On perturbations of boundary value problems for nonlinear elliptic equations on unbounded domains

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Introduction

In [1] it has been proved the existence of variational solutions of boundary value problems for the elliptic equation

$$\sum_{|\alpha|\leqq m} (-1)^{|\alpha|} D^\alpha f_\alpha(x, u, \dots, D^\beta u, \dots) +$$
$$+ \sum_{|\alpha|\leqq l} (-1)^{|\alpha|} D^\alpha g_\alpha(x, u, \dots, D^\beta u, \dots) = F, \quad x\in\Omega$$

where Ω is a possibly unbounded domain in $\mathbf{R}^n$; $|\beta|\leqq m$; l is an integer with the property $l<m-(n/p)(1-p+\varrho)$; p and ϱ are real numbers such that $1<p<\infty$, $p-1<\varrho\leqq p$. Functions f_α satisfy the same conditions as in [2] and g_α satisfy (essentially)

$$g_\alpha(x, \xi)\xi_\alpha \geqq 0,$$
$$|g_\alpha(x, \xi)| \leqq K(\xi')(C_1(x)+|\xi''|^\varrho)$$

where $\xi=(\xi', \xi'')$ and ξ' contains those coordinates ξ_β of ξ for which $|\beta|<m-(n/p)$, $C_1\in L^{p/\varrho}(\Omega)$.

In the present paper we give some stability results for solutions of the above problem. These results are connected with [3] and with several works referred in [3] where perturbation of other boundary value problems and variational inequalities has been considered.

1. Preliminaries

Let $\Omega\subset\mathbf{R}^n$ be a (possibly unbounded) domain, $p>1$, m a positive integer. Assume that Ω has the weak cone property (see [4]), and for all sufficiently large μ, there exists a bounded $\Omega_\mu\subset\Omega$ with the weak cone property such that $\Omega_\mu\supset\{x\in\Omega: |x|<\mu\}$. Denote by $W_p^m(\Omega)$ the usual Sobolev space of real valued functions u

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whose distributional derivatives of order $\leqq m$ belong to $L^p(\Omega)$. The norm on $W_p^m(\Omega)$ is defined by

$$\|u\| = \Big\{\sum_{|\alpha|\leqq m} \int_\Omega |D^\alpha u|^p \, dx\Big\}^{1/p}$$

where

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad |\alpha| = \sum_{j=1}^n \alpha_j, \quad D_j = \frac{\partial}{\partial x_j},$$

$$D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}.$$

Let N and M be the number of multiindices α satisfying $|\alpha|\leqq m$ and $|\alpha|\leqq m-1$, respectively. The vectors $\xi=(\xi_0, \ldots, \xi_\beta, \ldots)\in\mathbf{R}^N$ will be written in the form $\xi=(\eta, \zeta)$, where $\eta\in\mathbf{R}^M$ consists of those ξ_β for which $|\beta|\leqq m-1$. Assume that:

I. Functions $f_{\alpha,j}$: $\Omega\times\mathbf{R}^N\to\mathbf{R}$ ($|\alpha|\leqq m$; $j=0, 1, 2, \ldots$) satisfy the Carathéodory conditions, i.e. they are measurable with respect to x for each fixed $\xi\in\mathbf{R}^N$ and continuous with recpect to ξ for almost all $x\in\Omega$.

II. There exist a constant $c_1>0$ and a function $K_1\in L^q(\Omega)$ (where $1/p+1/q=1$) such that

$$|f_{\alpha,j}(x, \xi)| \leqq c_1|\xi|^{p-1}+K_1(x)$$

for all $|\alpha|\leqq m$, $j=0, 1, 2, \ldots$, a.e. $x\in\Omega$ and all $\xi\in\mathbf{R}^N$.

III. For all $(\eta, \zeta), (\eta, \zeta')\in\mathbf{R}^N$ with $\eta\in\mathbf{R}^M$, $\zeta\neq\zeta'$ and a.e. $x\in\Omega$ ($j=0, 1, 2, \ldots$)

$$\sum_{|\alpha|=m} [f_{\alpha,j}(x, \eta, \zeta)-f_{\alpha,j}(x, \eta, \zeta')](\xi_\alpha-\xi'_\alpha) > 0.$$

IV. There exist a constant $c_2>0$ and a function $K_2\in L^1(\Omega)$ such that for a.e. $x\in\Omega$ and all $\xi\in\mathbf{R}^N$

$$\sum_{|\alpha|\leqq m} f_{\alpha,j}(x, \xi)\xi_\alpha \geqq c_2|\xi|^p-K_2(x) \quad (j = 0, 1, 2, \ldots).$$

V. $\lim_{j\to\infty} \xi^{(j)} = \xi^{(0)}$ implies

$$\lim_{j\to\infty} f_{\alpha,j}(x, \xi^{(j)}) = f_{\alpha,0}(x, \xi^{(0)})$$

for a.e. $x\in\Omega$ and all $|\alpha|\leqq m$.

