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Varieties and quasivarieties, generated by two-element preprimal algebras, and their equivalences

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Dedicated to Professor H.-J. Hoehnke on his 63rd birthday

1. Introduction

The subsequent considerations on universal algebras are stimulated by the following situation in the variety of Boolean algebras: It is generated by the two-element Boolean algebra **2** which has the property that every function defined on the two-element set $\{0, 1\}$ is a term function of **2**. This property corresponds to the functional completeness of classical propositional calculus since the class of Boolean algebras constitutes a semantical basis for classical logics. As a generalization one defines a finite nontrivial algebra $\mathbf{A}=\langle A; F\rangle$ to be primal if every function on A is a term function of $\mathbf{A}$. Then many properties of Boolean algebras carry over immediately to varieties generated by a primal algebra. This is already implied by the categorical equivalence between any variety which is generated by a primal algebra and the variety of Boolean algebras.

This equivalence is generalized now in two directions: firstly to preprimal algebras and secondly to quasivarieties. The term functions of a preprimal algebra $\mathbf{A}=\langle A; F\rangle$ constitute a dual atom in the lattice of closed classes of functions defined on A. All two-element preprimal algebras were determined by E. L. Post [11]. Identifying algebras with the same term functions we obtain exactly the following two-element preprimal algebras (up to isomorphisms):

$$\mathbf{C}_3 = \langle\{0, 1\};\ \wedge,\ +,0\rangle,\quad \mathbf{A}_1 = \langle\{0, 1\};\ \wedge, \vee, 0, 1\rangle,$$
$$\mathbf{D}_3 = \langle\{0, 1\};\ d, x+y+z, N\rangle,\quad \mathbf{L}_1 = \langle\{0, 1\};\ +, N, 0, 1\rangle.$$

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Here $\wedge, \vee, +, N$ are the Boolean operations conjunction, disjunction, addition mod 2, and negation. Further d is the ternary operation with $d(x, y, z)=(x\wedge y)\vee \vee(x\wedge z)\vee(y\wedge z)$. Our main result is the following: A quasivariety is equivalent to the quasivariety generated by one of the two-element preprimal algebras if and only if it is generated by a preprimal algebra of a special form. The result can be applied in non-classical logics and in electrical circuit theory. Consider a variety $V_{2'}$ generated by a two-element algebra and assume $V_{2'}=\mathrm{ISP}(\mathbf{2}')$ (I—isomorphisms, S—subalgebras, P—direct products), i.e., assume the quasivariety $QV_{2'}=\mathrm{ISP}(\mathbf{2}')$ generated by $\mathbf{2}'$ agrees with the variety generated by $\mathbf{2}'$. In [2] the algebras $B\in\mathrm{ISP}(\mathbf{2}')$ are called pure dyadic algebras. Boolean algebras and Boolean rings, distributive lattices, implication algebras, median algebras, and Boolean groups are well-known examples of pure dyadic algebras. Let $\mathbf{B}(X)\in V_{2'}$ be the free algebra freely generated by $X=\{x_1, \ldots, x_n\}$, and let $\mathbf{p}, \mathbf{q}$ be two terms of $\mathbf{B}(X)$. The fact that every algebra of $V_{2'}$ is isomorphic to a subdirect power of $\mathbf{2}'$ implies that $\mathbf{p}, \mathbf{q}\in\mathbf{B}(X)$ are identical if for all homomorphisms $h\colon \mathbf{B}(X)\to\mathbf{2}'$ one has $h(\mathbf{p})=h(\mathbf{q})$. In the case of Boolean algebras this property is meaningful in the complexity theory of Boolean functions and the truth table method of classical logics ([8]). Let $\mathscr{K}$ be a variety which, as a category, is equivalent to $V_{2'}$. Then there is a map t from the n-ary terms of $V_{2'}$ to the n-ary terms of $\mathscr{K}$ such that

(i) $t(\mathbf{x}_i)=\mathbf{x}_i$,

(ii) if α and β are self-maps of $\{1, \ldots, n\}$ and $V_{2'}$ satisfies $\mathbf{p}(x_{\alpha 1}, \ldots, x_{\alpha n})= =\mathbf{p}(x_{\beta 1}, \ldots, x_{\beta n})$, then $\mathscr{K}$ satisfies $(t\mathbf{p})(x_{\alpha 1}, \ldots, x_{\alpha n})=(t\mathbf{q})(x_{\beta 1}, \ldots, x_{\beta n})$.

It follows that $\mathscr{K}$ satisfies $(t\mathbf{p})(x_{\alpha 1}, \ldots, x_{\alpha n})=(t\mathbf{q})(x_{\beta 1}, \ldots, x_{\beta n})$ if $h(\mathbf{p})=h(\mathbf{q})$ holds for all homomorphisms $h\colon \mathbf{B}(X)\to\mathbf{2}'$.

2. Preliminaries

Let A be a nonempty finite set. The collection of n-ary operations on A will be denoted by $O_A^{(n)}$ $(n\geqq 1)$. We set $O_A=\bigcup_{n\geqq 1} O_A^{(n)}$. Let ϱ be an h-ary relation on A $(h\geqq 1)$, i.e. $\varrho\subseteqq A^h$. Let $\mathrm{Pol}\,\varrho$ denote the set of all operations from O_A preserving ϱ, i.e. all operations $f\in O_A$ such that ϱ is a subalgebra of $\langle A; f\rangle^h$. A ternary operation $d\in O_A^{(3)}$ is called a majority function if for all $x, y\in A$ we have

$$d(x, x, y) = d(x, y, x) = d(y, x, x) = x.$$

We adopt the terminology of [7] except that polynomials will be called term functions. $T(\mathbf{A})$ denotes the set of term functions of an algebra $\mathbf{A}=\langle A;\ F\rangle$. $\mathbf{A}$ is said to be primal if $T(\mathbf{A})=O_A$. $\mathbf{A}$ is order complete if there is a lattice order $\leqq$ on A such that $\mathrm{Pol}\leqq\,=T(\mathbf{A})$. $\mathbf{A}$ is said to be preprimal if $T(\mathbf{A})\neq O_A$ and the algebra

