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On the minimal ring containing the boundary of a convex body

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1. Let $K \subset \mathbf{R}^2$ be a convex compact set with boundary C. For each point $x \in K$ there exist a minimal circular disc $B(R(x), x)$ containing K and a maximal circular disc $B(r(x), x)$ contained in K, where $B(r, x)$ denotes the disc with radius r and center x.

The function $R(x) - r(x)$ attaines its minimal value in a unique point $x_0 \in K$. This was shown by BONNESEN [1], Bonnesen and FENCHEL [2]. So the circular ring around x_0 with radii $R(x_0)$ and $r(x_0)$, respectively, is the *minimal ring* containing the boundary C of K.

This result was used by Bonnesen and Fenchel [2] to sharpen the isoperimetric inequality in $\mathbf{R}^2$. Later I. VINCZE [7] showed that

$$\frac{\min\{R(x):\ x \in K\}}{R(x_0)} \geqq \frac{\sqrt{3}}{2} \tag{1}$$

and

$$\frac{\max\{r(x):\ x \in K\}}{r(x_0)} < 2 \tag{2}$$

and these inequalities are sharp.

Answering a question due to I. Vincze we generalize the inequalities (1) and (2) to arbitrary dimension. To do so we need a theorem characterizing the minimal ring in $\mathbf{R}^d$. For $d=2$ and $d=3$ such a theorem was found by Bonnesen [1] and by KRITIKOS [4]. The main tool in the proof of our results is the use of convex analysis (see: Йоффе — Тихомиров [3] and ROCKEFELLAR [5]).

2. Again, let $K \subset \mathbf{R}^d$ be a convex compact set with boundary C. $B(r, x)$ stands for the ball with radius r and center x. For $x \in K$ we define

$$R(x) = \min\{R:\ B(R, x) \supseteqq K\},$$
$$r(x) = \max\{r:\ B(r, x) \subseteqq K\}.$$

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It is easy to see that the maximum and minimum above exist, so the definition is correct. Moreover, this means that for each $x \in K$ there exist points p and q such that $p, q \in C$ and $\|x-p\| = R(x)$ and $\|x-q\| = r(x)$. In this case we say that p supports $R(x)$ and q supports $r(x)$.

Theorem 1. *There exists a point $x_0 \in K$ in which the function $R(x)-r(x)$ attaines its minimal value. This point x_0 is unique.*

The set $\{x \in \mathbf{R}^d : r(x_0) \leqq \|x-x_0\| \leqq R(x_0)\}$ is called the *minimal ring* containing C. The characterization theorem for the minimal ring is this:

Theorem 2. *The point $x_0 \in K$ is the center of the minimal ring if and only if there are points $p_1, \dots, p_k \in C$ supporting $R(x_0)$ and $q_1, \dots, q_l \in C$ supporting $r(x_0)$ $(k, l \geqq 1)$ such that*

$$\operatorname{conv}\left\{\frac{p_i - x_0}{R(x_0)} : i = 1, \dots, k\right\} \cap \operatorname{conv}\left\{\frac{q_i - x_0}{r(x_0)} : j = 1, \dots, l\right\} \neq \emptyset,$$

where conv *denotes the convex hull.*

There is a certain converse to this theorem. We describe it when $x_0 = 0$.

Theorem 3. *Let $p_1, \dots, p_k$, $q_1, \dots, q_l$ be vectors in $\mathbf{R}^d$ such that*
(i) $\|p_1\| = \dots = \|p_k\| = R \geqq r$,
(ii) $\|q_1\| = \dots = \|q_l\| = r > 0$,
(iii) $\{p_i/R : i = 1, \dots, k\} \cap \operatorname{conv}\{q_j/r : j = 1, \dots, l\} \neq \emptyset$,
(iv) *each p_i is contained in the halfspaces*

$$\{x \in \mathbf{R}^d : \langle q_j, q_j - x\rangle \geqq 0\} \quad (j = 1, \dots, l).$$

In this case there exists a convex compact set $K \subset \mathbf{R}^d$ for which $R(x)-r(x)$ attaines its minimal value at $x_0 = 0$, $R(0) = R$, $r(0) = r$ and $R(0)$ is supported by $p_1, \dots, p_k \in C$ and $r(0)$ is supported by $q_1, \dots, q_l \in C$.

Now we give the generalization of the inequalities (1) and (2).