VI. Functions $p_{\alpha,j}, r_{\alpha,j}$: $\Omega\times\mathbf{R}^N\to\mathbf{R}$

$$(|\alpha| \leqq l, j = 0, 1, 2, \ldots)$$

satisfy the Carathéodory conditions and

$$g_{\alpha,j} = p_{\alpha,j}+r_{\alpha,j}.$$

VII. $p_{\alpha,j}(x, \xi)\xi_\alpha\geqq 0$ and $|r_{\alpha,j}(x, \xi)|\leqq h_\alpha(x)$ for all $|\alpha|\leqq l$, $\xi\in\mathbf{R}^N$ and a.e. $x\in\Omega$ where $h_\alpha\in L^{p/q}(\Omega)$, $j=0, 1, 2, \ldots$.

VIII. There exist a continuous function K_3 and $C_1 \in L^{p/\varrho}(\Omega)$ such that

$$|p_{\alpha,j}(x,\xi)| \leqq K_3(\xi')(C_1(x)+|\xi''|^{\varrho}) \quad j=0,1,2,\dots$$

for all $|\alpha| \leqq l$, $\xi=(\xi',\xi'') \in \mathbf{R}^N$ and a.e. $x \in \Omega$ (ξ' contains those ξ_β for which $|\beta| < m-(n/p)$; $p-1<\varrho\leqq p$, $l<m-(n/p)(1-p+\varrho)$).

IX. $\lim\limits_{j\to\infty} \xi^{(j)} = \xi^{(0)}$ implies

$$\lim_{j\to\infty} p_{\alpha,j}(x,\xi^{(j)}) = p_{\alpha,0}(x,\xi^{(0)}), \quad \lim_{j\to\infty} r_{\alpha,j}(x,\xi^{(j)}) = r_{\alpha,0}(x,\xi^{(0)})$$

for a.e. $x \in \Omega$ and all $|\alpha| \leqq l$.

X. V is a closed subspace of $W_p^m(\Omega)$ with the property: $v \in V$, $\varphi \in C_0^\infty(\mathbf{R}^n)$ imply that $\varphi v \in V$. (By $C_0^\infty(G)$ is denoted the set of infinitely differentiable functions with compact support contained in G.)

XI. $F_j \in V'$ ($j=0,1,2,\dots$), i.e. F_j is a linear continuous functional on V and

$$\lim_{j\to\infty} \|F_j - F_0\|_{V'} = 0.$$

Remarks. 1. Assume that I—IV, VI—VIII are fulfilled for $j=0$, i.e. $f_{\alpha,0}$, $g_{\alpha,0}$ satisfy conditions of the existence theorem in [1]. Further suppose that $f_{\alpha,j}$, $g_{\alpha,j}$ ($j=1,2,\dots$) satisfy I, VI such that

$$\lim_{j\to\infty} \big[\sup_{\xi\in\mathbf{R}^N} |f_{\alpha,j}(x,\xi)-f_{\alpha,0}(x,\xi)|\big] = 0 \quad \text{for a.e.} \quad x\in\Omega,$$

$$\sup_{\xi\in\mathbf{R}^N} |f_{\alpha,j}(x,\xi)-f_{\alpha,0}(x,\xi)| \leqq \varphi(x) \quad \text{for a.e.} \quad x\in\Omega$$

where $\varphi \in L^q(\Omega)$, $j=1,2,\dots$;

$$\lim_{j\to\infty} \big[\sup_{\xi\in\mathbf{R}^N} |g_{\alpha,j}(x,\xi)-g_{\alpha,0}(x,\xi)|\big] = 0 \quad \text{for a.e.} \quad x\in\Omega,$$

$$\sup_{\xi\in\mathbf{R}^N} |g_{\alpha,j}(x,\xi)-g_{\alpha,0}(x,\xi)| \leqq \psi(x) \quad \text{for a.e.} \quad x\in\Omega$$

where $\psi \in L^{p/\varrho}(\Omega)$, $j=1,2,\dots$.

Then I, II, IV—VIII are satisfied for $f_{\alpha,j}$, $g_{\alpha,j}$ ($j=1,2,\dots$) with $p_{\alpha,j} := p_{\alpha,0}$, $r_{\alpha,j} := (g_{\alpha,j}-g_{\alpha,0})+r_{\alpha,0}$.

