon)$ and $n(\varepsilon)\in\mathbf{N}$ such that

$$P(Y_{nk}\geq\varepsilon)\leq\varepsilon, \quad \text{for} \quad n\geq n(\varepsilon). \tag{2.21}$$

To show (2.21) for $k=1$ let $|X|_{1:n}=\min\{|X_1^{(n)}|,\ldots,|X_n^{(n)}|\}$. Note that $P(|X|_{1:n}\leq\beta_n/\gamma)\to 1-\exp(-1/\gamma)$, as $n\to\infty$, so that $P(|X|_{1:n}\leq\beta_n/\gamma)\leq\varepsilon$ for γ sufficiently large. Under the condition $\sup_{0\leq|t|\leq\beta_n/\gamma}\hat{F}_n(t)=0$, which is fulfilled with probability $\geq 1-\varepsilon$ according to the remark just made, it is easy to see that

$$Y_{n1}\leq n^{1/2}\sup_{0\leq|t|\leq\beta_n/\gamma} |t|/q(|t|)\leq n^{1/2}(\beta_n/\gamma)^{1/2}\{g(\beta_n/\gamma)(\log n)^{(d-1)/2}\}^{-1}<\varepsilon, \tag{2.22}$$

for n sufficiently large. Hence it follows that

$$(2.23)\qquad P(Y_{n1} \geqq \varepsilon) \leqq P\Big(\sup_{0\leqq|t|\leqq\beta_n/\gamma} \hat{F}_n(t) > 0\Big)+$$
$$+P\Big(\sup_{0\leqq|t|\leqq\beta_n/\gamma} |U_n(t)|/q(|t|) \geqq \varepsilon \,\Big|\, \sup_{0\leqq|t|\leqq\beta_n/\gamma} \hat{F}_n(t) = 0\Big) \leqq \varepsilon,$$

for n sufficiently large.

For $k=2$ the left hand side of (2.21) is for any $\gamma_1 \in (0, \infty)$ bounded above by

$$(2.24)\qquad \begin{aligned} &P\Big(\sup_{\beta_n/\gamma\leqq|t|\leqq\alpha_n} |U_n(t)|/|t|^{1/2} \geqq \varepsilon g(\alpha_n)(\log 1/\alpha_n)^{(d-1)/2}\Big) \leqq \\ &\leqq P\Big(\sup_{\beta_n/\gamma\leqq|t|\leqq\alpha_n} |U_n(t)|/|t|^{1/2} \geqq \gamma_1(\log n)^{(d-1)/2}\Big), \end{aligned}$$

for $n \geqq n_1 = n_1(\gamma_1)$. Hence, applying Inequality 2.1 with $q(\cdot)=(\cdot)^{1/2}$, we see that there exist $c_1, \ldots, c_4 \in (0, \infty)$ such that the last expression in (2.24) is in turn bounded above by

$$(2.25)\qquad \begin{aligned} &c_1(\log n)^d \exp\big(-c_2\gamma_1^2(\log n)^{d-1}\psi(c_3\gamma_1\gamma^{1/2}(\log n)^{d-1})\big) \leqq \\ &\leqq c_1(\log n)^d \exp(-c_4\gamma_1\gamma^{-1/2}\log\log n) \leqq \varepsilon, \end{aligned}$$

provided γ_1 and n are chosen sufficiently large.

Inequality 2.1 may be directly applied to Y_{n3} with $\alpha=\alpha_n$ and $\beta=\delta$. The integral in the resulting upper bound decreases to 0 as $\delta \downarrow 0$, since $q \in \mathcal{Q}_{d-1}$ implies that

$$(2.26)\qquad \int_0^1 (1/\sigma^2)\exp\big(-\lambda q^2(\sigma)/\sigma\big)\, d\sigma < \infty, \quad \text{for all} \quad \lambda > 0;$$

see SHORACK and WELLNER [17], ((1.9), (1.15) and (1.26)).