Theorem 4. *For $d \geqq 3$, $\max r(x)/r(x_0)$ is not bounded from above. On the other hand, for $d \geqq 3$,*

$$\min R(x)/R(x_0) \geqq \frac{1}{2}\left(\cos^2\alpha_0 + \cos\alpha_0 - 1 + \frac{1}{\cos\alpha_0}\right) \approx 0.8054,$$

where $\alpha_0 \in (0, \pi/2)$ is the root of the equation $\sin^2\alpha - 2\cos^3\alpha = 0$. This inequality is sharp.

3. This section contains the proofs. We start with some simple facts and observations.

Claim 1.

$$R(x) = \max_{p\in K} \|x-p\| = \max_{p\in C} \|x-p\|,$$

$$r(x) = \inf_{p\notin K} \|x-p\| = \min_{p\in C} \|x-p\|,$$

and the points in which the maximum (minimum) is attained support $R(x)$ ($r(x)$, respectively).

Claim 2.

(a)
$$R\left(\frac{x_1+x_2}{2}\right) \leqq \frac{1}{2}(R(x_1)+R(x_2))$$

and if equality holds here, then there is a unique $p\in C$ supporting $R((x_1+x_2)/2)$ and this point lies on the straight line through x_1 and x_2, and p supports $R(x_1)$ and $R(x_2)$ as well.

(b)
$$r\left(\frac{x_1+x_2}{2}\right) \geqq \frac{1}{2}(r(x_1)+r(x_2))$$

Proof. (a) Let $p\in C$ be a point of support for $R((x_1+x_2)/2)$. Then $p\in B(R(x_1), x_1)\cap B(R(x_2), x_2)$ and the triangle-inequality proves the claim.

(b) Obviously conv $(B(r(x_1), x_1)\cup B(r(x_2), x_2))\subseteqq K$ and an easy calculation shows that

$$B\left(\frac{r(x_1)+r(x_2)}{2},\ \frac{x_1+x_2}{2}\right) \subseteqq \operatorname{conv}(B(r(x_1), x_1)\cup B(r(x_2), x_2)).$$

Proof of Theorem 1. By Claim 2, $R(x)$ is a convex, $r(x)$ is a concave function. So $R(x)-r(x)$ is convex and attaines its infimum. What we have to show is the uniqueness of the minimum. This will be done by showing that $x_1, x_2\in K$, $x_1\neq x_2$ and $R(x_1)-r(x_1)=R(x_2)-r(x_2)=h$ implies that $R((x_1+x_2)/2)-r((x_1+x_2)/2)<h$.

Convexity implies that $R((x_1+x_2)/2)-r((x_1+x_2)/2)\leqq h$, so assume, by way of contradiction, that $R((x_1+x_2)/2)-r((x_1+x_2)/2)=h$. Then by Claim 2, we have $R((x_1+x_2)/2)=1/2\,(R(x_1)+R(x_2))$ and a unique point $p\in C$ supporting $R(x_1)$, $R(x_2)$ and $R((x_1+x_2)/2)$ and p lies on the straight line through x_1 and x_2. Without loss of generality we suppose that x_2 lies between x_1 and p on this line. By our assumption $R(x_1)-r(x_1)=R(x_2)-r(x_2)$, so $B(r(x_2), x_2)\subseteqq B(r(x_1), x_1)$, and then there is a unique point $q\in C$ supporting $r(x_2)$ and this point lies on the line segment joining x_2 and p. But K contains the set conv $(B(r(x_1), x_1)\cup\{p\})$ and this set contains q in its interior. This contradicts the assumption

$$R\left(\frac{x_1+x_2}{2}\right)-r\left(\frac{x_1+x_2}{2}\right) = h. \quad \square$$

For fixed $p \in C$ define $Z(p)$ as the set of unit outer normals to K at p, i.e.,

$$Z(p) = \{z \in R^d \colon \|z\| = 1, \langle z, p\rangle = \max_{x \in K} \langle z, x\rangle\}.$$

Define now

$$\Gamma = \{(p, z) \in R^d \times R^d \colon z \in Z(p)\}.$$

It is clear that Γ is compact.

Claim 3. (a) $R(x) = \max \{\langle z, p-x\rangle \colon (p, z) \in \Gamma\}$,

(b) $r(x) = \min \{\langle z, p-x\rangle \colon (p, z) \in \Gamma\}$.