2. If there is a constant $c>0$ such that for a.e. $x\in\Omega$, all (η,ζ), $(\eta,\zeta')\in\mathbf{R}^N$

$$\sum_{|\alpha|=m} [f_{\alpha,0}(x,\eta,\zeta)-f_{\alpha,0}(x,\eta,\zeta')](\xi_\alpha-\xi'_\alpha) \geqq c|\zeta-\zeta'|^p$$

and

$$|[f_{\alpha,j}(x,\eta,\zeta)-f_{\alpha,j}(x,\eta,\zeta')]-[f_{\alpha,0}(x,\eta,\zeta)-f_{\alpha,0}(x,\eta,\zeta')]| \leqq$$
$$\leqq d_j|\zeta-\zeta'|^{p-1} \quad (j=1,2,\dots)$$

where $\lim\limits_{j\to\infty} d_j=0$, then $f_{\alpha,j}$ satisfy III for sufficiently large j.

Lemma 1. *Assume that* $u_j \to u$ *weakly in* V *and for any bounded domain* $\omega \subset \Omega$

$$\lim_{j\to\infty} \int_\omega h_j \, dx = 0, \tag{1.1}$$

where

$$h_j(x) = \sum_{|\alpha|=m} [f_{\alpha,j}(x, u_j, \ldots, D^\gamma u_j, \ldots, D^\beta u_j, \ldots) - \tag{1.2}$$
$$-f_{\alpha,j}(x, u_j, \ldots, D^\gamma u_j, \ldots, D^\beta u, \ldots)](D^\alpha u_j - D^\alpha u),$$

$|\gamma|<m$, $|\beta|=m$. *Then there is a subsequence* (u_{j_k}) *of* (u_j) *such that* $D^\beta u_{j_k} \to D^\beta u$ *a.e. in* Ω *for all* β *with* $|\beta|\leqq m$ *and for any bounded* $\omega\subset\Omega$, $u_{j_k}\to u$ *with respect to the norm of* $W_p^m(\omega)$.

Proof. Since $u_j \to u$ weakly in V there is a subsequence (u_{j_k}) of (u_j) such that for $|\gamma|<m$

$$D^\gamma u_{j_k} \to D^\gamma u \quad \text{a.e. in } \Omega$$

and

$$\lim_{k\to\infty} \|D^\gamma u_{j_k} - D^\gamma u\|_{L^p(\omega)} = 0 \tag{1.3}$$

for any bounded subdomain ω of Ω (see e.g. [5] and [4]). Further, by assumption III $h_j \geqq 0$ and so (1.1) and Fatou's lemma imply that $h_j \to 0$ a.e. in ω. Thus there exists $\omega_0 \subset \omega$ of measure 0 such that for $x \in \omega \setminus \omega_0$

$$|D^\beta u(x)| < \infty, \ |K_1(x)| < \infty, \ |K_2(x)| < \infty, \tag{1.4}$$

$$D^\gamma u_{j_k}(x) \to D^\gamma u(x) \quad (|\gamma| < m), \ h_{j_k}(x) \to 0, \quad k \to \infty. \tag{1.5}$$

Set

$$\xi^{(k)}(x) = (\ldots, D^\beta u_{j_k}(x), \ldots)$$

where $|\beta|=m$. By assumptions II, IV, V and (1.4), (1.5) we have

$$h_{j_k}(x) \geqq \sum_{|\alpha|=m} f_{\alpha,j_k}(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots) D^\alpha u_{j_k} - \tag{1.6}$$
$$- \sum_{|\alpha|=m} |f_{\alpha,j_k}(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots) D^\alpha u| -$$
$$- \sum_{|\alpha|=m} |f_{\alpha,j_k}(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u, \ldots)(D^\alpha u_{j_k} - D^\alpha u)| \geqq$$
$$\geqq c_2 |\xi^{(k)}(x)|^p - c_3(x)[1 + |\xi^{(k)}(x)|^{p-1} + |\xi^{(k)}(x)|]$$

if $x\in\omega\setminus\omega_0$ where $|\gamma|<m$, $|\beta|=m$. (For a fixed $x\in\omega\setminus\omega_0$, $D^\gamma u_{j_k}(x)$ and $f_{\alpha,j_k}(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u, \ldots)$ are convergent and thus they are bounded.) By (1.5) $(h_{j_k}(x))$ is bounded for a fixed $x\in\omega\setminus\omega_0$, thus (1.6) implies that $(\xi^{(k)}(x))$ is bounded, too. Consequently, for a fixed $x\in\omega\setminus\omega_0$, $(\xi^{(k)}(x))$ contains a subsequence which converges to a vector $\xi^*(x)$.