According to OREY and PRUITT [12] (Theorem 2.2) the function λq is point upper class for U, for all $\lambda > 0$. This yields

$$(2.27)\qquad \sup_{0\leqq|t|\leqq\delta} |U(t)|/q(|t|) \to_{\text{a.s.}} 0, \quad \text{as} \quad \delta \downarrow 0,$$

which entails (2.21) for $k=4$. The validity of (2.21) for $k=5$ is immediate from (1.9).

($\Rightarrow$) Let β_n be as before. We obviously have

$$(2.28)\qquad \sup_{0\leqq|t|\leqq 1} |U_n(t)-U(t)|/q(|t|) \geqq \sup_{0\leqq|t|\leqq\beta_n} |U_n(t)-U(t)|/q(|t|) = Y.$$

Using the remark below (2.21) we see that with probability larger than 1/2 we have

$$(2.29)\qquad \begin{aligned} Y &\geqq \Big\{n^{1/2}(n^{-1}-\beta_n) - \sup_{0\leqq|t|\leqq\beta_n} |U(t)|\Big\}/q(\beta_n) \leqq \\ &\geqq \big(2n^{1/2}q(\beta_n)\big)^{-1} \geqq \big(3((d-1)!)^{1/2}g(\beta_n)\big)^{-1} \end{aligned}$$

for all large n, where for the second inequality again Theorem 2.2 in OREY and PRUITT [12] is applied.

The assumption that $\sup_{0\leq|t|\leq 1} |U_n(t)-U(t)|/q(|t|)\to_p 0$, as $n\to\infty$, jointly with (2.28), (2.29) and the fact that q is nondecreasing, implies that $q\in\mathcal{Q}_{d-1}$.

Theorem 2.2. *Let* $F(t)=|t|$, $t\in[0,1]^d$, $d\in\mathbf{N}$ *and* $\tilde{q}\in\mathcal{Q}^*$. *Then we have*

$$\sup_{0\leq|t|\leq 1} |U_n(t)-U(t)|/\tilde{q}(1-|t|)\to_p 0, \quad \text{as} \quad n\to\infty, \tag{2.30}$$

if and only if $\tilde{q}\in\mathcal{Q}_0$.

Proof. Suppose $\tilde{q}\in\mathcal{Q}_0$. Starting with the equalities

$$U_n(t) = -U_n\{R(\langle 0\rangle, t)^c\} \quad \text{and} \quad U(t) = -U\{R(\langle 0\rangle, t)^c\} \tag{2.31}$$

we obtain using the union-intersection principle

$$|U_n(t)-U(t)| \leq \sum_{i\in\mathcal{I}} \left|U_n\{R_i(t)\}-U\{R_i(t)\}\right|, \tag{2.32}$$

where the $R_i(t)$'s are rectangles and $\mathcal{I}$ a finite index set. This yields

$$\sup_{0\leq|t|\leq 1} |U_n(t)-U(t)|/\tilde{q}(1-|t|) \leq \sum_{i\in\mathcal{I}} \sup_{0\leq|t|\leq 1} \left|U_n\{R_i(t)\}-U\{R_i(t)\}\right|/\tilde{q}(1-|t|). \tag{2.33}$$

It turns out to be convenient to split this sum into two parts. Define $\mathcal{I}_0$ as the set of all $i\in\mathcal{I}$ such that $R_i(t)$ is $(0,1]^{j-1}\times(t_j,1]\times(0,1]^{d-j}$ for some $1\leq j\leq d$. Write $\mathcal{I}_1=\mathcal{I}\setminus\mathcal{I}_0$. For $i\in\mathcal{I}_0$ we have

$$\sup_{0\leq|t|\leq 1} \left|U_n\{R_i(t)\}-U\{R_i(t)\}\right|/\tilde{q}(1-|t|) \leq \sup_{0\leq|t|\leq 1} \left|U_n\{R_i(t)\}-U\{R_i(t)\}\right|/\tilde{q}(|R_i(t)|). \tag{2.34}$$

Application of Theorem 2.1 with $d=1$ (the case $d=1$ is symmetrical) completes the proof for this part of the sum.