$\langle A; F\cup\{f\}\rangle$ is primal for every operation $f\in O_A\setminus T(\mathbf{A})$. By a compatible relation of an algebra $\mathbf{A}=\langle A; F\rangle$ we mean a relation ϱ on A such that $F\subseteq \mathrm{Pol}\,\varrho$. The compatible binary reflexive and symmetric relations on $\mathbf{A}$ are called tolerance relations of $\mathbf{A}$. We say a relation ϱ generates an algebra $\mathbf{A}$ if $T(\mathbf{A})=\mathrm{Pol}\,\varrho$, and we write $\mathbf{A}\varrho$ for any such algebra.

For $2\leqq h<\infty$ let $\sigma_h=\{(a_1, \dots, a_h)\in A^h\colon a_i\neq a_j,\ 1\leqq i<j\leqq h\}$. Furthermore, we set $\iota_h=A^h\setminus\sigma_h$. An h-ary relation ϱ on A $(h\geqq 3)$ is totally reflexive if $\varrho\supseteq\iota_h$. A binary relation on A is called trivial if $\varrho=\iota_2$ or $\varrho=A^2$.

We say that an algebra is tolerance-free if it has no nontrivial tolerance relation. An algebra $\mathbf{A}=\langle A; F\rangle$ is said to be semiprimal if every operation on A admitting all subalgebras of $\mathbf{A}$ is a term function of $\mathbf{A}$ and demiprimal if $\mathbf{A}$ has no proper subalgebra and every operation on A admitting all automorphisms of $\mathbf{A}$ is a term function of $\mathbf{A}$. We need the following result from [1].

Theorem 2.1. *Let* $\mathbf{A}=\langle A; F\rangle$ *be a finite algebra with a majority term function. Then an operation on A is a term function of* $\mathbf{A}$ *iff it preserves all compatible binary relations of* $\mathbf{A}$.

From Theorem 2.1 we obtain immediately the following

Corollary 2.2. *Let* $\mathbf{A}=\langle A; F\rangle$ *be a finite algebra with a majority term function. Then* $\mathbf{A}$ *is primal iff it has no nontrivial compatible binary relation. Moreover,* $\mathbf{A}$ *is preprimal iff it has a nontrivial compatible binary relation and for any two nontrivial compatible relations* ϱ_1 *and* ϱ_2 *of* $\mathbf{A}$ *we have* $\mathrm{Pol}\,\varrho_1=\mathrm{Pol}\,\varrho_2$.

We need the following list of preprimal algebras ([12], [5]):

$\mathbf{A}_\leqq$, where $\leqq$ is a lattice order on A, hence $\mathbf{A}_\leqq$ is order complete,

$\mathbf{A}_{\{b\}}$, where $\{b\}$ is a one-element subalgebra of $\mathbf{A}_{\{b\}}$, hence $\mathbf{A}_{\{b\}}$ is semiprimal,

$\mathbf{A}_{s_2}$, where s_2 is a permutation on A without invariant elements and with cycles of the same length 2, hence $\mathbf{A}_{s_2}$ is demiprimal, $|A|=2m$, $m\in N$,

$\mathbf{A}_{\alpha_m}$, where $\alpha_m=\{(x, y, z, e)\colon e=x+y+z\}$, $x+y+z$ is the operation of a Boolean 3-group $\mathbf{G}_3^m=\langle A; x+y+z\rangle$ with $|A|=2^m$, $m\in N$, $m\geqq 1$.

Clearly, $\mathbf{A}_1$, $\mathbf{C}_3$, $\mathbf{D}_3$ and $\mathbf{L}_1$ are preprimal algebras of these forms with $|A|=2$.

Let $\mathscr{L}$ and $\mathscr{K}$ be quasivarieties which are equivalent as categories, i.e., there are functors $G\colon \mathscr{K}\to\mathscr{L}$ and $H\colon \mathscr{L}\to\mathscr{K}$, and for each $\mathbf{A}\in\mathscr{K}$ and $\mathbf{B}\in\mathscr{L}$ there are isomorphisms $\alpha_\mathbf{A}\colon \mathbf{A}\to HG(\mathbf{A})$ and $\beta_\mathbf{B}\colon \mathbf{B}\to GH(\mathbf{B})$ such that for each $g\colon \mathbf{A}\to\mathbf{A}'$ in $\mathscr{K}$ and each $h\colon \mathbf{B}\to\mathbf{B}'$ in $\mathscr{L}$ the following diagrams commute:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{g} & \mathbf{A}' \\ \downarrow{\scriptstyle\alpha_\mathbf{A}} & & \downarrow{\scriptstyle\alpha_{\mathbf{A}'}} \\ HG(\mathbf{A}) & \xrightarrow{HG(g)} & HG(\mathbf{A}') \end{array} \qquad \begin{array}{ccc} \mathbf{B} & \xrightarrow{h} & \mathbf{B}' \\ \downarrow{\scriptstyle\beta_\mathbf{B}} & & \downarrow{\scriptstyle\beta_{\mathbf{B}'}} \\ GH(\mathbf{B}) & \xrightarrow{GH(h)} & GH(\mathbf{B}') \end{array}$$

The question arises, which properties of a quasivariety carry over to equivalent quasivarieties? Necessary conditions are given by

Theorem 2.2. [3] *Let $\mathcal{L}$ and $\mathcal{K}$ be quasivarieties which are equivalent as categories via the functors $G\colon \mathcal{K}\to\mathcal{L}$ and $H\colon \mathcal{L}\to\mathcal{K}$.*