Now we show that

$$\xi^*(x) = \xi(x) = (\ldots, D^\beta u(x), \ldots). \tag{1.7}$$

Indeed, applying (1.2) to the subsequence of $(h_{j_k}(x))$ with $k\to\infty$, by (1.5) and assumption V we obtain

$$0 = \sum_{|\alpha|=m} [f_{\alpha,0}(x, u(x), \ldots, D^\gamma u(x), \ldots, \xi^*(x))-$$
$$-f_{\alpha,0}(x, u(x), \ldots, D^\gamma u(x), \ldots, \xi(x))][\xi^*_\alpha(x)-\xi_\alpha(x)]$$

which implies (1.7) in virtue of assumption III.

So we have shown that all convergent subsequences of the bounded sequence $(\xi^{(k)}(x))$ tend to $\xi(x)$. Therefore, $\lim_{k\to\infty} \xi^{(k)}(x)=\xi(x)$ if $x\in\omega\setminus\omega_0$ and thus, by (1.5) $D^\beta u_{j_k}\to D^\beta u$ a.e. in ω for all β satisfying $|\beta|\leqq m$. Since ω is an arbitrary bounded subset of Ω we have

$$D^\beta u_{j_k} \to D^\beta u \quad \text{a.e. in} \quad \Omega \quad \text{if} \quad |\beta| \leqq m. \tag{1.8}$$

By using notations

$$F_k(x) = \sum_{|\alpha|=m} f_{\alpha,j_k}(x, u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots)D^\alpha u_{j_k},$$
$$F_0(x) = \sum_{|\alpha|=m} f_{\alpha,0}(x, u, \ldots, D^\beta u, \ldots)D^\alpha u,$$

from (1.1) one obtains that

$$\int_\omega F_k\,dx - \sum_{|\alpha|=m}\int_\omega f_{\alpha,j_k}(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots)D^\alpha u\,dx-$$
$$-\sum_{|\alpha|=m}\int_\omega f_{\alpha,j_k}(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u, \ldots)D^\alpha(u_{j_k}-u)\,dx \to 0,$$

i.e.

$$\int_\omega F_k\,dx - \int_\omega F_0\,dx - \tag{1.9}$$
$$-\sum_{|\alpha|=m}\int_\omega [f_{\alpha,j_k}(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u_{j_k}, \ldots)-$$
$$-f_{\alpha,0}(x, u, \ldots, D^\gamma u, \ldots, D^\beta u, \ldots)]D^\alpha u\,dx-$$
$$-\sum_{|\alpha|=m}\int_\omega f_{\alpha,j_k}(x, u_{j_k}, \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u, \ldots)D^\alpha(u_{j_k}-u)\,dx \to 0.$$

By assumptions II, V, (1.8), Hölder's inequality and Vitali's theorem the third term in (1.9) converges to 0. Furthermore, (1.8), assumptions II, V, (1.3) and Vitali's theorem imply that

$$f_{\alpha,j_k}(x, u_{j_k} \ldots, D^\gamma u_{j_k}, \ldots, D^\beta u, \ldots) \to f_{\alpha,0}(x, u, \ldots, D^\gamma u, \ldots, D^\beta u, \ldots)$$

in the norm of $L^q(\omega)$. Since $\lim_{k\to\infty} D^\alpha(u_{j_k}-u)\to 0$ weakly in $L^p(\Omega)$ one finds that the fourth term in (1.9) converges to 0, too.

Therefore, from (1.9) it follows that

$$\lim_{k\to\infty} \int_\omega F_k\,dx = \int_\omega F_0\,dx. \tag{1.10}$$

By assumption IV

$$F_k(x) \geqq c_2 \sum_{|\beta|=m} |D^\beta u_{j_k}(x)|^p - K_2(x).$$

Thus for functions $G_k=F_k+K_2$, $G_0=F_0+K_2$ we have

$$G_k(x) \geqq c_2 \sum_{|\beta|=m} |D^\beta u_{j_k}(x)|^p \geqq 0, \tag{1.11}$$

and by (1.10)

$$\lim_{k\to\infty} \int_\omega G_k\,dx = \int_\omega G_0\,dx. \tag{1.12}$$

(1.8) and assumption V imply that $G_k\to G_0$ a.e. in ω, thus from (1.11), (1.12) it follows that

$$G_k \to G_0 \quad \text{in} \quad L^1(\omega) \tag{1.13}$$

(see [6]). Consequently, (1.8), (1.11) and Vitali's theorem imply that, for $|\beta|=m$, $D^\beta u_{j_k}\to D^\beta u$ in $L^p(\omega)$, and the proof of Lemma 1 is complete.