Now let $i\in\mathcal{I}_1$. Define dimension $(R_i(t))=\#\{j\colon R_i(t)$ depends on $t_j\}$. Suppose dimension $(R_i(t))=l$, $2\leq l\leq d$. By symmetry considerations, studying

$$\sup_{0\leq|t|\leq 1} \left|U_n\{R_i(t)\}-U\{R_i(t)\}\right|/\tilde{q}(1-|t|)$$

is equivalent with studying

$$\sup_{0\leq|t|\leq 1} |U_n(t')-U(t')|/\tilde{q}(1-|\langle 1\rangle-t|),$$

where t' is t restricted to $[0,1]^l$ in the way suggested above.

Define $\xi=\max_{1\leq j\leq d} t_j$. We have

$$\tilde{q}(1-|\langle 1\rangle-t|)\geq\tilde{q}(\xi), \tag{2.35}$$

and for small values of ξ

$$\tilde{q}(\xi)\geq\sqrt{\xi}, \tag{2.36}$$

because $\tilde{q}\in\mathcal{Q}_0$, using an argument similar to SHORACK and WELLNER [17] ((a) on p. 648). Define $q\in\mathcal{Q}_{l-1}$ in the following way:

$$q(\sigma) = \sup_{0\leq\tau\leq\sigma} \sqrt{\tau(\log 1/\tau)^l}.$$

Using $\xi^l \geqq |t'|$, it is easy to see that

$$\sqrt{\xi} \geqq \sqrt{|t'|(\log 1/|t'|)^l} = q(|t'|) \tag{2.37}$$

for small values of $|t'|$. The assertions (2.35)—(2.37) entail that

$$\sup_{t'\in[0,1]^l} |U_n(t')-U(t')|/q(|t'|) \to_p 0, \quad \text{as} \quad n\to\infty,$$

implies

$$\sup_{0\leq|t|\leq1} |U_n(t')-U(t')|/\tilde{q}(1-|\langle1\rangle-t|) \to_p 0, \quad \text{as} \quad n\to\infty.$$

Combining this with Theorem 2.1 completes the "if" part of the proof.

The "only if" part is clear from the "only if" part in the one-dimensional case by restricting the supremum e.g. to points of the form $t=\langle t_1, 1, ..., 1\rangle$.

Combining Theorems 2.1 and 2.2 yields

Corollary 2.1. *Let* $F(t)=|t|$, $t\in[0,1]^d$, $d\in\mathrm{N}$ *and* $q, \tilde{q}\in\mathcal{Q}^*$. *Then we have*

$$\sup_{0\leq|t|\leq1} |U_n(t)-U(t)|/q(|t|)\tilde{q}(1-|t|) \to_p 0, \quad as \quad n\to\infty, \tag{2.38}$$

if and only if both $q\in\mathcal{Q}_{d-1}$ *and* $\tilde{q}\in\mathcal{Q}_0$.

3. Weight functions for rectangles

Extending an example in SHORACK and WELLNER [17] to the multivariate case we have

$$\sup_{R\in\mathcal{R}} |U_n\{R\}|/q(|R|) = \infty, \quad \text{a.s.} \tag{3.1}$$

for any $q\in\mathcal{Q}^*$ with $q(0)=0$. For this reason $|R|$ should be bounded away from 0 when the growth of the empirical process for small rectangles $|R|$ is studied.

Our first goal is to obtain a suitable modification of Inequality 2.1. The special countably infinite partition of $(0, 1]^{2d}\setminus\{\bar{F}=0\}$ that will be used now becomes arbitrarily fine near the lower boundary of this set; for $d=1$ this boundary is the line segment joining (0, 1) and (1, 0). This partition cannot be written as a product of a partition of (0, 1] like (2.1), but it can be written as a product of a partition of a subset of $(0, 1]^2$, namely the set $A=\{(x, y)\in(0, 1]^2: x+y>1\}$. So we know the partition completely if we define it on A.