(1) *If $\mathbf{A}\in\mathcal{L}$ is a finite algebra, then $H(\mathbf{A})$ is a finite algebra.*

(2) *For all $\mathbf{A}\in\mathcal{L}$ the subalgebra lattices of $\mathbf{A}$ and $H(\mathbf{A})$ are isomorphic. Therefore the subalgebra lattices of $\mathbf{A}^2$ and $H(\mathbf{A}^2)$ are isomorphic and since $H(\mathbf{A}^2)$ is isomorphic to $H(\mathbf{A})^2$, the subalgebra lattices of $\mathbf{A}^2$ and $H(\mathbf{A})^2$ are isomorphic.*

(3) *H maps subdirectly irreducible algebras to subdirectly irreducible algebras, simple algebras to simple algebras, and tolerance-free algebras to tolerance-free algebras.*

(4) *If $\mathcal{L}$ is the variety generated by some algebra $\mathbf{A}$, then $\mathcal{K}$ is the variety generated by $H(\mathbf{A})$.*

(5) *If $\mathcal{L}$ and $\mathcal{K}$ are varieties and if in $\mathcal{L}$ there exists a majority term then in $\mathcal{K}$ there also exists a majority term; i.e. if $\mathcal{L}$ is the variety generated by $\mathbf{A}$ and $\mathbf{A}$ has a majority function among its term functions then $H(\mathbf{A})$ also has a majority function among its term functions.*

3. Tolerance-free algebras having majority term functions

The two-element preprimal algebras $\mathbf{C}_3$, $\mathbf{A}_1$ and $\mathbf{D}_3$ have majority functions among their algebraic functions ([4]) and admit no nontrivial tolerance relation. By [4] the quasivarieties generated by $\mathbf{C}_3$, $\mathbf{A}_1$ and $\mathbf{D}_3$ agree with the varieties generated by these algebras. Therefore, by Theorem 2.2 (3), (4), (5), varieties equivalent as categories to $V_{\mathbf{C}_3}, V_{\mathbf{A}_1}, V_{\mathbf{D}_3}$ are generated by tolerance-free algebras $H(\mathbf{C}_3)$, $H(\mathbf{A}_1)$, and $H(\mathbf{D}_3)$ having majority functions among their term functions. In order to characterize varieties equivalent to $V_{\mathbf{C}_3}, V_{\mathbf{A}_1}, V_{\mathbf{D}_3}$ we give some properties for tolerance-free algebras having majority term functions.

For a binary relation on A define two n-ary relations ϱ_n and ϱ_n' $(2\leqq n\leqq |A|)$ as follows:

$$\varrho_n = \{(a_1, \ldots, a_n)\in A^n\colon (a_i, u)\in\varrho,\ i = 1, \ldots, n, \text{ for some } u\in A\},$$
$$\varrho_n' = \{(a_1, \ldots, a_n)\in A^n\colon (o, a_i)\in\varrho,\ i = 1, \ldots, n, \text{ for some } o\in A\}.$$

Lemma 3.1. *Let ϱ be a binary relation on A preserved by a majority function $d\in O_A^{(3)}$. If $\varrho\circ\varrho^{-1}=A^2$ $(\varrho^{-1}\circ\varrho=A^2)$, then $\varrho_n=A^n$ $(\varrho_n'=A^n)$ for every $n=2, \ldots, |A|$.*

Proof. We prove the lemma by induction on n. Clearly, $\varrho_2=\varrho\circ\varrho^{-1}=A^2$. Suppose that $\varrho_{n-1}=A^{n-1}$, $2\leqq n\leqq |A|$. From the definition of ϱ_n it follows that $\varrho_n\supseteq\iota_n$, i.e. ϱ_n is totally reflexive. Now, if $(a_1, \ldots, a_n)\in A^n$ then $(a_2, a_2, a_3, a_4, \ldots, a_n)\in\varrho_n$, $(a_1, a_1, a_3, a_4, \ldots, a_n)\in\varrho_n$ and $(a_1, a_2, a_2, a_4, \ldots, a_n)\in\varrho_n$. Therefore

$(a_1, ..., a_n)=(d(a_2, a_1, a_1), d(a_2, a_1, a_2), d(a_3, a_3, a_2), d(a_4, a_4, a_4), ..., d(a_n, a_n, a_n))\in$ $\in \varrho_n$. Hence $\varrho_n=A^n$. (Similarly, we can prove that $\varrho^{-1}\circ\varrho=A^2$ implies $\varrho_n'=A^n$, $n=2, ..., |A|$.)

Lemma 3.2. *Let* $\mathbf{A}=\langle A;\ F\rangle$ *be a tolerance-free algebra admitting a majority term function, and let* ϱ *be a binary nontrivial reflexive compatible relation of* $\mathbf{A}$. *Then* ϱ *is a lattice order.*

Proof. $\varrho\cap\varrho^{-1}$ $(\subseteq\varrho)$ is a tolerance relation of $\mathbf{A}$ distinct from A^2. Therefore $\varrho\cap\varrho^{-1}=\iota_2$, i.e. ϱ is antisymmetric. $\varrho\circ\varrho^{-1}$ and $\varrho^{-1}\circ\varrho$ are tolerance relations distinct from ι_2. Therefore, $\varrho\circ\varrho^{-1}=\varrho^{-1}\circ\varrho=A^2$, which by Lemma 3.1 implies that $\varrho_{|A|}=\varrho_{|A|}'=A^{|A|}$. Hence there are elements $0, 1\in A$ such that $(a, 1)\in\varrho$ and $(0, a)\in\varrho$ for every $a\in A$. Let d be a majority term function of $\mathbf{A}$. It is known [6] that $d(0, a, b)=a\wedge b$ and $d(1, a, b)=a\vee b$ are the infimum and supremum of a and b with respect to ϱ. Finally we show that ϱ is transitive. Let $(a, b)\in\varrho$ and $(b, c)\in\varrho$. Then $d(0, a, b)=a\wedge b=a$ and $d(1, b, c)=b\vee c=c$. Therefore $(a, c)=(d(0, a, b), d(1, b, c))\in\varrho$, which completes the proof.