Assume that instead of III condition

III'. $\sum_{|\alpha|\leqq m} [f_{\alpha,j}(x,\xi)-f_{\alpha,j}(x,\xi')](\xi_\alpha-\xi'_\alpha) > 0$

is fulfilled if $\xi\neq\xi'$.

An easy modification of the proof of Lemma 1 gives

Lemma 2. *Suppose that* $u_j\to u$ *weakly in* V *and*

$$\lim_{j\to\infty} \int_\Omega \tilde{h}_j\,dx = 0,$$

where

$$\tilde{h}_j(x) = \sum_{|\alpha|\leqq m} [f_{\alpha,j}(x,u_j,\dots,D^\beta u_j,\dots)-f_{\alpha,j}(x,u,\dots,D^\beta u,\dots)](D^\alpha u_j - Du).$$

Then there is a subsequence (u_{j_k}) *of* (u_j) *such that* $u_{j_k}\to u$ *with respect to the norm of* $W_p^m(\Omega)$.

2. Stability results

Theorem 1. *Assume that conditions I—XI are fulfilled and* $u_j \in V$ *is a solution of*

(2.1)
$$\sum_{|\alpha|\leq m} \int_\Omega f_{\alpha,j}(x, u_j, \dots, D^\beta u_j, \dots) D^\alpha v\, dx +$$
$$+ \sum_{|\alpha|\leq l} \int_\Omega g_{\alpha,j}(x, u_j, \dots, D^\beta u_j, \dots) D^\alpha v\, dx = \langle F_j, v\rangle$$

for all $v \in V$ $(j = 1, 2, \dots)$.

Then there is a subsequence (u_{j_k}) *of* (u_j) *which converges weakly in* V *to a solution* $u \in V$ *of* (2.1) *for* $j=0$. *Moreover,* $D^\beta u_{j_k} \to D^\beta u$ *a.e. in* Ω *if* $|\beta| \leq m$, *and for arbitrary bounded* $\omega \subset \Omega$, $u_{j_k} \to u$ *strongly in* $W_p^m(\omega)$.

If solution u *of* (2.1) *for* $j=0$ *is unique then* $u_j \to u$ *weakly in* V *and strongly in* $W_p^m(\omega)$ *for any bounded* $\omega \subset \Omega$.

Remark. According to [1], for any $F_j \in V'$ there exists at least one solution $u_j \in V$ of (2.1).

Proof of Theorem 1. Applying (2.1) to $v = u_j$, by assumptions IV, VI, VII we obtain that

(2.2)
$$c_2 \|u_j\|_V^p - \int_\Omega K_2(x)\, dx - \sum_{|\alpha|\leq l} \|h_\alpha\|_{L^{p/\varrho}(\Omega)} \|D^\alpha u_j\|_{L^{q_1}(\Omega)} \leq \|F_j\|_{V'} \|u_j\|_V$$

where q_1 is defined by $1/(p/\varrho) + 1/q_1 = 1$.
By an imbedding theorem (see e.g. [4]) for

$$|\alpha| \leq l(< m - (n/p)(1 - p + \varrho)), \quad v \in W_p^m(\Omega) \quad \text{we have}$$

(2.3)
$$\|D^\alpha v\|_{L^{q_1}(\Omega)} \leq c \|v\|_{W_p^m(\Omega)}$$

(c is a constant) because $q_1 < np/(n-(m-l)p)$. Thus (2.2) and $p > 1$ imply that (u_j) is bounded in V. Therefore, there exist a subsequence (u_{j_k}) of (u_j) and $u \in V$ such that

(2.4)
$$u_{j_k} \to u \quad \text{weakly in} \quad V,$$

(2.5)
$$D^\gamma u_{j_k} \to D^\gamma u \quad \text{a.e. in} \quad \Omega \quad \text{for} \quad |\gamma| \leq m-1$$

(see [5]).

Consider an arbitrary bounded domain $\omega \subset \Omega$ and a function $\Theta \in C_0^\infty(\mathbf{R}^n)$ such that $\Theta \geq 0$ and $\Theta(x) = 1$ for $x \in \omega$. By the theorems on compact imbedding (see e.g. [4]) it may be supposed that

(2.6)
$$D^\gamma u_{j_k} \to D^\gamma u \text{ in } L^p(\Omega \cap \operatorname{supp} \Theta) \quad \text{for} \quad |\gamma| \leq m-1$$

and

$$D^\gamma u_{j_k} \to D^\gamma u \quad \text{in} \quad L^{q_1}(\Omega \cap \operatorname{supp} \Theta) \quad \text{for} \quad |\gamma| \leqq l, \tag{2.7}$$

where q_1 is defined by $1/(p/\varrho)+1/q_1=1$ $(l<m-(n/p)(1-p+\varrho))$. By a "diagonal process" the subsequence (u_{j_k}) can be chosen so that (2.6), (2.7) are true for any fixed $\Theta \in C_0^\infty(\mathbf{R}^n)$.