Proof. (a) Clearly for each $(p, z) \in \Gamma$

$$\langle z, p-x\rangle \leqq \|z\| \cdot \|p-x\| = \|p-x\| \leqq R(x).$$

If p_0 supports $R(x)$, then $(p_0, ((p_0-x)/\|p_0-x\|) \in \Gamma$ and

$$\left\langle \frac{p_0-x}{\|p_0-x\|}, p_0-x \right\rangle = R(x).$$

(b) Trivially $\langle z, p-x\rangle \geqq r(x)$ for each $(p, z) \in \Gamma$. On the other hand it is easy to check that if p_0 supports $r(x)$, then $Z(p_0) = \{p_0 - x/\|p_0-x\|\}$ and

$$\left\langle \frac{p_0-x}{\|p_0-x\|}, p_0-x \right\rangle = r(x). \quad \square$$

Using Claim 3 the function $r \colon K \to R^1$ can be extended over the whole space $\mathbf{R}^d$. It is again easy to see that the extended $r(x)$ is concave, and so the function $R(x) - r(x)$ $(x \in \mathbf{R}^d)$ attaines its minimal value at $x_0 \in K$ only.

To prove Theorem 2 we need some definitions and theorem from convex analysis.

Definition. Let $f \colon \mathbf{R}^d \to \mathbf{R}$ be a convex function. The set

$$\partial f(x) = \{x^* \in \mathbf{R}^d \colon \langle x^*, z-x\rangle \leqq f(z) - f(x) \text{ (for every } z \in \mathbf{R}^d)\}$$

is the *subgradient of f at x*.

It is well-known that the subgradient of a finite convex function is nonempty, convex and compact.

Theorem A (Fenchel, Rockafellar—Moreau, see [5]). *Let* $f \colon \mathbf{R}^d \to \mathbf{R}$ *be convex,* $g \colon \mathbf{R}^d \to \mathbf{R}$ *concave functions, finite over the whole space. Then* $f(x) - g(x)$ *attains its minimum at* x_0 *if and only if*

$$0 \in \partial f(x_0) + \partial(-g)(x_0).$$

Here the last addition is meant in the Minkowski sense; $(-g)$ is a convex function so $\partial(-g)(x_0)$ is its subgradient at x_0.

Theorem B (Йоффе — Тихомиров [3]). *Assume Γ is compact and the map $\gamma\mapsto(x_\gamma^*, a_\gamma)\in\mathbf{R}^d\times\mathbf{R}$ is continuous. Let $f(x)=\sup\{\langle x_\gamma^*, x\rangle+a_\gamma:\ \gamma\in\Gamma\}$. Then f: $\mathbf{R}^d\to\mathbf{R}$ is a finite convex function and $\partial f(x_0)=\operatorname{conv}\{x_\gamma^*:\ \gamma\in\Gamma$ and $\langle x_\gamma^*, x_0\rangle+a_\gamma=f(x_0)\}$.*

Now we are ready to prove Theorem 2.

Proof of Theorem 2. First by Theorem B

$$\partial R(x_0) = \operatorname{conv}\{-z:\ (p, z)\in\Gamma, \langle z, p-x_0\rangle = R(x_0)\},$$

$$\partial(-r)(x_0) = \operatorname{conv}\{z:\ (p, z)\in\Gamma, \langle z, p-x_0\rangle = r(x_0)\}.$$

By Theorem A, $R(x)-r(x)$ is minimal at x_0 if and only if for some $x^*\in\mathbf{R}^d$, $x^*\in\partial R(x_0)$ and $-x^*\in\partial(-r)(x_0)$. But $x^*\in\partial R(x_0)$ is the same as $x^*=-\sum_{i=1}^k\alpha_i z_i$ for some $\alpha_i\geqq 0$, $\sum_1^k\alpha_i=1$ and z_i with $(p_i, z_i)\in\Gamma$, $\langle z_i, p_i-x_0\rangle=R(x_0)$.