Let us first introduce a sequence $\bar{\mathscr{P}}_1''$, $\bar{\mathscr{P}}_2''$, ... of partitions of $(0, 1]^2$ consisting of a finite number of half-open squares. More specifically, let

$$\text{(3.2)} \quad \bar{\mathscr{P}}_n'' = \{R(\langle(1/2)^{n+1}(k(j)-1)\rangle, \langle(1/2)^{n+1}k(j)\rangle), \langle k(j)\rangle \in \{1, \dots, 2^{n+1}\}^2\}.$$

Let us next define recursively

$$\text{(3.3)} \quad \begin{aligned} &\bar{\mathscr{P}}_1' = \{R\in\bar{\mathscr{P}}_1'' : R \subset \{(x, y)\in(0, 1]^2 : (1/2) < x+y-1 \le 1\}\}, \\ &\bar{\mathscr{P}}_n' = \{R\in\bar{\mathscr{P}}_n'' : R \subset [\{(x, y)\in(0, 1]^2 : (1/2)^n < x+y-1 \le 3\cdot(1/2)^n\} \setminus \bigcup_{R\in\bar{\mathscr{P}}_{n-1}'} R]\}, \\ &\qquad \text{for } n \ge 2, \end{aligned}$$

and finally the desired partition of A by

$$\text{(3.4)} \quad \bar{\mathscr{P}}' = \bigcup_{n=1}^{\infty} \bar{\mathscr{P}}_n'.$$

We now obtain the partition of $(0, 1]^{2d} \setminus \{\bar{F}=0\}$ by taking the product of $\bar{\mathscr{P}}'$ taking the co-ordinates s_j and t_j together to form $(0, 1]^2$, $1 \le j \le d$. Denote this partition as $\bar{\mathscr{P}}$.

For any $R(a, b)\in\bar{\mathscr{P}}$ we have the property

$$\text{(3.5)} \quad \bar{F}(a)/\bar{F}(b) \ge (1/2)^d = \bar{\theta}(d) = \bar{\theta}\in(0, 1).$$

Again for $0<\alpha\le\beta\le 1$ we introduce

$$\text{(3.6)} \quad \bar{\mathscr{P}}_{\alpha,\beta} = \{R(a, b)\in\bar{\mathscr{P}} : \bar{F}(b) \ge \alpha, \bar{F}(a) < \beta\}$$

and remark

$$\text{(3.7)} \quad \{\alpha \le \bar{F}(\bar{s}, t) \le \beta\} \subset \bigcup_{R\in\bar{\mathscr{P}}_{\alpha,\beta}} R \subset \{\bar{\theta}\alpha \le \bar{F}(\bar{s}, t) \le \beta/\bar{\theta}\}.$$

Inequality 3.1. *Let us choose any* $0<\alpha\le\beta\le\bar{\theta}/2=(1/2)^{d+1}$. *For any* $q\in\mathscr{Q}$ *and* $\lambda\ge 0$ *we have*

$$\text{(3.8)} \quad P(\sup_{\alpha\le\bar{F}(\bar{s},t)\le\beta} |\bar{U}_n(\bar{s}, t)|/q(|t-s|) \ge \lambda) \le$$

$$\le 2^{4d+4}\cdot 3^{2d} \int_{\bar{\theta}\alpha}^{\beta/\bar{\theta}} \frac{(\log 1/\sigma)^{d-1}}{\sigma^2} \exp\left(\frac{-\lambda^2\bar{\theta}}{32}\frac{q^2(\sigma)}{\sigma}\psi\left(\frac{\lambda q(\alpha)}{4\alpha n^{1/2}}\right)\right) d\sigma.$$

Proof. The same reasoning as in the proof of Inequality 2.1. yields

$$\text{(3.9)} \quad P(\sup_{\alpha\le\bar{F}(\bar{s},t)\le\beta} |\bar{U}_n(\bar{s}, t)|/q(|t-s|) \ge \lambda) \le$$

$$\le 2^{2d+4} \sum_{R(a,b)\in\bar{\mathscr{P}}_{\alpha,\beta}} \exp\left(\frac{-\lambda^2 q^2(\bar{F}(a))}{32\bar{F}(b)}\psi\left(\frac{\lambda q(\bar{F}(a))}{4\bar{F}(b)n^{1/2}}\right)\right).$$