In virtue of assumption X $\Theta(u_{j_k}-u)\in V$ and thus from (2.1) one obtains

$$\sum_{|\alpha|\leqq m} \int_\Omega f_{\alpha, j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots) D^\alpha[(\Theta(u_{j_k}-u)]\, dx + \tag{2.8}$$
$$+\sum_{|\alpha|\leqq l} \int_\Omega g_{\alpha, j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots) D^\alpha[\Theta(u_{j_k}-u)]\, dx =$$
$$= \langle F_{j_k}, \Theta(u_{j_k}-u)\rangle.$$

Since $(u_{j_k}-u) \to 0$ weakly in V

$$\Theta(u_{j_k}-u) \to 0 \quad \text{weakly in } V. \tag{2.9}$$

From (2.8) it follows that

$$\sum_{|\alpha|=m} \int_\Omega [f_{\alpha, j_k}(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u_{j_k}, \dots)- \tag{2.10}$$
$$-f_{\alpha, j_k}(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u, \dots]\, \Theta D^\alpha(u_{j_k}-u)\, dx =$$
$$= \sum_{|\alpha|=m} \int_\Omega f_{\alpha, j_k}(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u, \dots)\, \Theta D^\alpha(u-u_{j_k})\, dx +$$
$$+\sum_{|\alpha|=m} \int_\Omega f_{\alpha, j_k}(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u_{j_k}, \dots) \sum_{|\gamma|\leqq m-1} c_\gamma D^\gamma(u-u_{j_k}) D^{\alpha-\gamma}\Theta\, dx +$$
$$+\sum_{|\alpha|\leqq m-1} \int_\Omega f_{\alpha, j_k}(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u_{j_k}, \dots) D^\alpha[\Theta(u-u_{j_k})]\, dx +$$
$$+\sum_{|\alpha|\leqq l} \int_\Omega g_{\alpha, j_k}(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u_{j_k}, \dots) D^\alpha[\Theta(u-u_{j_k})]\, dx +$$
$$+\langle F_{j_k}, \Theta(u_{j_k}-u)\rangle \quad (|\gamma|<m, |\beta|=m).$$

Now we show that all the terms on the right-hand side of (2.10) converge to 0 as $k\to\infty$. By (2.4), $D^\alpha(u_{j_k}-u)\to 0$ weakly in $L^p(\Omega)$. Furthermore, from (2.5) and assumption V we get

$$\Theta f_{\alpha, j_k}(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u, \dots) \to \tag{2.11}$$
$$\to \Theta f_{\alpha, 0}(x, u, \dots, D^\gamma u, \dots, D^\beta u, \dots)$$

a.e. in Ω, and, consequently, by assumption II, (2.6) and Vitali's theorem (2.11) is valid in $L^q(\Omega)$ norm, too. Thus the first term in (2.10) converges to 0.

By assumptions I, II the functions

$$f_{\alpha, j_k}(x; u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u_{j_k}, \dots)$$

are bounded in $L^q(\Omega)$, hence (2.6) implies that the second and third terms in (2.10) converge to 0 as $k\to\infty$.

From assumptions VI—VIII it follows that

$$g_{\alpha,j_k}(x, u_{j_k}, \dots, D^\gamma u_{j_k}, \dots, D^\beta u_{j_k}, \dots)$$

is bounded in $L^{p/q}(\Omega\cap\operatorname{supp}\Theta)$, thus (2.7) implies that the fourth term in (2.10) converges to 0 as $k\to\infty$. Finally, for the last term we have

$$|\langle F_{j_k}, \Theta(u_{j_k}-u)\rangle| \leqq |\langle F_{j_k}-F_0, \Theta(u_{j_k}-u)\rangle| +$$
$$+|\langle F_0, \Theta(u_{j_k}-u)\rangle| \leqq \|F_{j_k}-F_0\|_{V'}\|\Theta(u_{j_k}-u)\|_V+|\langle F_0, \Theta(u_{j_k}-u)\rangle|,$$

thus assumption XI, (2.9) imply that also the last term in (2.10) converges to 0 as $k\to\infty$.