Lemma 3.3. *Let* $\mathbf{A}=\langle A;\ F\rangle$ *be a tolerance-free algebra with a majority term function admitting no proper subalgebra. Let* ϱ *be a binary nontrivial symmetric compatible relation of* $\mathbf{A}$ *with* $\varrho\cap\iota_2=\emptyset$. *Then* $\varrho=\{(a, s(a)):\ a\in A\}$ *where* s *is an automorphism of* $\mathbf{A}$ *without fixed points and with cycles of equal length* 2.

Proof. Since $\varrho\circ\varrho^{-1}$ and $\varrho^{-1}\circ\varrho$ are tolerance relations of $\mathbf{A}$ it follows that $\varrho\circ\varrho^{-1}, \varrho^{-1}\circ\varrho\in\{\iota_2, A^2\}$. If $\varrho\circ\varrho^{-1}=A^2$, then by Lemma 3.1 $\varrho_{|A|}=A^{|A|}$. Thus there is a $u\in A$ such that $(a, u)\in\varrho$ for every $a\in A$, implying that $(u, u)\in\varrho$, a contradiction. Similarly we can prove that $\varrho^{-1}\circ\varrho\neq A^2$. Hence $\varrho\circ\varrho^{-1}=\varrho^{-1}\circ\varrho=\iota_2$, which implies that $\varrho=\{(a, s(a)):\ a\in A\}$ for a permutation s on A. Clearly, s has no fixed point $(\varrho\cap\iota_2=\emptyset)$. From $\varrho=\varrho^{-1}$ one gets $\varrho^2=\iota_2$. Therefore each cycle of s has length 2.

The proof of the next lemma is given in [6].

Lemma 3.4. *Let* $\mathbf{A}=\langle A;\ F\rangle$ *be a tolerance-free algebra having a majority term function. Then* $\mathbf{A}$ *has at most two compatible lattice orders* ϱ *and* ϱ^{-1}.

Lemma 3.5. *Let* $\mathbf{A}=\langle A;\ F\rangle$ *be an algebra with a majority term function and exactly one proper subalgebra which moreover has exactly one element. Let* $\{b\}$ *be the one-element subalgebra of* $\mathbf{A}$. *Suppose* $\mathbf{A}$ *has exactly three nontrivial binary compatible relations. Then* $\mathbf{A}$ *is a semiprimal algebra of the form* $\mathbf{A}_{\{b\}}$ *and thus preprimal.*

Proof. $\{b\}\times\{b\}$, $A\times\{b\}$, and $\{b\}\times A$ are all nontrivial compatible binary relations of $\mathbf{A}$. Therefore, by Theorem 2.1, $T(\mathbf{A})=\mathrm{Pol}\,(\{b\}\times\{b\})\cap\mathrm{Pol}\,(A\times\{b\})\cap$

$\cap \text{Pol}(\{b\}\times A)=\text{Pol}(\{b\})$, i.e. $\mathbf{A}$ is a semiprimal algebra of the form $\mathbf{A}_{\{b\}}$ and thus preprimal.

We are ready to formulate and prove our first theorem.

Theorem 3.6. *Let* $\mathbf{P}$ *be one of the two-element algebras* $\mathbf{A}_1, \mathbf{C}_3, \mathbf{D}_3$, *and let* $V_{\mathbf{P}}$ *be the variety generated by* $\mathbf{P}$. *Let* $\mathscr{K}$ *be a variety equivalent as a category to* $V_{\mathbf{P}}$. *Then* $\mathscr{K}$ *is generated by one of the preprimal algebras* $\mathbf{A}_{\leqq}$, $\mathbf{A}_{\{b\}}$ *or* A_{s_2}.

Proof. Let $\mathscr{K}$ be a quasivariety which is equivalent as a category to the quasivariety $QV_{\mathbf{P}}$ via some functors $G\colon \mathscr{K}\to QV_{\mathbf{P}}$ and $H\colon QV_{\mathbf{P}}\to\mathscr{K}$. Since $\mathbf{P}$ has a term function which is a majority function, by a result of Jónsson [10], we have $QV_{\mathbf{P}}=V_{\mathbf{P}}$. By Theorem 2.2, $\mathscr{K}$ is the variety generated by the finite algebra $H(\mathbf{P})$ and $H(\mathbf{P})$ is tolerance-free, having a term function which is a majority function. $H(\mathbf{A}_1)$ and $H(\mathbf{D}_3)$ have no proper subalgebras and $H(\mathbf{C}_3)$ has exactly one (one-element) subalgebra. By Theorem 2.2 (2), the subalgebra lattices of $\mathbf{P}^2$ and $H(\mathbf{P})^2$ are isomorphic. Therefore $H(\mathbf{D}_3)$ has exactly one nontrivial compatible binary relation ϱ and $\varrho\cap\iota_2=\emptyset$ holds. By Lemma 3.3, Theorem 2.1, and Corollary 2.2 $H(\mathbf{D}_3)$ is a demiprimal preprimal algebra of the form $\mathbf{A}_{s_2}$. Further, $H(\mathbf{A}_1)$ has exactly two binary nontrivial compatible relations which are reflexive. By Lemma 3.2, Lemma 3.4, Theorem 2.1, and Corollary 2.2 $H(\mathbf{A}_1)$ is an order-complete preprimal algebra $\mathbf{A}_{\leqq}$. $H(\mathbf{C}_3)$ has exactly three nontrivial binary compatible relations. By Lemma 3.5, $H(\mathbf{C}_3)$ is a semiprimal preprimal algebra of the form $\mathbf{A}_{\{b\}}$.