In this case we have, moreover, that

$$\lambda^2 q^2(\bar{F}(a))/32\bar{F}(b) \geqq \bar{\theta}\lambda^2 q^2(\bar{F}(t))/32\bar{F}(t), \quad \text{for} \quad t\in R(a, b); \tag{3.10}$$

$$\psi(\lambda q(\bar{F}(a))/4\bar{F}(b)n^{1/2}) \geqq \psi(\lambda q(\alpha)/4n^{1/2}\alpha), \quad \text{for} \quad R(a, b)\in\bar{\mathscr{P}}_{\alpha,\beta}. \tag{3.11}$$

The way of construction of $\bar{\mathscr{P}}$ entails

$$1 \leqq 2^{2d}\cdot 3^{2d} \int_R (1/\bar{F}(t))^2\, |dt| \quad \text{for} \quad R\in\bar{\mathscr{P}}. \tag{3.12}$$

Combination of (3.7) and (3.9)—(3.12) yields

$$P\Big(\sup_{\alpha\leqq F(\bar{s},t)\leqq\beta} |\bar{U}_n(\bar{s}, t)|/q(|t-s|) \geqq \lambda\Big) \leqq \tag{3.13}$$

$$\leqq 2^{4d+4}\cdot 3^{2d} \int_{\{\bar{\theta}\alpha\leqq|t-s|\leqq\beta/\bar{\theta}\}} (1/\bar{F}(\bar{s}, t))^2 \exp\left(\frac{-\bar{\theta}^2 q^2(\bar{F}(\bar{s}, t))}{32\bar{F}(\bar{s}, t)}\psi\left(\frac{\lambda q(\alpha)}{4n^{1/2}\alpha}\right)\right) |d(\bar{s}, t)|.$$

To complete the proof let us recall formula (1.12) for $\bar{F}(\bar{s}, t)$. The change of variables $u_j=t_j+\bar{s}_j-1$ and $v_j=t_j-\bar{s}_j$ for $1\leqq j\leqq d$, with Jacobian $(1/2)^d$, yields as upper bound for the integral in (3.13)

$$\int_{\{\bar{\theta}\alpha\leqq|u|\leqq\beta/\bar{\theta}\}} \frac{1}{|u|^2}\cdot \exp\left(\frac{-\bar{\theta}\lambda^2 q^2(|u|)}{32\,|u|}\psi\left(\frac{\lambda q(\alpha)}{4n^{1/2}\alpha}\right)\right) |du|. \tag{3.14}$$

Another change of variables, similar to the one above (2.11), completes the proof.

Theorem 3.1. *Let* $F(t)=|t|$, $t\in[0, 1]^d$, $d\in\mathbf{N}$, *and* $q\in\mathscr{Q}^*$. *For any fixed* $\gamma\in(0, \infty)$ *we have*

$$\sup_{\gamma \log n/n\leqq|R|\leqq1} \big|U_n\{R\} - U\{R\}\big|/q(|R|) \to_p 0, \quad as \quad n\to\infty, \tag{3.15}$$

if and only if $q\in\mathscr{Q}_1$.

Proof. ($\Leftarrow$) Suppose that $q\in\mathscr{Q}_1$. Like in the proof of Theorem 2.1 the notation

$$g(\sigma) = q(\sigma)/\sqrt{\sigma\log 1/\sigma}, \quad \sigma > 0, \tag{3.16}$$

will be used. We can and will assume without loss of generality that (2.14) holds true (for q as in (3.16)) with $\sqrt{(\log 1/(\cdot))^{d-1}}$ replaced by $\sqrt{\log 1/(\cdot)}$. We have for any $0<\delta\leqq(1/2)^{d+1}$ that

$$\sup_{\gamma\log n/n\leqq|R|\leqq1} \big|U_n\{R\} - U\{R\}\big|/q(|R|) \leqq \sum_{k=1}^{4} Z_{nk}, \tag{3.17}$$

where with $\alpha_n = q^2(n^{-1})$ and $\beta_n = \gamma \log n/n$ the r.v.'s Z_{nk} are given by