Thus we have shown that the term on the left-hand side of (2.10) converges to 0 as $k\to\infty$. By assumption III and $\Theta\geqq 0$ we find that (1.1) is valid for a subsequence of (h_j). Consequently, from Lemma 1 we obtain that (u_{j_k}) contains a subsequence $(u_{j'_k})$ such that

$$D^\beta u_{j'_k} \to D^\beta u \quad \text{a.e. in } \Omega \tag{2.12}$$

if $|\beta|\leqq m$, and for any bounded $\omega\subset\Omega$

$$(u_{j'_k}) \to u \quad \text{in} \quad W_p^m(\omega). \tag{2.13}$$

(2.12) and assumption V implies that

$$f_{\alpha,j'_k}(x, u_{j'_k}, \dots, D^\beta u_{j'_k}, \dots) \to f_{\alpha,0}(x, u, \dots, D^\beta u, \dots)$$

a.e. in Ω. Therefore, assumption II the boundedness of $\|u_{j'_k}\|_V$, Hölder's inequality and Vitali's theorem imply that for any $v\in V$

$$\lim_{k\to\infty}\sum_{|\alpha|\leqq m}\int_\Omega f_{\alpha,j'_k}(x, u_{j'_k}, \dots, D^\beta u_{j'_k}, \dots)D^\alpha v\,dx = \tag{2.14}$$
$$=\sum_{|\alpha|\leqq m}\int_\Omega f_{\alpha,0}(x, u, \dots, D^\beta u, \dots)D^\alpha v\,dx.$$

By using assumption IX and (2.12) we find $g_{\alpha,j'_k}(x, u_{j'_k}, \dots, D^\beta u_{j'_k}, \dots)\to$ $\to g_{\alpha,0}(x, u, \dots, D^\beta u, \dots)$ a.e. in Ω and thus, by assumptions VI—VIII, (2.3), Hölder's inequality and Vitali's theorem we find that for any $v\in V$

$$\lim_{k\to\infty}\sum_{|\alpha|\leqq l}\int_\Omega g_{\alpha,j'_k}(x, u_{j'_k}, \dots, D^\beta u_{j'_k}, \dots)D^\alpha v\,dx =$$
$$=\sum_{|\alpha|\leqq l}\int_\Omega g_{\alpha,0}(x, u, \dots, D^\beta u, \dots)D^\alpha v\,dx.$$

Thus from (2.1), (2.14), assumption XI it follows that u is a solution of (2.1) for $j=0$ and, by (2.4), (2.12), (2.13), the proof of the first statement of Theorem 1 is complete.

If solution u of problem (2.1) for $j=0$ is unique but "$u_j \to u$ weakly in V" is not true then there are $G \in V'$, a positive number ε and a subsequence (u_j') of (u_j) such that

$$|Gu_j' - Gu| > \varepsilon, \quad j = 1, 2, \dots . \tag{2.15}$$

Applying the first statement of Theorem 1 to (u_j') instead of (u_j) we find that there is a subsequence (u_j'') of (u_j') which converges weakly in V to a solution of (2.1) for $j=0$, i.e. $u_j'' \to u$ weakly in V (because the solution of (2.1) for $j=0$ is unique). But this is impossible because of (2.15). It can be proved similarly that then $u_j \to u$ strongly in $W_p^m(\omega)$ for any bounded $\omega \subset \Omega$.

Theorem 2. *Assume that conditions I—II, III′, IV—XI are fulfilled and $u_j \in V$ is a solution of (2.1). Then there is a subsequence (u_{j_k}) of (u_j) which converges strongly in V to a solution $u \in V$ of (2.1) for $j=0$. If the solution u of (2.1) for $j=0$ is unique then (u_j) also converges to u strongly in V.*

Proof. Assumption III′ implies III thus all conditions of Theorem 1 are fulfilled. Consequently, by Theorem 1 there is a subsequence (u_{j_k}) of (u_j) such that

$$u_{j_k} \to u \quad \text{weakly in } V \tag{2.16}$$

and

$$D^\beta u_{j_k} \to D^\beta u \quad \text{a.e. in } \Omega \qquad \text{for} \quad |\beta| \leq m, \tag{2.17}$$

where u is a solution of (2.1) for $j=0$.