4. Dualities and full dualities of quasivarieties

The next statements concern the category equivalence of a quasivariety generated by any preprimal algebra of the form $\mathbf{A}_{\leqq}$, $\mathbf{A}_{\{b\}}$, $\mathbf{A}_{s_2}$, $\mathbf{A}_{\alpha_m}$ to the quasivariety generated by a two-element preprimal algebra $\mathbf{A}_1, \mathbf{C}_3, \mathbf{D}_3, \mathbf{L}_1$. These considerations rest upon concepts and results of Davey—Werner [3] on dualities and equivalences of quasivarieties.

Let $\mathbf{C}=\langle C;\ F\rangle$ be a finite algebra and let $\mathscr{L}=\text{ISP}(\mathbf{C})$ be the quasivariety generated by $\mathbf{C}$. Let $\mathsf{C}=\langle C;\ \tau, R\rangle$ be a topological relational structure where R is a set of compatible relations of $\mathbf{C}$, and τ is the discrete topology on C. Let $\mathscr{X}$ be the class of all topological relational structures of the same type as C. For $\mathsf{X}, \mathsf{Y}\in\mathscr{X}$ a morphism $X\to Y$ is a map between the carrier sets of X, Y, which preserves the defining relations of X, Y. Let $\mathscr{X}(X, Y)$ denote the set of all continuous morphisms $X\to Y$. A mapping $\Phi\in\mathscr{X}(X, Y)$ is an embedding if it is one-to-one, closed, and for each relation $r\in R$ and $x_1, \ldots, x_n\in X$ we have

$$(\Phi(x_1), \ldots, \Phi(x_n))\in r \Rightarrow (x_1, \ldots, x_n)\in r.$$

An onto-embedding is an isomorphism in $\mathscr{X}$. Let $\mathbf{X}\in\mathscr{X}$ and $Y\subseteq X$. $\mathbf{Y}$ is a closed substructure if the inclusion map $Y\to X$ is an embedding. A power of $\mathbf{C}$ is always endowed with the product topology and the pointwise relations, i.e. the sets

$$\langle i;\, p\rangle := \{x\in C^I : x(i) = p\} \quad \text{with } i\in I \text{ and } p\in C$$

form a subbasis for the topology on C^I. For $x_1, \ldots, x_n\in C^I$ one has

$$(x_1, \ldots, x_n)\in r \Leftrightarrow (\forall i\in I)\big(x_1(i), \ldots, x_n(i)\big)\in r.$$

The subclass of $\mathscr{X}$ consisting of all members isomorphic to a closed substructure of a power of $\mathbf{C}$ is denoted by $\mathscr{R}$. Symbolically, we write $\mathscr{R}=\mathrm{ISP}(\mathbf{C})$.

The following lemma shows the interconnection between the categories $\mathscr{L}$ and $\mathscr{R}$.

Lemma 4.1. *There exists a pair of adjoint contravariant functors* $D\colon \mathscr{L}\to\mathscr{R}$, $E\colon \mathscr{R}\to\mathscr{L}$.

A pair (D, E) as in Lemma 4.1 is called a protoduality. The protoduality is called a duality if for each algebra $\mathbf{A}$ in $\mathscr{L}$ the embedding $e_A\colon \mathbf{A}\to ED(\mathbf{A})$ is an isomorphism.

Let $\mathscr{R}_0\subseteq\mathscr{R}$ be the subcategory consisting of all structures isomorphic to some closed substructure of a power of $\mathbf{C}$. Then the duality (D, E) is called a full duality between $\mathscr{L}$ and $\mathscr{R}_0$ if for all $\mathbf{X}\in\mathscr{R}_0$ the embedding $\varepsilon_X\colon \mathbf{X}\to DE(\mathbf{X})$ is an isomorphism. $\mathbf{C}$ is said to be injective in $\mathscr{R}_0$ (with respect to some class $\mathscr{I}$ of embeddings) if for each embedding $\sigma\colon \mathbf{X}\to\mathbf{Y}$ in $\mathscr{R}_0$ $(\sigma\in\mathscr{I})$, every continuous morphism $\varphi\colon \mathbf{X}\to\mathbf{C}$ extends to a continuous morphism $\psi\colon \mathbf{Y}\to\mathbf{C}$ with $\psi\circ\sigma=\varphi$.

The next statements rest upon the following two conditions (IB) and (EF).

(IB) For every substructure $\mathbf{X}$ of a finite power $\mathbf{C}^n$ of $\mathbf{C}$, each morphism $\varphi\colon \mathbf{X}\to\mathbf{C}$ extends to a term function $\bar{\varphi}\colon C^n\to C$ of $\mathbf{C}$.

(EF) If $\mathbf{X}$ is a proper substructure of some finite $\mathbf{Y}\in\mathscr{R}_0$ then there exist two different morphisms $\varphi, \psi\colon \mathbf{Y}\to\mathbf{C}$ such that $\varphi/X=\psi/X$.

Lemma 4.2. *Let* $\mathscr{L}=\mathrm{ISP}(\mathbf{C})$ *for a finite algebra* $\mathbf{C}=\langle C;\, F\rangle$. *Let* $\mathbf{C}=\langle C;\, \tau, R\rangle$ *be a (finite) relational structure where R is a finite set of compatible relations on* $\mathbf{C}$ *and* $\mathscr{R}=\mathrm{ISP}(\mathbf{C})$. *Suppose the conditions* (IB) *and* (EF) *hold. Then the protoduality* (D, E) *is a full duality between* $\mathscr{L}$ *and* $\mathscr{R}_0$, *and* $\mathbf{C}$ *is injective in* $\mathscr{R}_0$.