$$Z_{n1} = \sup_{\beta_n \leqq |R| \leqq \alpha_n} |U_n\{R\}|/q(|R|), \tag{3.18}$$

$$Z_{n2} = \sup_{\alpha_n \leqq |R| \leqq \delta} |U_n\{R\}|/q(|R|), \tag{3.19}$$

$$Z_{n3} = \sup_{0 \leqq |R| \leqq \delta} |U\{R\}|/q(|R|), \tag{3.20}$$

$$Z_{n4} = \sup_{0 \leqq |R| \leqq 1} |U_n\{R\} - U\{R\}|/q(\delta). \tag{3.21}$$

Again it will be shown that for any $\varepsilon > 0$ and each $k = 1, 2, 3, 4$ there exist $\delta = \delta(\varepsilon)$ and $n(\varepsilon) \in \mathbf{N}$ such that

$$P(Z_{nk} \geqq \varepsilon) \leqq \varepsilon \quad \text{for} \quad n \geqq n(\varepsilon). \tag{3.22}$$

For $k=1$ the left-hand side of (3.22) is bounded above by

$$\begin{aligned} &P\Big(\sup_{\beta_n \leqq |R| \leqq \alpha_n} |U_n\{R\}|/|R|^{1/2} \geqq \varepsilon g(\alpha_n)(\log 1/\alpha_n)^{1/2}\Big) \leqq \\ &\leqq P\Big(\sup_{\beta_n \leqq |R| \leqq \alpha_n} |U_n\{R\}|/|R|^{1/2} \geqq \gamma_1 (\log n)^{1/2}\Big) \end{aligned} \tag{3.23}$$

for $\gamma_1 \in (0, \infty)$ arbitrary and $n \geqq n_1 = n_1(\gamma_1)$. Using the point representation for rectangles we can apply Inequality 3.1. This yields the existence of $c_1, \ldots, c_4 \in (0, \infty)$ such that the last expression of (3.23) is bounded above by

(3.24)
$$c_1 \cdot n (\log n)^{d-2} \exp\left(-c_2 \gamma_1^2 \log n \psi(c_3 \gamma_1)\right) \leqq c_1 n (\log n)^{d-2} \exp\left(-c_4 \gamma_1 \log \gamma_1 \log n\right) \leqq \varepsilon,$$

provided γ_1 and n are chosen sufficiently large.

To handle Z_{n2} we can again use Inequality 3.1. The integral in the resulting upper bound decreases to 0 as $\delta \downarrow 0$ since $q \in \mathcal{Q}_1$ implies

$$\int_0^1 \frac{(\log 1/\sigma)^{d-1}}{\sigma^2} \exp\left(\frac{-\lambda q^2(\sigma)}{\sigma}\right) d\sigma < \infty, \quad \text{for all} \quad \lambda > 0, \quad d \in \mathbf{N}, \tag{3.25}$$

by a slight modification of the proof of Proposition 3.1 in Shorack and Wellner [17].

Using Theorem 2.1 in Orey and Pruitt [12] we can treat Z_{n3} in the same way as Y_{n4} in the preceding section. We also have similarity between Z_{n4} and Y_{n5} using (1.10) instead of (1.9).

($\Rightarrow$) For this half of the proof we refer to Csörgő, Csörgő, Horváth and Mason [7] (pp. 87—89) where the proof is given for the quantile process and the one-dimensional empirical process. Their proof immediately carries over to the multivariate empirical process; the generalizations of the results required in that paper can be found in Einmahl [8] (p. 2) and Pyke [13] (p. 340) respectively.

We note in passing that the analogue for rectangles of Proposition 2.1 in O'Reilly [11] can be obtained using some of the ideas in the proof of Theorem 3.1: *Let* $d\in\mathbf{N}$ *and* $q\in\mathcal{Q}^*$. *Then we have*

(3.26) $$\lim_{\delta\downarrow 0}\sup_{|R|\le\delta}|U\{R\}|/q(|R|)=0 \quad a.s.$$

if and only if $q\in\mathcal{Q}_1$.