This is true if and only if $z_i=p_i-x_0/\|p_i-x_0\|$, i.e., if p_i supports $R(x_0)$. Similarly $-x^*\in\partial(-r)(x_0)$ is equivalent to $-x^*=\sum_{j=1}^l\beta_j w_j$ for some $\beta_j\geqq 0$, $\sum_1^l\beta_j=1$ and w_j with $(q_j, w_j)\in\Gamma$, $\langle w_j, q_j-x_0\rangle=r(x_0)$. In this case, again $w_j=(q_j-x_0)/\|q_j-x_0\|$ and q_j supports $r(x_0)$. These conditions imply that $R(x)-r(x)$ is minimal at x_0 if and only if there exist points $p_1, \dots, p_k\in C$ supporting $R(x_0)$ and $q_1, \dots, q_l\in C$ supporting $r(x_0)$ such that

$$\operatorname{conv}\left\{\frac{p_i-x_0}{R(x_0)}:\ i = 1, \dots, k\right\}\cap\operatorname{conv}\left\{\frac{q_j-x_0}{r(x_0)}:\ j = 1, \dots, l\right\} \neq \emptyset.$$

So we are finished with the proof. We mention that $k=1$ (or $l=1$) implies that K is a ball. Further, it can be shown that if $\operatorname{conv} P\cap\operatorname{conv} Q\neq\emptyset$ for some $P, Q\in\mathbf{R}^d$, then there are subsets $P'\subseteqq P$ and $Q'\subseteqq Q$ such that $\operatorname{conv} P'\cap\operatorname{conv} Q'\neq\emptyset$ and $|P'|+|Q|'\leqq d+2$. This means that we can suppose $k+l\leqq d+2$ in Theorem 2.

I mention here that the "only if" part of Theorem 2 can be proved in a simpler way: Set $P=\{(p_i-x_0)/R(x_0):\ i=1, \dots, k\}$ and $Q=\{(q_j-x_0)/r(x_0):\ j=1, \dots, l\}$. If $\operatorname{conv} P\cap\operatorname{conv} Q=\emptyset$, then there is a hyperplane separating P and Q with normal $a\in\mathbf{R}^d$, say. One can easily see that $R(x_0)>R(x_0+a)$ and $r(x_0)<r(x_0+a)$ which shows that $R(x)-r(x)$ cannot attain its minimal value at x_0.

Proof of Theorem 3. Set

$$K_{\min} = \operatorname{conv}(B(r, 0)\cup\{p_1, \dots, p_k\}).$$

$$K_{\max} = B(R, 0)\cap\bigcap_{j=1}^{l}\{x:\ \langle q_j, q_j-x\rangle \geqq 0\}.$$

It is easy to see that both $K_{\min}$ and $K_{\max}$ satisfy the conditions of Theorem 2 with $x_0=0$ and $p_1, \ldots, p_k, q_1, \ldots, q_l$. Moreover, any convex compact set K with $K_{\min}\subseteq \subseteq K \subseteq K_{\max}$ will do the same.

Proof of Theorem 4. *First part.* We construct a convex compact set $K\subset \mathbf{R}^d$ for each $d\geqq 3$ such that $\max r(x)/r(x_0)$ is "large".

Let $\bar{p}_1, \bar{p}_2, q_1, q_2$ be the vertices of a square such that $\|\bar{p}_1\|=\|\bar{p}_2\|=\|q_1\|=\|q_2\|=1$ and the length of the diagonals $\bar{p}_1\bar{p}_2$ and q_1q_2 is $2-\varepsilon$ (where $\varepsilon>0$ is small). The hyperplanes $\langle q_1, q_1-x\rangle=0$ and $\langle q_2, q_2-x\rangle=0$ meet in an affine flat A. The halflines starting from the origin in directions $\bar{p}_1$ and $\bar{p}_2$ meet A in the points $p_1=R\bar{p}_1$ and $p_2=R\bar{p}_2$. Consider the set $K_{\max}$ from Theorem 3 with p_1, p_2 and q_1, q_2. A simple calculation shows that

$$R(0)=\left(\varepsilon-\frac{\varepsilon^2}{4}\right)^{-1}, \quad r(0)=1, \quad \text{and} \quad \max r(x)=\left(\varepsilon-\frac{\varepsilon^2}{4}\right)^{-1/2}.$$

So we have

$$\frac{\max r(x)}{r(x_0)}=\left(\varepsilon-\frac{\varepsilon^2}{4}\right)^{-1/2}$$

which indeed tends to infinity as $\varepsilon\to 0$.