Now we assume that $\mathbf{C}$ admits a majority term function.

Lemma 4.3. *Let* $\mathbf{C}=\langle C;\, F\rangle$ *be a finite algebra with a majority term function. Let R be the set of all binary compatible relations on* $\mathbf{C}$. *Then the protoduality* (D, E) *is a duality between* $\mathscr{L}$ *and* $\mathscr{R}_0$, *and* $\mathbf{C}$ *is injective in* $\mathscr{R}_0$. *If* (EF) *holds,* (D, E) *is a full duality between* $\mathscr{L}$ *and* $\mathscr{R}_0$.

We are ready to apply the preceding duality theory to obtain dualities or even full dualities for varieties (quasivarieties) generated by two-element preprimal algebras.

Theorem 4.4. *Let* $\mathbf{2}_P=\langle\{0,1\};\ F\rangle$ *be a two-element preprimal algebra* $(\mathbf{2}_P\in\{\mathbf{A}_1,\mathbf{C}_3,\mathbf{D}_3,\mathbf{L}_1\})$. *Let* $\mathbf{2}_P=\langle\{0,1\};\ \varrho\rangle$ *be a finite relational structure with* $F=\mathrm{Pol}\,\varrho$ *and* $\mathscr{R}=\mathrm{ISP}(\mathbf{2}_P)$. *Then the protoduality is a full duality between* $\mathscr{L}$ *and* $\mathscr{R}_0$ *and* $\mathbf{2}_P$ *is injective in* $\mathscr{R}_0$.

Proof. By Corollary 2.2 for any two nontrivial compatible relations ϱ_1,ϱ_2 of a preprimal algebra $\mathbf{A}=\langle A;\ F\rangle$ we have $F=\mathrm{Pol}\,\varrho_1=\mathrm{Pol}\,\varrho_2$. Therefore we can set $\mathbf{2}_P=\langle\{0,1\};\ \varrho\rangle$ with $F=\mathrm{Pol}\,\varrho$. The algebras $\mathbf{A}_1$, $\mathbf{C}_3$, and $\mathbf{D}_3$ have majority term functions. In view of Lemma 4.3 it is sufficient to prove that condition (EF) is satisfied. We define $\mathbf{A}_1=\langle\{0,1\};\ \leqq\rangle$, $\mathbf{C}_3=\langle\{0,1\};\ 0\rangle$, $\mathbf{D}_3=\langle\{0,1\};\ N\rangle$. In the first case, if $\mathbf{X}\subset\mathbf{Y}\in\mathscr{R}_0$, Y finite, and $a\in Y\setminus X$, then both $(a]=\{y\in Y:\ y\leqq a\}$ and $(a)=\{y\in Y:\ y<a\}$ are ideals such that $X\cap(a]=X\cap(a)$. Thus $\varphi,\psi\colon Y\to\{0,1\}$, $\varphi(x)=0\Leftrightarrow x\leqq a$, $\psi(x)=0\Leftrightarrow x<a$ are two order-preserving maps which agree on X. In the second case, let $\mathbf{X}\subset\mathbf{Y}$ be a substructure of a finite $\mathbf{Y}\in\mathscr{R}_0$, i.e. $0\in\mathbf{X}$ and let $\varphi,\psi\colon Y\to C_3$ with $\varphi(x)=0$ and

$$\psi(x)=\begin{cases}0 & \text{if } x\in X\\ 1 & \text{if } x\notin X.\end{cases}$$

Then φ and ψ are morphisms, $\varphi\neq\psi$ but $\varphi/X=\psi/X$.

Now we consider $\mathbf{D}_3$. Let $\mathbf{X}\subset\mathbf{Y}\in\mathscr{R}_0$, Y finite, i.e. $N\mathbf{X}\subseteq\mathbf{X}$ where N is a permutation on Y with cycles of the same length 2 and without fixed points. Then we consider two proper subsets $X_1,X_2\subset X$ with $X_1=\{x\in X:\ Nx\in X_2\}$, $X_2=\{x\in X:\ Nx\in X_1\}$, $0\in X_1$, $1\in X_2$, $N0=1$. From $Nx\neq x$, $x\in Y$ it follows $X_1\cap X_2=\emptyset$. Further, we have $X_1\cup X_2=X$, X_1 and X_2 can be extended to Y_1 and Y_2, respectively, such that $Y_1=\{x\in Y:\ Nx\in Y_2\}$, $Y_2=\{x\in Y:\ Nx\in Y_1\}$, $Y_1\cap Y_2=\emptyset$, $Y_1\cup Y_2=Y$. We choose

$$\varphi(x)=\begin{cases}0 & \text{if } x\in X_1\\ 1 & \text{if } x\in X_2\\ 0 & \text{if } x\in Y_1\setminus X_1\\ 1 & \text{if } x\in Y_2\setminus X_2\end{cases},\qquad \psi(x)=\begin{cases}0 & \text{if } x\in X_1\\ 1 & \text{if } x\in X_2\\ 1 & \text{if } x\in Y_1\setminus X_1\\ 0 & \text{if } x\in Y_2\setminus X_2\end{cases}.$$

φ and ψ are two distinct morphisms which agree on X.