For any $\gamma\in(0,\infty)$, define $U_{n,\gamma}$, a process indexed by rectangles, by

(3.27) $$U_{n,\gamma}\{R\}=U_n\{R\}\,1_{[\gamma\log n/n,1]}(|R|),\quad R\in\mathcal{R}.$$

Combining Theorem 3.1 and (3.26) yields

Corollary 3.1. *Let* $F(t)=|t|$, $t\in[0,1]^d$, $d\in\mathbf{N}$ *and* $q\in\mathcal{Q}^*$. *For any fixed* $\gamma\in(0,\infty)$ *we have*

(3.28) $$\sup_{R\in\mathcal{R}}|U_{n,\gamma}\{R\}-U\{R\}|/q(|R|)\to_p 0,\quad as\quad n\to\infty,$$

if and only if $q\in\mathcal{Q}_1$.

Theorem 3.2. *Let* $F(t)=|t|$, $t\in[0,1]^d$, $d\in\mathbf{N}$ *and* $\tilde q\in\mathcal{Q}^*$. *Then we have*

(3.29) $$\sup_{0\le|R|\le 1}|U_n\{R\}-U\{R\}|/\tilde q(1-|R|)\to_p 0,\quad as\quad n\to\infty,$$

if and only if $\tilde q\in\mathcal{Q}_0$.

Proof. ($\Leftarrow$) To avoid difficulties with notations and technicalities we restrict ourselves to the case $d=2$. Without any mathematical problems the proof can be extended to arbitrary d. (See also the proof of Theorem 2.2.)

Let us first remark that for $0<\delta<1$

(3.30) $$\sup_{0\le|R|\le 1}|U_n\{R\}-U\{R\}|/\tilde q(1-|R|)\le$$
$$\le\sup_{0\le|R|\le 1}|U_n\{R\}-U\{R\}|/\tilde q(\delta)+\sup_{1-\delta\le|R|\le 1}|U_n\{R\}-U\{R\}|/\tilde q(1-|R|).$$

The first term of the last expression causes no problems, so we focus on the second term. Let us choose R with $|R|\ge 1-\delta$ and angular points a_1, a_2, a_3, a_4 starting at the upper vertex and moving clockwise. Remark that $|a_1|\ge 1-\delta$ and $|a_2|, |a_3|, |a_4|<\delta$. Using the inequality

(3.31) $$|U_n\{R\}-U\{R\}|/\tilde q(1-|R|)\le\sum_{i=1}^{4}|U_n(a_i)-U(a_i)|/\tilde q(1-|R|)$$

we see that we only have to handle $\sup_{1-\delta\le|R|\le 1}|U_n(a_i)-U(a_i)|/\tilde q(1-|R|)$ for $i=1, 2, 3, 4$. Using $\tilde q(1-|R|)\ge\tilde q(1-|a_1|)$ we can apply Theorem 2.2 to handle the case $i=1$. With the same technique as used in the proof of this theorem we can also treat the cases $i=2, 3, 4$.

($\Rightarrow$) Theorem 2.2 together with the remark that (3.29) implies (2.30) yields this part of the proof.

Combining Theorem 3.1, Theorem 3.2 and Corollary 3.1 yields

Corollary 3.2. *Let* $F(t)=|t|$, $t\in[0, 1]^d$, $d\in\mathbf{N}$ *and* $q, \tilde{q}\in\mathcal{Q}^*$. *For any fixed* $\gamma\in(0, \infty)$ *the following three statements are equivalent:*

(3.32) $$\sup_{\gamma \log n/n \leq |R| \leq 1} |U_n\{R\}-U\{R\}|/q(|R|)\tilde{q}(1-|R|) \to_p 0, \quad as \quad n\to\infty,$$

(3.33) $$\sup_{R\in\mathcal{R}} |U_{n,\gamma}\{R\}-U\{R\}|/q(|R|)\tilde{q}(1-|R|) \to_p 0, \quad as \quad n\to\infty,$$

(3.34) $$q\in\mathcal{Q}_1 \quad and \quad \tilde{q}\in\mathcal{Q}_0.$$

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