Now we show that the sequence (u_{j_k}) satisfies the condition of Lemma 2. Since u_{j_k} is a solution of (2.1) with $j=j_k$, $v=u_{j_k}$ and u is a solution of (2.1) with $j=0$, $v=u$, we have

(2.18)

$$\sum_{|\alpha| \leq m} \int_\Omega [f_{\alpha, j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots) - f_{\alpha, j_k}(x, u, \dots, D^\beta u, \dots)](D^\alpha u_{j_k} - D^\alpha u)\, dx =$$
$$= \sum_{|\alpha| \leq m} \int_\Omega f_{\alpha, j_k}(x, u, \dots, D^\beta u, \dots)(D^\alpha u - D^\alpha u_{j_k})\, dx +$$
$$+ \sum_{|\alpha| \leq m} \int_\Omega [f_{\alpha,0}(x, u, \dots, D^\beta u, \dots) - f_{\alpha, j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots)] D^\alpha u\, dx +$$
$$+ \sum_{|\alpha| \leq l} \int_\Omega [g_{\alpha,0}(x, u, \dots, D^\beta u, \dots) D^\alpha u - g_{\alpha, j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots) D^\alpha u_{j_k}]\, dx +$$
$$+ \{\langle F_{j_k}, u_{j_k}\rangle - \langle F_0, u\rangle\}.$$

Applying Vitali's theorem, Hölder's inequality, assumptions I, II, V and (2.16), (2.17), we find that the first and second terms on the right-hand side of (2.18) converge to 0 as $k\to\infty$. By assumption XI and (2.16) we have

$$|\langle F_{j_k}, u_{j_k}\rangle-\langle F_0, u\rangle| \leqq |\langle F_{j_k}-F_0, u_{j_k}\rangle|+|\langle F_0, u_{j_k}-u\rangle| \leqq$$
$$\leqq \|F_{j_k}-F_0\|_{V'}\|u_{j_k}\|_V+|\langle F_0, u_{j_k}-u\rangle| \to 0 \quad \text{as} \quad k\to\infty.$$

Furthermore, (2.17) and assumption IX yield

$$p_{\alpha,j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots)D^\alpha u_{j_k} \to p_{\alpha,0}(x, u, \dots, D^\beta u, \dots)D^\alpha u$$

a.e. in Ω. In virtue of Fatou's lemma and assumption VII we get the inequality

$$\text{(2.19)} \qquad \int_\Omega p_{\alpha,0}(x, u, \dots, D^\beta u, \dots)D^\alpha u\, dx \leqq$$
$$\leqq \liminf_{k\to\infty} \int_\Omega p_{\alpha,j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots)D^\alpha u_{j_k}\, dx.$$

Assumptions VII, IX, (2.17), Hölder's inequality and Vitali's theorem imply that

$$\lim_{k\to\infty} \int_\Omega r_{\alpha,j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots)D^\alpha u_{j_k}\, dx =$$
$$= \int_\Omega r_{\alpha,0}(x, u, \dots, D^\beta u, \dots)D^\alpha u\, dx.$$

Hence and from (2.19) it follows that

$$\limsup_{k\to\infty} \sum_{|\alpha|\leqq l} \int_\Omega [g_{\alpha,0}(x, u, \dots, D^\beta u, \dots)D^\alpha u -$$
$$- g_{\alpha,j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots)D^\alpha u_{j_k}]\, dx =$$
$$= \limsup_{k\to\infty} \sum_{|\alpha|\leqq l} \int_\Omega [p_{\alpha,0}(x, u, \dots, D^\beta u, \dots)D^\alpha u -$$
$$- p_{\alpha,j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots)D^\alpha u_{j_k}]\, dx \leqq$$
$$\leqq \sum_{|\alpha|\leqq l} \int_\Omega p_{\alpha,0}(x, u, \dots, D^\beta u, \dots)D^\alpha u\, dx +$$
$$+ \sum_{|\alpha|\leqq l} \limsup_{k\to\infty} \int_\Omega [-p_{\alpha,j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots)D^\alpha u_{j_k}]\, dx \leqq 0.$$

In virtue of (2.18) we have shown that

$$\limsup_{k\to\infty} \sum_{|\alpha|\leqq m} \int_\Omega [f_{\alpha,j_k}(x, u_{j_k}, \dots, D^\beta u_{j_k}, \dots) -$$
$$- f_{\alpha,j_k}(x, u, \dots, D^\beta u, \dots)](D^\alpha u_{j_k} - D^\alpha u)\, dx \leqq 0.$$

Hence by assumption III′ it follows that (u_{j_k}) satisfies the conditions of Lemma 2 and there is a subsequence $(u_{j_k'})$ of (u_{j_k}) such that $u_{j_k'} \to u$ in $W_p^m(\Omega)$. This completes the proof of the first statement of Theorem 2. The case when the solution u of (2.1) for $j=0$ is unique can be treated in the same way as in the proof of Theorem 1.

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