Finally, we consider $\mathbf{L}_1=\langle\{0,1\},+,\mathrm{N},0,1\rangle$. Let $\mathscr{L}=\mathrm{ISP}(\mathbf{L}_1)$ be the quasivariety generated by $\mathbf{L}_1$ ($\mathscr{L}\neq V_{\mathbf{L}_1}$). The term functions of $\mathbf{L}_1$ are exactly all Boolean functions which preserve $\alpha=\{(x,y,z,e):\ e=x+y+z\}$. Here $x+y+z$ is the ternary operation of the Boolean 3-group $\mathbf{G}_3=\langle\{0,1\};\ x+y+z\rangle$. For $\mathbf{L}_1=\mathbf{G}_3$ condition (IB) is satisfied. $\mathrm{ISP}(\mathbf{G}_3)$ is the variety of Boolean 3-groups. $\mathbf{X}$ being a proper subalgebra of a finite Boolean 3-group $\mathbf{Y}\in\mathcal{R}_0$, we choose a maximal subgroup $\mathbf{Z}$ of $\mathbf{Y}$ containing $\mathbf{X}$. $\mathbf{Y}\setminus\mathbf{Z}$ is simple and thus isomorphic to $\mathbf{L}_1$. Hence we have two homomorphisms $\mathbf{Y}\to\mathbf{L}_1$ with kernels $\mathbf{Z}$ and $\mathbf{Y}$, respectively, which therefore agree on X. Thus condition (EF) is satisfied.

5. Application of the Equivalent Quasivarieties Theorem

In this section we prove that the quasivarieties generated by the preprimal algebras $\mathbf{A}_{\leqq}$, $\mathbf{A}_{\{b\}}$, $\mathbf{A}_{s_2}$, $\mathbf{A}_{\alpha_m}$, respectively, are equivalent as categories to the varieties (quasivarieties) generated by the two-element preprimal algebras $\mathbf{A}_1, \mathbf{C}_3, \mathbf{D}_3, \mathbf{L}_1$. We need the following Equivalent Quasivarieties Theorem [3].

Theorem 5.1. *Assume that the protoduality (D, E) is a full duality between $\mathcal{L}$ and $\mathcal{R}_0$ and assume further that $\mathbf{C}$ is injective in $\mathcal{R}_0$. Then a quasivariety $\mathcal{K}$ is equivalent as a category to the quasivariety $\mathcal{L}$ if and only if the following conditions are satisfied:*

(i) *there is a finite algebra* $\mathbf{Q}$ *in* $\mathcal{K}$ *and a family R of compatible relations on* $\mathbf{Q}$ *such that* $\mathbf{Q}=\langle Q;\ R\rangle$ *is an object of* $\mathcal{R}_0$,

(ii) (a) $\mathcal{K}=\mathrm{ISP}(\mathbf{Q})$,

(b) $\mathbf{C}$ *is isomorphic to a subalgebra of a power of* $\mathbf{Q}$,

(iii) $\mathbf{Q}$ *is injective in* $\mathcal{R}_0$ *(or equivalently,* $\mathbf{Q}$ *is a retract of a finite power of* $\mathbf{C}$*),*

(iv) *for each positive integer n every morphism* $\mathbf{Q}^n\to\mathbf{Q}$ *is a term function on* $\mathbf{Q}$.

If $\mathcal{K}$ is equivalent as a category to $\mathcal{L}$, then $\mathbf{Q}$ *above can be chosen to be* $H(\mathbf{C})$.

Let $\mathbf{2}_P=\langle\{0, 1\};\ F\rangle$ be a two-element preprimal algebra and let $\mathbf{2}_P=\langle\{0, 1\};\ \varrho\rangle$ be a relational structure with $F=\mathrm{Pol}\ \varrho$. We set $\mathcal{L}=\mathrm{ISP}(\mathbf{2}_P)$ and $\mathcal{R}=\mathrm{ISP}(\mathbf{2}_P)$. By Theorem 4.4 (D, E) is a full duality between $\mathcal{L}$ and $\mathcal{R}_0$ and $\mathbf{2}_P$ is injective in $\mathcal{R}_0$. In order to apply Theorem 5.1 for the proof that the quasivariety generated by one of the preprimal algebras $\mathbf{A}_{\leqq}$, $\mathbf{A}_{\{b\}}$, $\mathbf{A}_{s_2}$, $\mathbf{A}_{\alpha_m}$ is equivalent as a category to the quasivariety $\mathcal{L}$ one has to show that conditions (i)—(iv) are satisfied.

Lemma 5.2. *The variety generated by a preprimal algebra* $\mathbf{A}_{\leqq}$ *is category equivalent to* $V_{\mathbf{A}_1}$.

Proof. By Theorem 3.6 $\mathcal{K}=\mathrm{ISP}(\mathbf{A}_{\leqq})$ is the variety generated by $\mathbf{A}_{\leqq}$. It is clear that $\mathbf{C}=\mathbf{A}_1=\langle\{0, 1\};\ \leqq\rangle$, $\mathbf{Q}=\mathbf{A}_{\leqq}$, $\mathbf{Q}=\mathbf{A}_{\leqq}=\langle A;\ \leqq\rangle$ fulfil the conditions (i), (ii) (a), and (iv). $\mathbf{A}_1$ is isomorphic to the substructure of $\mathbf{A}_{\leqq}$ consisting of the least and the greatest element with respect to $\leqq$, i.e. (ii) (b) holds. Then the lattice $P(A)$ of all subsets of A is isomorphic to a finite power of $\mathbf{A}_1$, and the maps σ and τ

given by

$$\sigma\colon \mathbf{A}_{\leqq} \to P(A), \quad \sigma(a) = \{x \in A\colon (x, a) \in \leqq \text{ for all } a \in A\},$$

$$\tau\colon P(A) \to \mathbf{A}_{\leqq}, \quad \tau(B) = \sup B \text{ for all } B \subseteqq A,$$

are order preserving and such that $\sigma\circ\tau = 1_{\mathbf{A}_{\leqq}}$. Hence (iii) holds.