Second part. Let $K\subset \mathbf{R}^d$ $(d\geqq 3)$ be convex compact body and suppose that $R(x)-r(x)$ attaines its minimal value at $x_0=0$ and $r(x_0)=1$, $R(x_0)=R$. By Theorem 2 there exist points $p_1, \ldots, p_l$ supporting $R(x_0)$ and $q_1, \ldots, q_l$ supporting $r(x_0)$ with

$$\operatorname{conv}\{p_i/R\colon i=1, \ldots, k\}\cap \operatorname{conv}\{q_j\colon j=1, \ldots, l\}\neq\emptyset,$$

and we may assume $k, l\geqq 2$, $k+l\leqq d+2$. Then $\operatorname{conv}\{p_1, \ldots, p_k\}$ is a simplex whose nearest point to the origin is p_0 say. Clearly $\|p_1-p_0\|=\ldots=\|p_k-p_0\|$ and the angle between the vectors p_i and p_0 is the same for each i. Denote this angle by α.

Now the halfspaces $\langle q_j, q_j-x\rangle\geqq 0$ $(j=1, \ldots, l)$ have to contain the simplex $\operatorname{conv}\{p_1, \ldots, p_k\}$ and so the point p_0 as well. On the other hand, for some $j=1, \ldots, l$ the angle between the vectors q_j and p_0 is not larger than α for otherwise

$$\operatorname{conv}\{p_i/R\colon i=1, \ldots, k\}\cap \operatorname{conv}\{q_j\colon j=1, \ldots, l\}=\emptyset.$$

This implies that

$$0\leqq\langle q_j, q_j-p_0\rangle=1-\langle q_j, p_0\rangle=$$
$$=1-\|q_j\|\cdot\|p_0\|\cos(\sphericalangle q_j 0 p_0)\leqq 1-R\cos^2\alpha.$$

Consider now $\min_x R(x)=\varrho$ and set $R(\bar{x})=\varrho$, $\bar{x}\in K$. Then $B(\varrho, \bar{x})$ contains the points $p_1, \ldots, p_k$ and the ball $B(1, 0)$, so it contains the point $\bar{p}_0=-p_0/\|p_0\|$ as well. We are going to give an estimation from below for the radius of the smallest ball containing the points $\bar{p}_0, p_1, \ldots, p_k$. It is clear that the smallest ball containing

$p_1, \dots, p_k$ is $B(R\sin\alpha, p_0)$ and so $R\sin\alpha \leqq \varrho$. However if $\|\bar{p}_0 - p_0\| = R\cos\alpha + 1 > R\sin\alpha$, then $B(R\sin\alpha, p_0)$ does not contain $\bar{p}_0$. In this case, using some elementary geometry, we get the estimation

$$\varrho \geqq \frac{1+2R\cos\alpha + R^2}{2(1+R\cos\alpha)}.$$

Define now

$$f(R,\alpha) = \begin{cases} \sin\alpha & \text{if} \quad R\sin\alpha \geqq R\cos\alpha + 1, \\ \dfrac{1+2R\cos\alpha + R^2}{2R(1+R\cos\alpha)} & \text{otherwise} \end{cases}$$

where $R \geqq 1$, $0 \leqq \alpha \leqq \pi/2$ and $R\cos^2\alpha \leqq 1$.

What we have to do is to find the minimum of f in the domain determined by these inequalities. This is a routine calculation. The main steps are:

1) for R fixed $f(R,\alpha)$ is monotone non-decreasing, so the minimum is attained on the curve $R\cos^2\alpha = 1$;

2) on this curve the minimum of f is equal to

$$\frac{1}{2}(\cos^2\alpha_0 + \cos\alpha_0 - 1 + \cos^{-1}\alpha_0)$$

where α_0 is the solution of the equation $\sin^2\alpha - 2\cos^3\alpha = 0$ with $0 \leqq \alpha_0 \leqq \pi/2$.

This proves that

$$\frac{\min R(x)}{R(x_0)} \geqq \frac{1}{2}\left(\cos^2\alpha_0 + \cos\alpha_0 - 1 + \frac{1}{\cos\alpha_0}\right). \tag{4}$$

Finally we give an example showing that equality can occur here for $d = 3, 4, \dots$. Again, let $\bar{p}_1, \bar{p}_2, q_1, q_2$ be the vertices of a square such that the diagonals $\bar{p}_1, \bar{p}_2$ and q_1, q_2 meet in a point q and the angle between q and $\bar{p}_1, \bar{p}_2, q_1, q_2$ equals α_0. Now set $p = \cos^{-2}\alpha_0\,\bar{p}_1$ and $p_2 = \cos^{-2}\alpha_0\,\bar{p}_2$ and apply Theorem 3 with the vectors p_1, p_2, q_1, q_2 to get the convex compact set $K_{\min}$. An easy calculation shows that for $K_{\min}$ (4) holds with equality.

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