Lemma 5.3. *The variety generated by a preprimal algebra $\mathrm{A}_{\{b\}}$ is category equivalent to V_{C_3}.*

Proof. By Theorem 3.6 we have $\mathscr{K} = \mathrm{ISP}(\mathrm{A}_{\{b\}}) = V_{\mathrm{A}_{\{b\}}}$. For $\mathbf{C} = \mathbf{C}_3 = \langle\{0, 1\}; 0\rangle$, $\mathbf{Q} = \mathrm{A}_{\{b\}}$, $\mathbf{Q} = \mathbf{A}_{\{b\}} = \langle A; b\rangle$, conditions (i), (ii) (a), and (iv) hold. $\mathbf{C}_3$ is isomorphic to a substructure of $\mathbf{A}_{\{b\}}$ consisting of b and any other element of A. Hence (ii) (b) holds. We choose a positive integer n such that $|A| \leqq 2^n$. Then there exist a monomorphism $\sigma\colon \mathbf{A}_{\{b\}} \to \langle\{0, 1\}^n; 0\rangle$ and an epimorphism $\tau\colon \langle\{0, 1\}^n; 0\rangle \to \mathbf{A}_{\{b\}}$ such that $\sigma\circ\tau = 1_{\mathbf{A}_{\{b\}}}$. Hence (iii) holds.

Lemma 5.4. *The variety generated by a preprimal algebra A_{s_2} is category equivalent to V_{D_3}.*

Proof. By Theorem 3.6, we have $\mathscr{K} = \mathrm{ISP}(\mathrm{A}) = V_{\mathrm{A}_{s_2}}$. For $\mathbf{C} = \mathbf{D}_3 = \langle\{0, 1\}; N\rangle$, $\mathbf{Q} = \mathrm{A}_{s_2}$, $\mathbf{Q} = \mathbf{A}_{s_2} = \langle A; N\rangle$, conditions (i), (ii) (a), and (iv) hold. $\mathbf{C}_3$ is isomorphic to a substructure of $\mathbf{A}_{s_2}$ consisting of any two elements a, b, $a \neq b$, of A with $Na = b$, $Nb = a$ ($|A| = 2k$). Hence (ii) (b) holds. We choose n such that $|A| \leqq 2^n$. Without restriction of generality we choose $\mathbf{A}_{s_2} = \langle\{0, 1, \ldots, 2k-1\}; N\rangle$ with $N = (01)(23)\ldots(2k-1\ 2k)$, and $\mathbf{2}^n = \langle\{a_0, a_1, \ldots, a_{2^n-1}\}, N\rangle$. Then we can define a monomorphism $\sigma\colon \mathbf{A}_{s_2} \to \mathbf{2}^n$ by $\sigma(i) = a_i$, $i = 0, \ldots, 2k-1$, and an epimorphism $\tau\colon \mathbf{2}^n \to \mathbf{A}_{s_2}$ by $\tau(a_i) = i$ for $i = 0, \ldots, 2k-1$ and $\tau(a_{2k+i}) = i$ for $i = 0, \ldots, 2^n - 2k$ such that $\sigma\circ\tau = 1_{\mathbf{A}_{s_2}}$. Hence (iii) holds.

Lemma 5.5. *A quasivariety $\mathscr{K}$ is category equivalent to the quasivariety generated by L_1 if and only if it is generated by a preprimal algebra A_{α_m}.*

Proof. Let $\mathscr{L} = \mathrm{ISP}(\mathrm{L}_1)$ be the quasivariety generated by L_1. By Theorem 4.4, for $\mathbf{C} = \mathbf{L}_1 = \mathrm{G}_3 = \langle\{0, 1\}; x+y+z\rangle$, $\mathscr{R} = \mathrm{ISP}(\mathrm{L}_1)$ the protoduality (D, E) is a full duality between $\mathscr{L}$ and $\mathscr{R}_0$, and $\mathbf{L}_1$ is injective in $\mathscr{R}_0$.

Let $\mathscr{K}$ be equivalent to $\mathscr{L} = \mathrm{ISP}(\mathrm{L}_1)$. Then by Theorem 5.1 (i), there exist a finite algebra $\mathbf{Q}$ in $\mathscr{K}$ and a family R of compatible relations of $\mathbf{Q}$ such that $\mathbf{Q} = \langle Q; R\rangle$ is an object of $\mathscr{R}_0$, i.e. $\mathbf{Q}$ is a Boolean 3-group and therefore $\mathbf{Q}$ is a finite power of the two-element Boolean 3-group. By (iv), $\mathbf{Q}$ is a preprimal algebra of the form A_{α_m} with $\alpha_m = \{(x, y, z, e)\colon e = x+y+z\}$ and $x+y+z$ the operation of a Boolean 3-group $\mathrm{G}_3^m = \langle A; x+y+z\rangle$, $|A| = 2^m$, $m > 1$. Conversely, let $\mathrm{ISP}(\mathrm{A}_{\alpha_m})$ be the quasivariety generated by A_{α_m}. Taking $\mathbf{Q} = \mathrm{A}_{\alpha_m}$, $\mathbf{Q} = \mathrm{G}_3^m$, (i), (ii) (a), (b), and (iv) are satisfied. Since G_3 is injective in $\mathscr{R}_0$, $\mathbf{Q}=\mathbf{G}_3^m$ also is injective in $\mathscr{R}_0$. Hence (iii) holds and $\mathrm{ISP}(\mathbf{A}_{\alpha_m})$ is equivalent to $\mathrm{ISP}(\mathbf{L}_1)$.

Finally, by Lemmas 5.2—5.5 and Theorem 3.6 we obtain

Theorem 5.6. *A quasivariety is category equivalent to the quasivariety generated by a two-element preprimal algebra iff it is generated by a preprimal algebra of one of the forms* $\mathbf{A}_{\leqq}$, $\mathbf{A}_{\{b\}}$, $\mathbf{A}_{s_2}$ $(|A|=2k)$, $\mathbf{A}_{\alpha_m}$ $(|A|=2^m)$.

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